(1) $f'(x) : E \to F$ *is surjective for all* $x \in f^{-1}(y)$,
(2) $\operatorname{Ker} f'(x)$ *is complemented in* E *for all* $x \in f^{-1}(y)$.

Under the conditions of Corollary 2.5, one can prove that the manifold topology and the induced subspace topology of $f^{-1}(y)$ coincide. One can also see that the submanifold has an equivalent atlas modelled on closed subspaces of E having complements linearly homeomorphic to F.

References

[1] A. Frölicher and W. Bucher, *Calculus in Vector Spaces without Norm*, Lecture Notes in Math. 30, Springer, Berlin, 1966.
[2] A. Frölicher and A. Kriegl, *Linear Spaces and Differentiation Theory*, Pure Appl. Math., Wiley, Chichester, 1988.
[3] M. W. Hirsch, *Differential Topology*, Springer, New York, 1976.
[4] H. H. Keller, *Differential Calculus in Locally Convex Spaces*, Lecture Notes in Math. 417, Springer, Berlin, 1974.
[5] A. Kriegl, *Eine kartesisch abgeschlossene Kategorie glatter Abbildungen zwischen beliebigen lokalkonvexen Vektorräumen*, Monatsh. Math. 95 (1983), 287–309.
[6] A. Kriegl and P. W. Michor, *Aspects of the theory of infinite dimensional manifolds*, Differential Geom. Appl. 1 (1991), 159–176.
[7] S. Lang, *Differential Manifolds*, Springer, New York, 1988.
[8] U. Seip, *Kompakt erzeugte Vektorräume und Analysis*, Lecture Notes in Math. 273, Springer, Berlin, 1972.

Institute of Mathematics
Helsinki University of Technology
02150 Espoo, Finland
E-mail: shiltune@cc.hut.fi

Received December 3, 1997 (4014)
Revised version October 19, 1998

STUDIA MATHEMATICA 134 (3) (1999)

Embedding of random vectors into continuous martingales

by

E. DETTWEILER (Tübingen)

Abstract. Let E be a real, separable Banach space and denote by $L^0(\Omega, E)$ the space of all E-valued random vectors defined on the probability space Ω. The following result is proved. There exists an extension $\widetilde{\Omega}$ of Ω, and a filtration $(\widetilde{\mathfrak{F}}_t)_{t\ge 0}$ on $\widetilde{\Omega}$, such that for every $X \in L^0(\Omega, E)$ there is an E-valued, continuous $(\widetilde{\mathfrak{F}}_t)$-martingale $(M_t(X))_{t\ge 0}$ in which X is embedded in the sense that $X = M_\tau(X)$ a.s. for an a.s. finite stopping time τ. For $E = \mathbb{R}$ this gives a Skorokhod embedding for all $X \in L^0(\Omega, \mathbb{R})$, and for general E this leads to a representation of random vectors as stochastic integrals relative to a Brownian motion.

1. Introduction. In 1960 Skorokhod [11] proved that for any mean zero, square integrable random variable X there is a Brownian motion $(B_t)_{t\ge 0}$ and a stopping time τ such that X and B_τ have the same distribution. There now exist a series of different proofs of this so-called Skorokhod embedding (cf. [1], [2], [5] and [8]). One possibility to get Skorokhod's result is the following. First one constructs a continuous martingale $(M_s)_{0\le s\le 1}$ such that $X = M_1$ a.s. Then by the theorem of Dubins and Schwarz [6] there exists a Brownian motion $(B_t)_{t\ge 0}$ such that $M_t = B_{\tau_t}$, where $\tau_t = [M](t)$, the quadratic variation of (M_t) at time t. In particular, one gets $X = M_1 = B_{\tau_1}$ a.s.

The Skorokhod embedding is a result for one-dimensional random variables and in general one cannot expect similar results for random vectors with values in $\mathbb{R}^n$ or—even more general—in a Banach space. For this reason we consider another type of embedding, which is not so restricted to the real line. We embed random vectors into continuous vector-valued martingales with the additional nice property that they are stochastic integrals relative to a Brownian motion.

In §2 we first consider the case of integrable Banach space valued random vectors. If $L^1(\Omega, E)$ denotes the space of all integrable random vectors de-

1991 *Mathematics Subject Classification*: 60G44, 60H05, 60B11.
Key words and phrases: Skorokhod embedding, martingale, stochastic integral, Brownian motion.

fined on a probability space Ω and taking values in a real, separable Banach space E, then we prove the following result (Theorem 1). There is a universal extension $\overline{\Omega}$ of Ω and a universal filtration $(\overline{\mathfrak{F}}_t)_{0\le t\le 1}$ on $\overline{\Omega}$ such that every $X \in L^1(\Omega, E)$ can be embedded into an E-valued $(\overline{\mathfrak{F}}_t)$-martingale $M(X) = (M_t(X))_{0\le t\le 1}$ on $\overline{\Omega}$ in the sense that $M_1(X) = X$. The map $X \to M(X)$ is linear and has also nice continuity properties.

In §3 we analyse the structure of the constructed martingales $M(X)$. It is proved that there is one Brownian motion $(B_t)_{t\ge 0}$ such that

$$M_t(X) = \mathbb{E}X + \int_0^t F_s(X)\,dB_s$$

for every martingale $M(X)$, where $F(X)$ is a progressively measurable, E-valued function, which is explicitly constructed from X. In particular, for $t = 1$ we get

$$X = \mathbb{E}X + \int_0^1 F_s(X)\,dB_s.$$

In the last section we first show that the embedding of random vectors into continuous martingales can be done even for non-integrable random vectors (cf. [7] for a related result in a more special situation). If $L^0(\Omega, E)$ denotes the space of all E-valued random vectors, then we show that there is a universal extension $\widetilde{\Omega}$ of Ω, a universal filtration $(\widetilde{\mathfrak{F}}_t)$ on $\widetilde{\Omega}$, and for every $X \in L^0(\Omega, E)$ an $(\widetilde{\mathfrak{F}}_t)$-martingale $M(X) = (M_t(X))_{t\ge 0}$ such that

$$X = M_{\tau_X}(X)$$

for some a.s. finite stopping time τ_X (Theorem 3). Similarly to §3 this leads to a representation

$$X = \int_0^{\tau_X} F_s(X)\,dB_s$$

for every $X \in L^0(\Omega, E)$.

As a by-product we obtain for $E = \mathbb{R}$ a classical Skorokhod embedding for all random variables, integrable or not.

2. A universal embedding of random vectors into continuous martingales. Let $(E, \|\cdot\|)$ denote a real, separable Banach space provided with its Borel field $\mathfrak{B} = \mathfrak{B}(E)$. We write E' for the topological dual of E, and for $x \in E$ and $x' \in E'$ we denote by $\langle x', x\rangle$ or $\langle x, x'\rangle$ the value of the linear form x' at x. If $\Omega = (\Omega, \mathfrak{F}, P)$ is a fixed probability space, then $L^0(\Omega, E)$ denotes the space of all E-valued random vectors defined on Ω. $L^p(\Omega, E)$ $(1 \le p < \infty)$ is the subspace of all $X \in L^0(\Omega, E)$ such that $\|X\|_p := (\mathbb{E}\|X\|^p)^{1/p} < \infty$, and $L_0^p(\Omega, E)$ denotes all $X \in L^p(\Omega, E)$ with $\mathbb{E}X = 0$.

If $\Omega' = (\Omega', \mathfrak{F}', P')$ is another probability space, then the probability space

$$\overline{\Omega} := \Omega \times \Omega' = (\Omega \times \Omega', \mathfrak{F} \otimes \mathfrak{F}', P \otimes P')$$

is called an *extension* of Ω.

If $\overline{\Omega} = \Omega \times \Omega'$ is an extension of Ω, then every $X \in L^0(\Omega, E)$ can be canonically extended to an $\overline{X} \in L^0(\overline{\Omega}, E)$ by defining $\overline{X}(\overline{\omega}) := X(\omega)$ for $\overline{\omega} = (\omega, \omega') \in \overline{\Omega}$. If there is no danger of confusion, we will write again X for the canonical extension $\overline{X}$ of X.

If $\overline{\Omega}$ is an extension of Ω of the form $\overline{\Omega} = \Omega \times \ldots \times \Omega' \times \ldots$ with $\Omega' = \Omega$, then for every $X \in L^0(\Omega, E)$ the random vector $\overline{X}' \in L^0(\overline{\Omega}, E)$, given by

$$\overline{X}'(\overline{\omega}) := X(\omega') \quad \text{for } \overline{\omega} = (\omega, \ldots, \omega', \ldots) \in \overline{\Omega},$$

is independent of the canonical extension $\overline{X}$ of X, and we will call $\overline{X}'$ the *canonical independent copy of X on Ω'*.

For our results we need an additional probability space (S, Σ, Q) with a filtration $(\Sigma_t)_{t\ge 0}$ such that there exists a (Σ_t)-Brownian motion $(B_t)_{t\ge 0}$ on S. We assume that $B_0 = 0$. Then the hitting time τ of $\{-1/2, 1/2\}$ is an a.s. finite stopping time and the stopped Brownian motion $(B_{t\wedge\tau}, \Sigma_t)_{t\ge 0}$ is a bounded, continuous martingale such that

$$Q[B_\tau = -1/2] = Q[B_\tau = 1/2] = 1/2.$$

Now we define

$$\begin{aligned} L_s &:= B_{(-\log(1-s))\wedge\tau}, \\ \mathcal{H}_s &:= \Sigma_{(-\log(1-s))} \quad (\text{for } 0 \le s < 1), \\ L_1 &:= B_\tau \quad \text{and} \quad \mathcal{H}_1 := \sigma(\Sigma_t;\ t \ge 0). \end{aligned} \tag{2.1}$$

Then $(L_s, \mathcal{H}_s)_{0\le s\le 1}$ is a bounded continuous martingale with the quadratic variation

$$\begin{aligned} [L](s) &= (-\log(1-s)) \wedge \tau \\ &= \int_0^s 1_{[0,1-\exp(-\tau)[}(r)\frac{1}{1-r}\,dr \quad (0 \le s \le 1). \end{aligned} \tag{2.2}$$

For a given filtration $(\mathfrak{F}_t)_{t\in I}$ $(I \subset \mathbb{R}_+)$ on a probability space Ω, we denote by $\mathcal{M}(\Omega, (\mathfrak{F}_t)_{t\in I}, E)$ the space of all $(\mathfrak{F}_t)_{t\in I}$-martingales on Ω with values in E.

THEOREM 1. *Let Ω and E be given. Then there exists an extension $\overline{\Omega}$ of Ω, a filtration $(\overline{\mathfrak{F}}_t)_{0\le t\le 1}$ on $\overline{\Omega}$, and a map*

$$M : L^1(\Omega, E) \to \mathcal{M}(\overline{\Omega}, (\overline{\mathfrak{F}}_t)_{0\le t\le 1}, E)$$

with the properties:

(1) *For every* $X \in L^1(\Omega, E)$ *the martingale* $M(X) = (M_t(X))_{0\le t\le 1}$ *is continuous on* $]0,1]$ *and a.s. continuous at* 0.

(2) $M_1(X) = X$ (*more precisely*: $M_1(X) = \overline{X}$, *the canonical extension of* X *to* $\overline{\Omega}$) *and* $M_0(X) = \mathbb{E}X$.

Proof. For every $k \ge 0$ we set $\Omega_k = \Omega$ and $S_k = S$ and define

$$\overline{\Omega} := \prod_{k\ge 0} \Omega_k \times \prod_{k\ge 0} S_k. \tag{2.3}$$

Obviously, $\overline{\Omega}$ is an extension of Ω, and we identify every $X \in L^1(\Omega, E)$ with its canonical extension to $\overline{\Omega}$. We set $Z_0 = X$ and denote by Z_k for $k \ge 1$ the canonical independent copy of X on Ω_k. Similarly, we define for every $k \ge 0$ the process $L^k = (L^k_s)_{0\le s\le 1}$ on $\overline{\Omega}$ by

$$L^k_s(\overline{\omega}) := L_s(\xi_k)$$

for $0 \le s \le 1$ and $\overline{\omega} = ((\omega_j)_{j\ge 0}, (\xi_j)_{j\ge 0})$. If π_{S_k} denotes the projection from $\overline{\Omega}$ onto S_k, then $\mathcal{H}^k_s := \pi^{-1}_{S_k}(\mathcal{H}_s)$ defines a filtration on $\overline{\Omega}$ such that every process L^k is an $(\mathcal{H}^k_s)$-martingale and that the sequence $(L^k)_{k\ge 0}$ is independent. By definition of the process L we have

$$\overline{P}[L^k_1 = -1/2] = \overline{P}[L^k_1 = 1/2] = 1/2,$$

and hence the sequence $(\varepsilon_k)_{k\ge 0}$, given by $\varepsilon_k := 2L^k_1$ for every $k \ge 0$, is an independent sequence of Bernoulli random variables. Now we define, for every $n \ge 0$,

$$\begin{aligned} X_{2^{-n}} &:= 2^{-n} \sum_{k=0}^{2^n-1} Z_k, \\ X^c_{2^{-n}} &:= 2^{-n} \sum_{k=2^n}^{2^{n+1}-1} Z_k, \\ \Delta X_{2^{-n}} &:= \varepsilon_n (X_{2^{-n}} - X^c_{2^{-n}}), \quad \text{and finally} \\ M_t(X) &:= X_{2^{-(n+1)}} + \Delta X_{2^{-n}} \cdot L^n_{2^{n+1}t-1} \quad (\text{for } 2^{-(n+1)} < t \le 2^{-n}). \end{aligned} \tag{2.4}$$

For $t = 2^{-n}$ we obtain from the last definition

$$M_{2^{-n}}(X) = X_{2^{-(n+1)}} + \tfrac{1}{2}\varepsilon_n^2 (X_{2^{-n}} - X^c_{2^{-n}}) = X_{2^{-n}},$$

and the continuity of the processes L^n implies that the process $(M_t(X))_{0<t\le 1}$ is continuous. By the strong law of large numbers for Banach space valued random vectors (cf. [9], p. 131) we have $X_{2^{-n}} \to \mathbb{E}X$ a.s. for $n \to \infty$, and so the definition $M_0(X) = \mathbb{E}X$ gives a process $M(X) = (M_t(X))_{0\le t\le 1}$ which is continuous on $]0,1]$, a.s. continuous at 0, and has the properties $M_1(X) = X$ and $M_0(X) = \mathbb{E}X$.

It remains to prove that there is a filtration $(\overline{\mathfrak{F}}_t)_{0\le t\le 1}$ on $\overline{\Omega}$ such that for every $X \in L^1(\Omega, E)$ the process $M(X)$ is a martingale relative to $(\overline{\mathfrak{F}}_t)_{0\le t\le 1}$.

For every $n \ge 0$ we first define

$$\mathcal{G}_{n+1} := \sigma(\Delta X_{2^{-m}}, X_{2^{-(m+1)}}; X \in L^1(\Omega, E), m \ge n). \tag{2.5}$$

Then we claim that $\mathcal{G}_{n+1}$ and the process L^n are independent.

Indeed, we have

$$\begin{aligned} \mathcal{G}_{n+1} &= \sigma\big(X_{2^{-(m+1)}} + \tfrac{1}{2}\Delta X_{2^{-m}}, X_{2^{-(m+1)}} - \tfrac{1}{2}\Delta X_{2^{-m}}; \\ &\qquad\qquad m \ge n,\ X \in L^1(\Omega, E)\big) \\ &\subset \sigma\big(X_{2^{-(n+1)}} + \tfrac{1}{2}\Delta X_{2^{-n}}, X_{2^{-(n+1)}} - \tfrac{1}{2}\Delta X_{2^{-n}}, X^c_{2^{-m}}, L^m; \\ &\qquad\qquad m \ge n+1,\ X \in L^1(\Omega, E)\big), \end{aligned}$$

and it follows that it is sufficient to prove that L^n and $\sigma\big(X_{2^{-(n+1)}} + \frac{1}{2}\Delta X_{2^{-n}}, X_{2^{-(n+1)}} - \frac{1}{2}\Delta X_{2^{-n}}; X \in L^1(\Omega, E)\big)$ are independent.

Take $X^{(1)}, \ldots, X^{(m)} \in L^1(\Omega, E)$. Then (2.4) gives the E^m-valued random vectors

$$\widetilde{X}_{2^{-n}} = (X^{(1)}_{2^{-n}}, \ldots, X^{(m)}_{2^{-n}}) \quad \text{and} \quad \widetilde{X}^c_{2^{-n}} = ((X^{(1)})^c_{2^{-n}}, \ldots, (X^{(m)})^c_{2^{-n}})$$

such that

$$\Delta\widetilde{X}_{2^{-n}} := (\Delta(X^{(1)})_{2^{-n}}, \ldots, \Delta(X^{(m)})_{2^{-n}}) = \varepsilon_n(\widetilde{X}_{2^{-n}} - \widetilde{X}^c_{2^{-n}}).$$

For $0 \le t_1 < \ldots < t_k \le 1$ we write $\widetilde{L}^n := (L^n_{t_1}, \ldots, L^n_{t_k})$. Then for any $A, B \in \mathfrak{B}(E^m)$ and $C \in \mathfrak{B}(\mathbb{R}^k)$ we get

$$\begin{aligned} \overline{P}\big[\widetilde{X}_{2^{-(n+1)}} &+ \tfrac{1}{2}\Delta\widetilde{X}_{2^{-n}} \in A,\ \widetilde{X}_{2^{-(n+1)}} - \tfrac{1}{2}\Delta\widetilde{X}_{2^{-n}} \in B,\ \widetilde{L}^n \in C\big] \\ &= \overline{P}[\widetilde{X}_{2^{-n}} \in A,\ \widetilde{X}^c_{2^{-n}} \in B,\ \varepsilon_n = 1,\ \widetilde{L}^n \in C] \\ &\quad + \overline{P}[\widetilde{X}^c_{2^{-n}} \in A,\ \widetilde{X}_{2^{-n}} \in B,\ \varepsilon_n = -1,\ \widetilde{L}^n \in C] \\ &= \overline{P}[\widetilde{X}_{2^{-n}} \in A]\overline{P}[\widetilde{X}^c_{2^{-n}} \in B]\overline{P}[\varepsilon_n = 1,\ \widetilde{L}^n \in C] \\ &\quad + \overline{P}[\widetilde{X}^c_{2^{-n}} \in A]\overline{P}[\widetilde{X}_{2^{-n}} \in B]\overline{P}[\varepsilon_n = -1,\ \widetilde{L}^n \in C] \\ &= \overline{P}[\widetilde{X}_{2^{-n}} \in A]\overline{P}[\widetilde{X}_{2^{-n}} \in B]\overline{P}[\widetilde{L}^n \in C], \end{aligned}$$

since $\widetilde{X}_{2^{-n}}$ and $\widetilde{X}^c_{2^{-n}}$ are independent and have the same distribution. For $C = \mathbb{R}^k$ we have in particular

$$\begin{aligned} \overline{P}[\widetilde{X}_{2^{-n}} &\in A]\overline{P}[\widetilde{X}_{2^{-n}} \in B] \\ &= \overline{P}\big[\widetilde{X}_{2^{-(n+1)}} + \tfrac{1}{2}\Delta\widetilde{X}_{2^{-n}} \in A,\ \widetilde{X}_{2^{-(n+1)}} - \tfrac{1}{2}\Delta\widetilde{X}_{2^{-n}} \in B\big], \end{aligned}$$

and the independence of $\widetilde{L}^n$ and $\widetilde{X}_{2^{-(n+1)}} + \frac{1}{2}\Delta\widetilde{X}_{2^{-n}},\ \widetilde{X}_{2^{-(n+1)}} - \frac{1}{2}\Delta\widetilde{X}_{2^{-n}}$ is proved.

Now we define the filtration $(\overline{\mathfrak{F}}_t)_{0\le t\le 1}$ as follows. For $t=0$ we set $\overline{\mathfrak{F}}_0 = \{\emptyset, \overline{\Omega}\}$, and for $t \in]2^{-(n+1)}, 2^{-n}]$ $(n \ge 0)$ we set

$$\overline{\mathfrak{F}}_t := \sigma\Big(\mathcal{G}_{n+1} \cup \bigcup_{m>n} \mathcal{H}_1^m \cup \mathcal{H}^n_{2^{n+1}t-1}\Big). \tag{2.6}$$

The definition of $\mathcal{G}_{n+1}$ shows that $M_t(X)$ is $\overline{\mathfrak{F}}_t$-measurable, hence $M(X)$ is $(\overline{\mathfrak{F}}_t)$-adapted for every $X \in L^1(\Omega, E)$. For the proof of the martingale property it is sufficient to show that for every $n\ge 0$ and $2^{-(n+1)} \le s < t \le 2^{-n}$ we have

$$\mathbb{E}[M_t(X) - M_s(X) \,|\, \overline{\mathfrak{F}}_s] = 0 \quad \text{a.s.}$$

From (2.4) and the $\mathcal{G}_{n+1}$-measurability of $\Delta X_{2^{-n}}$ we get

$$\begin{aligned}\mathbb{E}[M_t(X) - M_s(X) \,|\, \overline{\mathfrak{F}}_s] &= \mathbb{E}[\Delta X_{2^{-n}}(L^n_{2^{n+1}t-1} - L^n_{2^{n+1}s-1}) \,|\, \overline{\mathfrak{F}}_s] \\ &= \Delta X_{2^{-n}} \mathbb{E}[L^n_{2^{n+1}t-1} - L^n_{2^{n+1}s-1} \,|\, \overline{\mathfrak{F}}_s].\end{aligned}$$

By the proof above, L^n is independent of $\overline{\mathfrak{F}}_{2^{-(n+1)}}$, and hence

$$\begin{aligned}&\mathbb{E}[L^n_{2^{n+1}t-1} - L^n_{2^{n+1}s-1} \,|\, \overline{\mathfrak{F}}_s] \\ &\qquad = \mathbb{E}[L^n_{2^{n+1}t-1} - L^n_{2^{n+1}s-1} \,|\, \mathcal{H}^n_{2^{n+1}s-1}] = 0 \quad \text{a.s.,}\end{aligned}$$

since L^n is an $(\mathcal{H}^n_t)$-martingale. This finishes the proof of Theorem 1. ∎

(2.7) REMARKS. (1) A closer look at the construction of the universal map M shows that M is linear. Moreover, it follows from the Doob inequalities that M also has nice continuity properties. Let us define on $\mathcal{M}(\overline{\Omega}, (\overline{\mathfrak{F}}_t)_{0\le t\le 1}, E)$ the following functional: For every $(\overline{\mathfrak{F}}_t)$-martingale $Y = (Y_t)_{0\le t\le 1}$ we set

$$\lambda_1(Y) = \inf\{a \ge 0 \mid s\overline{P}[\sup_{0\le t\le 1} \|Y_t\| > as] \le a \text{ for all } s \ge 0\}.$$

Then $d_1(Y,Z) = \lambda_1(Y-Z)$ for $Y, Z \in \mathcal{M}(\overline{\Omega}, (\overline{\mathfrak{F}}_t)_{0\le t\le 1}, E)$ defines a metric on $\mathcal{M}(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ (cf. [9], p. 12) and we write $\mathcal{M}^{(1)}(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ for $\mathcal{M}(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ provided with this metric. Now Doob's inequality together with Theorem 1 gives

$$s\overline{P}[\sup_{0\le t\le 1} \|M_t(X)\| > s] \le \mathbb{E}\|M_1(X)\| = \mathbb{E}\|X\|,$$

which implies $\lambda_1(M(X))^2 \le \mathbb{E}\|X\|$ (cf. [9], p. 13) for every $X \in L^1(\Omega, E)$. Hence $M : L^1(\Omega, E) \to \mathcal{M}^{(1)}(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ is a continuous linear operator.

Similarly, let $\mathcal{M}^p(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ denote the space of all p-integrable $(\overline{\mathfrak{F}}_t)$-martingales $(p > 1)$ provided with the norm

$$\|Y\|_p := (\mathbb{E} \sup_{0\le t\le 1} \|Y_t\|^p)^{1/p}.$$

The construction in Theorem 1 shows that

$$M(L^p(\Omega, E)) \subset \mathcal{M}^p(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E),$$

and Doob's maximal inequality gives

$$\|M(X)\|_p \le \frac{p}{p-1}\|X\|_p.$$

Hence also the restriction of M to $L^p(\Omega, E)$, which we again denote by M, gives a continuous linear operator

$$M : L^p(\Omega, E) \to \mathcal{M}^p(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$$

in addition to the properties stated in Theorem 1.

(2) If $E = \mathbb{R}$, then the embedding of X into the martingale $M(X)$ can be used to obtain a so-called Skorokhod embedding of X into a Brownian motion. For every single $X \in L^1_0(\Omega, E)$ we may assume that $M(X)$ is continuous on $[0,1]$. From the theorem of Dubins and Schwarz (cf. e.g. [10]) we get the existence of a Brownian motion $(B^X_t, \mathfrak{B}^X_t)_{t\ge 0}$ such that $M_t(X) = B^X_{\tau_t}$ a.s. with $\tau_t = [M(X)](t)$. In particular a.s. $X = B^X_\tau$ for $\tau = [M(X)](1)$. For $X \in L^p_0(\Omega, E)$ $(p > 1)$, the Burkholder–Davis–Gundy inequality (cf. [10]) then implies that $\mathbb{E}\tau^{p/2} \le C\mathbb{E}|X|^p$, where C is a constant not depending on τ and X.

3. Integrable random vectors as stochastic integrals. In this section we use Theorem 1 to show that every $X \in L^1_0(\Omega, E)$ can be represented as a stochastic integral relative to a Brownian motion. We denote by

$$\mathcal{P}(\Omega, (\mathfrak{F}_t)_{t\in I}, E) \quad (I \subset \mathbb{R}_+)$$

the space of all functions $f : I \times \Omega \to E$ which are progressively measurable relative to the filtration $(\mathfrak{F}_t)_{t\in I}$.

THEOREM 2. *There exist an extension $\widetilde{\Omega}$ of Ω, a filtration $(\widetilde{\mathfrak{F}}_t)_{0\le t\le 1}$ on $\widetilde{\Omega}$, an $(\widetilde{\mathfrak{F}}_t)$-Brownian motion $(B_t)_{t\ge 0}$, and a linear map*

$$F : L^1_0(\Omega, E) \to \mathcal{P}(\widetilde{\Omega}, (\widetilde{\mathfrak{F}}_t)_{0\le t\le 1}, E)$$

such that the following properties hold for every $X \in L^1_0(\Omega, E)$. If we denote by $F_t(X) : \widetilde{\Omega} \to E$ the random vector given by $F_t(X)(\widetilde{\omega}) = F(X)(t, \widetilde{\omega})$ for $\widetilde{\omega} \in \widetilde{\Omega}$, then for every $x' \in E'$,

$$\int_0^1 \langle F_t(X), x'\rangle^2\, dt < \infty \quad \textit{a.s., and} \tag{1}$$

$$\langle X, x'\rangle = \int_0^1 \langle F_t(X), x'\rangle\, dB_t. \tag{2}$$

Proof. Let $M : L^1(\Omega, E) \to \mathcal{M}(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ denote the universal map of Theorem 1, and let $(L^k)_{k\ge 0}$ be the independent sequence of stopped

Brownian motions introduced in the proof of Theorem 1. Recall that

$$L_s^k = B^k_{(-\log(1-s))\wedge\tau^k} \quad (0\le s\le 1),$$

where τ^k is the hitting time of $\{-1/2,1/2\}$ of the Brownian motion B^k. Let $(a_k)_{k\ge 0}$ be a fixed sequence of positive numbers such that $\sum_{k\ge 0} a_k < \infty$. Since $L_1^k = B^k_{\tau^k} \in \{-1/2,1/2\}$, the series $\sum_{k\ge 0} a_k L_1^k$ converges everywhere. Now for every $n\ge 0$ and $2^{-(n+1)} < t \le 2^{-n}$ we define

$$N_t = \sum_{k>n} a_k L_1^k + a_n L^n_{2^{n+1}t-1} \quad \text{and} \quad N_0 = 0.$$

Then $(N_t,\overline{\mathfrak{F}}_t)_{0\le t\le 1}$ is a continuous, bounded martingale on $\overline{\Omega}$.

For a given $X\in L_0^1(\Omega,E)$ we define a function $G(X):[0,1]\times\overline{\Omega}\to E$ by

$$G_t(X) := \begin{cases} a_n^{-1}\Delta X_{2^{-n}} & \text{for } 2^{-(n+1)} < t\le 2^{-n},\\ 0 & \text{for } t=0.\end{cases}$$

Then $G(X)$ is left continuous and $(\overline{\mathfrak{F}}_t)$-adapted, and therefore progressively measurable. From (2.4) we obtain

$$X - M_{2^{-n}}(X) = M_1(X) - M_{2^{-n}}(X) = \sum_{k=0}^{n-1} \Delta X_{2^{-k}} L_1^k$$
$$= \sum_{k=0}^{n-1} a_k^{-1}\Delta X_{2^{-k}} a_k L_1^k = \int_{2^{-n}}^{1} G_s(X)\,dN_s,$$

and $\lim_{n\to\infty} M_{2^{-n}}(X) = 0$ a.s. shows that

$$X = \int_0^1 G_s(X)\,dN_s. \tag{3.1}$$

Next we compute the quadratic variation of (N_t). From

$$N_t - N_{2^{-(n+1)}} = a_n L^n_{2^{n+1}t-1} \quad \text{for } 2^{-(n+1)} < t\le 2^{-n}$$

and

$$[L^n](s) = \int_0^s 1_{[0,1-\exp(-\tau^n)[}(r)\frac{1}{1-r}\,dr$$

we obtain

$$[N](t) - [N](2^{-(n+1)}) \tag{3.2}$$
$$= a_n^2 \int_{2^{-(n+1)}}^{t} 1_{[2^{-(n+1)},2^{-n}-2^{-(n+1)}e^{-\tau^n}[}(u)\frac{1}{2^{-n}-u}\,du.$$

Now let $\psi:[0,1]\times\overline{\Omega}\to\mathbb{R}_+$ be the function defined by

$$\psi(t) = a_n^2 1_{]2^{-(n+1)},2^{-n}-2^{-(n+1)}e^{-\tau^n}]}(t)\frac{1}{2^{-n}-t}$$

for $2^{-(n+1)} < t\le 2^{-n}$ $(n\ge 0)$ and $\psi(t)=0$ for $t=0$. Then ψ is $(\overline{\mathfrak{F}}_t)$-adapted and left continuous, and hence progressively measurable, and it follows from (3.2) that

$$[N](t) = \int_0^t \psi(s)\,ds. \tag{3.3}$$

By a theorem of Doob (cf. [10], p. 170), (3.3) implies that there exists an extension $\widetilde{\Omega}$ of $\overline{\Omega}$, an extended filtration $(\widetilde{\mathfrak{F}}_t)_{0\le t\le 1}$, and an $(\widetilde{\mathfrak{F}}_t)$-Brownian motion $(B_t)_{0\le t\le 1}$ such that

$$N_t = \int_0^t \sqrt{\psi(s)}\,dB_s \quad \text{for } 0\le t\le 1. \tag{3.4}$$

Now we define

$$F_t(X) := G_t(X)\sqrt{\psi(t)} \tag{3.5}$$
$$= \Delta X_{2^{-n}} 1_{]0,\tau^n]}(-\log(2-2^{n+1}t))\frac{1}{\sqrt{2^{-n}-t}}$$

for $n\ge 0$ and $2^{-(n+1)} < t\le 2^{-n}$, and $F_0(X)=0$. Then it follows from (3.1) and (3.4) that

$$X = \int_0^1 F_t(X)\,dB_t. \tag{3.6}$$

Assertion (1) follows from

$$\int_0^1 \langle F_t(X),x'\rangle^2\,dt = \sum_{n\ge 0}\langle \Delta X_{2^{-n}},x'\rangle^2\tau^n \tag{3.7}$$
$$= [\langle M(X),x'\rangle](1) < \infty \quad \text{a.s.},$$

where $[\langle M(X),x'\rangle]$ denotes the quadratic variation of the one-dimensional continuous martingale $\langle M(X),x'\rangle$. This finishes the proof of Theorem 2. ■

(3.8) Remarks. (1) The proof of Theorem 2 shows that the stochastic integral $\int_0^1 F_t(X)\,dB_t$ exists as an element of $L^1(\widetilde{\Omega},E)$, whereas in general for a given progressively measurable function $G:[0,1]\times\widetilde{\Omega}\to E$ the stochastic integral $\int_0^1 G_s\,dB_s$ need not exist even under strong integrability conditions such as $\mathbb{E}\int_0^1 \|G_t\|^2\,dt < \infty$ (cf. [3] and [4]). The reason for the existence of $\int_0^1 F_t(X)\,dB_t$ is that $F(X)$ is a very special limit of elementary functions.

(2) The universal map F of Theorem 2 can easily be extended to all of $L^1(\Omega,E)$. Just define $\widetilde{F}(X) := F(X-\mathbb{E}X)$. Then for every $X\in L^1(\Omega,E)$ we have the representation

$$X = \mathbb{E}X + \int_0^1 \widetilde{F}_t(X)\,dB_t.$$

(3) Since $\mathbb{E}\tau^n = \mathbb{E}(B^n_{\tau^n})^2 = 1/4$ and $\mathbb{E}\langle \Delta X_{2^{-n}}, x'\rangle^2 = 2^{-n+1}\mathbb{E}\langle X,x'\rangle^2$ for $X\in L^2(\Omega,E)$ by the definition of $\Delta X_{2^{-n}}$, it follows from (3.7) that in that case we have

$$\mathbb{E}\int_0^1 \langle F_t(X),x'\rangle^2\,dt = \mathbb{E}\langle X,x'\rangle^2 < \infty.$$

Similar to M, also the universal map F has certain continuity properties.

(3.9) COROLLARY. *Let F be the universal map of Theorem 2. Then F has the following continuity properties.*

(1) *For every $0<r<1$ and every $\delta>0$ there exists a constant $C>0$ such that for all $\lambda>0$ and all $X\in L^1_0(\Omega,E)$,*

$$\widetilde{P}[\sup_{r\le t\le 1}\|F_t(X)\|>\lambda] \le \delta + \frac{C}{\lambda}\mathbb{E}\|X\|.$$

(2) *If E is a Banach space of type p $(1\le p\le 2)$, then there exists a constant C_p such that for all $X\in L^p_0(\Omega,E)$,*

$$\mathbb{E}\int_0^1 \|F_t(X)\|^p\,dt \le C_p\mathbb{E}\|X\|^p.$$

Proof. Let $0<r<1$ and $\delta>0$ be given. If $2^{-(m+1)}<r$, then it follows from (3.5) that

$$\sup_{r\le t\le 1}\|F_t(X)\| \le \sup_{0\le n\le m}\|\Delta X_{2^{-n}}\|\sqrt{2^{n+1}e^{\tau^n}}. \tag{3.10}$$

The definition of $\Delta X_{2^{-n}}$ shows that

$$\|\Delta X_{2^{-n}}\| = 2\|X_{2^{-n}} - X_{2^{-(n+1)}}\|. \tag{3.11}$$

We set $\xi_n := 2\sqrt{2^{n+1}e^{\tau^n}}$ $(0\le n\le m)$ and choose the constant $C>0$ such that

$$\widetilde{P}[\max_{0\le n\le m}\xi_n > C/4] < \delta/2. \tag{3.12}$$

Then for every $\lambda>0$ we get

$$\begin{aligned}
&\widetilde{P}[\sup_{r\le t\le 1}\|F_t(X)\|>\lambda]\\
&\quad\le \widetilde{P}[\max_{0\le n\le m}\|\Delta X_{2^{-n}}\|\sqrt{2^{n+1}e^{\tau^n}}>\lambda] \quad \text{(by (3.10))}\\
&\quad\le \widetilde{P}[\max_{0\le n\le m}\|X_{2^{-n}} - X_{2^{-(n+1)}}\|\xi_n>\lambda] \quad \text{(by (3.11))}\\
&\quad\le \widetilde{P}[\max_{0\le n\le m}\|X_{2^{-n}}\|\xi_n>\lambda/2] + \widetilde{P}[\max_{0\le n\le m}\|X_{2^{-(n+1)}}\|\xi_n>\lambda/2]\\
&\quad\le \delta + 2P[\max_{0\le n\le m+1}\|X_{2^{-n}}\|>2\lambda/C] \quad \text{(by (3.12))}\\
&\quad\le \delta + (C/\lambda)\mathbb{E}\|X\|,
\end{aligned}$$

by Doob's inequality applied to the martingale $(X_{2^{-n}})_{n\ge 0}$ $(X = X_{2^{-0}})$. This proves (1).

From (3.5) we obtain, for $1\le p\le 2$,

$$\int_0^1 \|F_t(X)\|^p\,dt = \sum_{n\ge 0}\|\Delta X_{2^{-n}}\|^p\int_{a_n}^{b_n}(2^{-n}-t)^{-p/2}\,dt, \tag{3.13}$$

where $a_n = 2^{-(n+1)}$ and $b_n = 2^{-n}-2^{-(n+1)}\exp(-\tau^n)$. For $1\le p<2$ we get

$$\int_{a_n}^{b_n}(2^{-n}-t)^{-p/2}\,dt \le \int_{2^{-(n+1)}}^{2^{-n}}(2^{-n}-t)^{-p/2}\,dt = \frac{2}{2-p}2^{-\frac{2-p}{2}(n+1)},$$

and for $p=2$ we have

$$\int_{a_n}^{b_n}(2^{-n}-t)^{-1}\,dt = \tau^n.$$

If E is of type p, then

$$\mathbb{E}\|\Delta X_{2^{-n}}\|^p = \mathbb{E}\|X_{2^{-n}} - Y_{2^{-n}}\|^p \le 2^{-np}2^{n+1}C_p\mathbb{E}\|X\|^p,$$

where C_p denotes the type p constant (cf. [9]).

Hence for $1\le p<2$ we have

$$\begin{aligned}
\mathbb{E}\int_0^1\|F_t(X)\|^p\,dt &\le \frac{2}{2-p}C_p\mathbb{E}\|X\|^p\sum_{n\ge 0}2^{-np}2^{n+1}2^{-\frac{2-p}{2}(n+1)}\\
&= \frac{2^{1+p}}{(2-p)(2^{p/2}-1)}C_p\mathbb{E}\|X\|^p,
\end{aligned}$$

and for $p=2$ we have

$$\mathbb{E}\int_0^1\|F_t(X)\|^2\,dt \le C_2\mathbb{E}\|X\|^2\sum_{n\ge 0}2^{-2n}2^{n+1}\mathbb{E}\tau^n = C_2\mathbb{E}\|X\|^2,$$

and the corollary is proved. ∎

4. Non-integrable random vectors as stochastic integrals. In this section we show that Theorem 2 can be extended to the space of all random vectors. The lemma below is the essential step for this result.

(4.1) LEMMA. *Let Ω and E be given. Then there exists an extension $\widehat{\Omega}$ of Ω, a filtration $(\widehat{\mathfrak{F}}_n)_{n\geq 0}$ on $\widehat{\Omega}$, and a map*

$$\widehat{M} : L^0(\Omega, E) \to \mathcal{M}_0(\widehat{\Omega}, (\widehat{\mathfrak{F}}_n)_{n\geq 0}, E),$$

where $\mathcal{M}_0(\widehat{\Omega}, (\widehat{\mathfrak{F}}_n)_{n\geq 0}, E)$ denotes the space of all $(\widehat{\mathfrak{F}}_n)$-martingales $(M_n)_{n\geq 0}$ on $\widehat{\Omega}$ with values in E with $M_0 = 0$, such that for every $X \in L^0(\Omega, E)$,

$$\lim_{n\to\infty} \widehat{M}_n(X) = X \quad \text{eventually a.s.},$$

where $\lim x_n = x$ eventually ($x_n, x \in E$) means convergence relative to the discrete topology.

Proof. We take a probability space Ω' such that on Ω' there exists an independent triangular array $(\varepsilon_{k,j})_{k\geq 1, 1\leq j\leq k}$ of Bernoulli random variables. Then we define

$$\widehat{\Omega} = \Omega \times \Omega',$$
$$\widehat{\mathfrak{F}}_n = \sigma(X \in L^0(\Omega, E),\ \varepsilon_{k,j}\ (k \leq n,\ 1 \leq j \leq k)), \text{ and} \tag{4.2}$$
$$\widehat{\mathfrak{F}}_0 = \{\emptyset, \widehat{\Omega}\}.$$

For every $X \in L^0(\Omega, E)$ we set $X_j = 1_{[j-1\leq \|X\| < j]} X$. Then we define an E-valued process $M^{(j)} = (M_k^{(j)})_{k\geq 0}$ by

$$M_k^{(j)} = 0 \qquad \text{for } k < j,$$
$$M_k^{(j)} = \varepsilon_{j,j} X_j \qquad \text{for } k = j, \text{ and} \tag{4.3}$$
$$M_k^{(j)} = M_{k-1}^{(j)} + 1_{[\varepsilon_{j,j}=-1,\ldots,\varepsilon_{k-1,j}=-1]} \varepsilon_{k,j} 2^{k-j} X_j \qquad \text{for } k > j.$$

Then $M^{(j)} \in \mathcal{M}_0(\widehat{\Omega}, (\widehat{\mathfrak{F}}_k)_{k\geq 0}, E)$ and $\lim_{k\to\infty} M_k^{(j)} = X_j$ eventually a.s. Indeed, for $k > j$ we obtain

$$\mathbb{E}[M_k^{(j)} - M_{k-1}^{(j)} \mid \widehat{\mathfrak{F}}_{k-1}] = \mathbb{E}[1_{[\varepsilon_{j,j}=-1,\ldots,\varepsilon_{k-1,j}=-1]} \varepsilon_{k,j} 2^{k-j} X_j \mid \widehat{\mathfrak{F}}_{k-1}]$$
$$= 1_{[\varepsilon_{j,j}=-1,\ldots,\varepsilon_{k-1,j}=-1]} 2^{k-j} X_j \mathbb{E}[\varepsilon_{k,j} \mid \widehat{\mathfrak{F}}_{k-1}] = 0,$$

because of the obvious independence of $\varepsilon_{k,j}$ and $\widehat{\mathfrak{F}}_{k-1}$. Similarly, we have $\mathbb{E}[M_j^{(j)} \mid \widehat{\mathfrak{F}}_{j-1}] = 0$, and we have proved that $M^{(j)}$ is an $(\widehat{\mathfrak{F}}_k)$-martingale. Now we define

$$\sigma_j := \inf\{k \geq j \mid \varepsilon_{k,j} = +1\}.$$

Then it is easily proved by induction that

$$M_k^{(j)} = -(2^{k-j} - 1) X_j \quad \text{for } k < \sigma_j,$$

and it follows that

$$M_{\sigma_j}^{(j)} = X_j \quad \text{if } \sigma_j < \infty.$$

But $P[\sigma_j = \infty] = P[\varepsilon_{k,j} = -1 \text{ for all } k \geq j] = 0$. This shows that $\lim_{k\to\infty} M_k^{(j)} = X_j$ eventually a.s.

Now we define $M_k = \sum_{j=1}^k M_k^{(j)}$ for $k \geq 1$ and $M_0 = 0$. Then $(M_k)_{k\geq 0} \in \mathcal{M}_0(\widehat{\Omega}, (\widehat{\mathfrak{F}}_k)_{k\geq 0}, E)$. For $N = \bigcup_{j\geq 1} [\sigma_j = \infty]$ we have $P(N) = 0$ and on $[j-1 \leq \|X\| < j] \setminus N$ we have

$$M_k = M_k^{(j)} = X \quad \text{for all } k \geq \sigma_j.$$

So we have proved the lemma. ∎

(4.4) REMARK. The universal map $\widehat{M}$ is no longer linear as the universal maps defined before. The reason is the splitting of X into the integrable parts X_j. For $L^1(\Omega, E)$ there is indeed a linear universal map into $\mathcal{M}_0(\widehat{\Omega}, (\widehat{\mathfrak{F}}_k), E)$ such that the assertion of the lemma holds.

THEOREM 3. *Let Ω and E be given as before. Then there exist an extension $\overline{\Omega}$ of Ω, a filtration $(\overline{\mathfrak{F}}_t)_{t\geq 0}$ on $\overline{\Omega}$, and a map $M : L^0(\Omega, E) \to \mathcal{M}_0(\overline{\Omega}, (\overline{\mathfrak{F}}_t)_{t\geq 0}, E)$ such that for all $X \in L^0(\Omega, E)$,*

(1) *the martingale $M(X) = (M_t(X))_{t\geq 0}$ is continuous on $\bigcup_{n\geq 0}]n, n+1[$ and a.s. continuous at the time points $t \in \mathbb{Z}_+$,*

(2) *$\lim_{t\to\infty} M_t(X) = X$ eventually a.s.*

Proof. Let $\widehat{M}$ be the map of Lemma 4.1 and denote by $\Phi_n : L^0(\Omega, E) \to L_0^1(\widehat{\Omega}, E)$ the map given by

$$\Phi_n(X) := \widehat{M}_n(X) - \widehat{M}_{n-1}(X)$$

for $n \geq 1$. For every $n \geq 1$ we set

$$\Omega^{(n)} := \widehat{\Omega} \times \prod_{k\geq 1} \Omega_{n,k} \times \prod_{k\geq 0} S_{n,k}$$

with $\Omega_{n,k} = \widehat{\Omega}$ for $k \geq 1$ and $S_{n,k} = S$ for $k \geq 0$ (for the following, cf. the proof of Theorem 1), and we denote by $L^{n,k}$ the copy of L on $S_{n,k}$ and by $(\mathcal{H}_t^{n,k})_{0\leq t\leq 1}$ the corresponding filtration. Furthermore, we set

$$\mathcal{G}_{n,k+1} := \sigma(\Delta X_{2^{-j}}, X_{2^{-(j+1)}}; j \geq k,\ X \in \Phi_n(L^0(\Omega, E)))$$

and define

$$\mathfrak{F}_t^{(n)} := \sigma\Big(\mathcal{G}_{n,k+1} \cup \bigcup_{m>k} \mathcal{H}_1^{n,m} \cup \mathcal{H}_{2^{k+1}t-1}^{n,k}\Big)$$

for $2^{-(k+1)} < t \leq 2^{-k}$ $(k \geq 0)$ and $\mathfrak{F}_0^{(n)} = \{\emptyset, \Omega^{(n)}\}$.

From Theorem 1 we get a map

$$M^{(n)} : \Phi_n(L^0(\Omega, E)) \to \mathcal{M}_0(\Omega^{(n)}, (\mathfrak{F}_t^{(n)})_{0\leq t\leq 1}, E)$$

such that for every $Z \in \Phi_n(L^0(\Omega, E))$ the martingale $M^{(n)}(Z)$ is continuous on $]0,1]$, a.s. continuous at 0, and has the property $M_1^{(n)}(Z) = Z$. Finally, we set

$$\overline{\Omega} := \widehat{\Omega} \times \prod_{n\ge1} \Big\{ \prod_{k\ge1} \Omega_{n,k} \times \prod_{k\ge0} S_{n,k} \Big\}$$

and use the same notation $(\mathfrak{F}_t^{(n)})_{0\le t\le1}$ for the filtrations on $\overline{\Omega}$ canonically induced by the filtrations $(\mathfrak{F}_t^{(n)})_{0\le t\le 1}$ on $\Omega^{(n)}$. This gives a sequence of maps

$$M^{(n)} : \Phi_n(L^0(\Omega, E)) \to \mathcal{M}_0(\overline{\Omega}, (\mathfrak{F}_t^{(n)})_{0\le t\le1}, E)$$

with the properties stated above.

Now we can define the filtration $(\overline{\mathfrak{F}}_t)_{t\ge0}$. We set

$$\overline{\mathfrak{F}}_0 := \sigma(L^0(\Omega, E)) \quad \text{and} \quad \overline{\mathfrak{F}}_t := \sigma\Big(\overline{\mathfrak{F}}_0 \cup \bigcup_{m\le n} \mathfrak{F}_1^{(m)} \cup \mathfrak{F}_{t-n}^{(n+1)}\Big)$$

for $n < t \le n+1$ $(n \ge 0)$.

For a given $X \in L^0(\Omega, E)$ we now define $\widetilde{M}_0(X) = 0$ and

$$\widetilde{M}_t(X) := \widetilde{M}_n(X) + \widetilde{M}_{t-n}^{(n+1)}(\Phi_n(X)) \quad \text{for } n < t \le n+1 \ (n \ge 0). \tag{4.5}$$

By the definition of the filtration $(\overline{\mathfrak{F}}_t)_{t\ge0}$ and Theorem 1, every process $\widetilde{M}(X)$ is $(\overline{\mathfrak{F}}_t)$-adapted, continuous on every open interval $]n, n+1[$ $(n \ge 0)$, left continuous at every $n \in \mathbb{N}$ and a.s. right continuous at every $n \in \mathbb{Z}_+$. We now show that $\widetilde{M}(X)$ is also a martingale relative to the filtration $(\overline{\mathfrak{F}}_t)$.

For the martingale property it is sufficient to prove

$$\mathbb{E}[\widetilde{M}_{n+1}(X) - \widetilde{M}_n(X) \,|\, \overline{\mathfrak{F}}_n] = 0 \quad \text{a.s.} \tag{4.6}$$

for $n \ge 1$. Indeed, let $\mathfrak{F}_{m,j}$ denote the σ-algebra on the probability space $\Omega_{m,j}$ $(m \ge 1, j \ge 1)$. From the definition of the filtration $(\overline{\mathfrak{F}}_t)_{t\ge0}$ it follows that

$$\begin{aligned}\overline{\mathfrak{F}}_n &\subset \sigma\Big(\overline{\mathfrak{F}}_0 \cup \sigma\Big(\bigcup_{m\le n} \Phi_m(L^0(\Omega, E))\Big) \cup \bigcup_{\substack{m\le n\\ j\ge1}} \mathfrak{F}_{m,j} \cup \bigcup_{\substack{m\le n\\ k\ge0}} \mathcal{H}_1^{m,k}\Big)\\ &\subset \sigma\Big(\widehat{\mathfrak{F}}_n \cup \bigcup_{\substack{m\le n\\ j\ge1}} \mathfrak{F}_{m,j} \cup \bigcup_{\substack{m\le n\\ k\ge0}} \mathcal{H}_1^{m,k}\Big) =: \mathcal{A}_n.\end{aligned}$$

Since $\widetilde{M}_{n+1}(X) - \widetilde{M}_n(X) = \widehat{M}_{n+1}(X) - \widehat{M}_n(X)$ is independent of

$$\bigcup_{\substack{m\le n\\ j\ge1}} \mathfrak{F}_{m,j} \cup \bigcup_{\substack{m\le n\\ k\ge0}} \mathcal{H}_1^{m,k},$$

we get

$$\begin{aligned}\mathbb{E}[\widetilde{M}_{n+1}(X) - \widetilde{M}_n(X) \,|\, \overline{\mathfrak{F}}_n] &= \mathbb{E}[\mathbb{E}[\widehat{M}_{n+1}(X) - \widehat{M}_n(X) \,|\, \mathcal{A}_n] \,|\, \overline{\mathfrak{F}}_n]\\ &= \mathbb{E}[\mathbb{E}[\widehat{M}_{n+1}(X) - \widehat{M}_n(X) \,|\, \widehat{\mathfrak{F}}_n] \,|\, \overline{\mathfrak{F}}_n] = 0 \quad \text{a.s.}\end{aligned}$$

by Lemma (4.1), and we have proved (4.6).

For a given $X \in L^0(\Omega, E)$, from Lemma (4.1) we have $\lim_{n\to\infty} \widehat{M}_n(X) = X$ eventually a.s. If we set

$$\widehat{\sigma}_X := \inf\{n \ge 1 \mid \widehat{M}_n(X) = X\},$$

then $\widetilde{M}_{\widehat{\sigma}_X}(X) = \widehat{M}_{\widehat{\sigma}_X}(X) = X$ a.s. Now suppose that we have enlarged every $\overline{\mathfrak{F}}_t$ by all $\overline{P}$-nullsets. We use the same notation $(\overline{\mathfrak{F}}_t)_{t\ge0}$ for this enlarged filtration. If we define

$$\sigma_X := \inf\{t \ge 0 \mid \widetilde{M}_t(X) = X\},$$

then the definition of $(\overline{\mathfrak{F}}_t)_{t\ge0}$ and the continuity properties of $\widetilde{M}(X)$ imply that σ_X is an $(\overline{\mathfrak{F}}_t)$-stopping time, and hence the definition

$$M_t(X) := \widetilde{M}_{t\wedge\sigma_X}(X) \quad (t \ge 0)$$

yields an $(\overline{\mathfrak{F}}_t)$-martingale $M(X)$. Since $\sigma_X \le \widehat{\sigma}_X$ we have $\lim_{t\to\infty} M_t(X) = X$ eventually a.s. ∎

For the case $E = \mathbb{R}$ Theorem 3 can be used to get a Skorokhod embedding for every random variable $X \in L^0(\Omega, \mathbb{R})$.

(4.7) COROLLARY. *For every $X \in L^0(\Omega, \mathbb{R})$ there exists an extension $\widetilde{\Omega}$ of $\overline{\Omega}$, a Brownian motion $(B_t, \mathfrak{B}_t)_{t\ge0}$ on $\widetilde{\Omega}$, and a $(\mathfrak{B}_t)$-stopping time τ_X such that $X = B_{\tau_X}$ a.s.*

Proof. For every fixed X we may suppose that the martingale $M(X)$ is continuous. The theorem of Dubins and Schwarz yields an extension $\widetilde{\Omega}$ of $\overline{\Omega}$, a Brownian motion $(B_t, \mathfrak{B}_t)_{t\ge0}$ on $\widetilde{\Omega}$ and an increasing family $(\tau_t)_{t\ge0}$ of $(\mathfrak{B}_t)_{t\ge0}$-stopping times $(\tau_t = [M](t))$ such that $M_t(X) = B_{\tau_t}$ a.s. for every $t \ge 0$. Define $\sigma := \inf\{t \ge 0 \mid M_t(X) = X\}$. Then σ is a stopping time for $(\mathfrak{B}_{\tau_t})$. Set

$$\sigma^{(n)} = \sum_{k\ge0} \frac{k+1}{2^n} 1_{[k/2^n \le \sigma < (k+1)/2^n]}.$$

Then

$$[\tau_{\sigma^{(n)}} \le t] = \bigcup_{k\ge0} [\tau_{(k+1)/2^n} \le t] \cup [k/2^n \le \sigma < (k+1)/2^n]$$

shows that $\tau_{\sigma^{(n)}}$ is a $(\mathfrak{B}_t)$-stopping time. Assuming that $(\mathfrak{B}_t)_{t\ge0}$ is right continuous, we deduce that τ_σ is a $(\mathfrak{B}_t)$-stopping time. Hence we have $X = M_\sigma(X) = B_{\tau_X}$ a.s. for $\tau_X = \tau_\sigma$. ∎

THEOREM 4. *Let Ω and E be given as before. Then there exist an extension $\widehat{\Omega}$ of Ω, a filtration $(\widehat{\mathfrak{F}}_t)_{t\geq 0}$ on $\widehat{\Omega}$, an $(\widehat{\mathfrak{F}}_t)$-Brownian motion $(B_t)_{t\geq 0}$, a map*

$$F : L^0(\Omega, E) \to \mathcal{P}(\widehat{\Omega}, (\widehat{\mathfrak{F}}_t)_{t\geq 0}, E),$$

and a map

$$\tau : L^0(\Omega, E) \to \zeta_{\mathrm{f}}(\widehat{\mathfrak{F}}_t)$$

(where we denote by $\zeta_{\mathrm{f}}(\widehat{\mathfrak{F}}_t)$ the family of all a.s. finite $(\widehat{\mathfrak{F}}_t)$-stopping times), such that for all $x' \in E'$,

$$\int_0^{\tau(X)} \langle F_t(X), x'\rangle^2\, dt < \infty \quad \text{a.s., and} \tag{1}$$

$$\langle X, x'\rangle = \int_0^{\tau(X)} \langle F_t(X), x'\rangle\, dB_t. \tag{2}$$

Proof. We proceed similarly to the proof of Theorem 2. Let $\widetilde{M} : L^0(\Omega, E) \to \mathcal{M}_0(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ denote the universal martingale map constructed in the proof of Theorem 3. Again, we take a fixed sequence $(a_k)_{k\geq 0}$ of positive numbers such that $\sum_{k\geq 0} a_k < \infty$ and start to define a one-dimensional continuous martingale $(N_t)_{t\geq 0}$ relative to the filtration $(\overline{\mathfrak{F}}_t)$. We set $N_0 = 0$. Now suppose that N_n is already defined for $n \geq 0$. Then for $t \in \,]n, n+1]$ with $n + 2^{-(k+1)} < t \leq n + 2^{-k}$ we set

$$N_t := N_n + \sum_{j>k} a_j L_1^{n,j} + a_k L^{n,k}_{2^{k+1}(t-n)-1}. \tag{4.8}$$

Then $(N_t)_{t\geq 0}$ is a continuous $(\overline{\mathfrak{F}}_t)$-martingale, and as in the proof of Theorem 2 one computes that the quadratic variation $[N]$ of N is of the form

$$[N](t) = \int_0^t \psi(s)\, ds,$$

where

$$\psi(t) = a_k^2 1_{]0,\tau^{n,k}]}(-\log(2 - 2^{k+1}(t-n)))\frac{1}{\sqrt{2^{-k} - (t-n)}}$$

for $n + 2^{-(k+1)} < t \leq n + 2^{-k}$.

Now we define $G : L^0(\Omega, E) \to \mathcal{P}(\overline{\Omega}, (\overline{\mathfrak{F}}_t), E)$ by $G_0(X) = 0$ and by

$$G_t(X) := a_k^{-1} \Delta(\widehat{M}_{n+1}(X) - \widehat{M}_n(X))_{2^{-k}}$$

for $n + 2^{-(k+1)} < t \leq n + 2^{-k}$. Similarly to (3.1) we then have

$$\widetilde{M}_t(X) = \int_0^t G_s(X)\, dN_s \quad \text{for all } t \geq 0, \tag{4.9}$$

and also

$$\begin{aligned}[\langle \widetilde{M}(X), x'\rangle](t) &= \int_0^t \langle G_s(X), x'\rangle^2\, d[N](s) \\ &= \int_0^t \langle G_s(X), x'\rangle^2 \psi(s)\, ds < \infty \quad \text{a.s.}\end{aligned} \tag{4.10}$$

for all $t \geq 0$ and $x' \in E'$.

Since $[N]$ has a density relative to the Lebesgue measure, it follows from the theorem of Doob (cf. [10]) that there is an extension $\widehat{\Omega}$ of $\overline{\Omega}$, an extension $(\widehat{\mathfrak{F}}_t)_{t\geq 0}$ of $(\overline{\mathfrak{F}}_t)_{t\geq 0}$, and an $(\widehat{\mathfrak{F}}_t)$-Brownian motion $(B_t)_{t\geq 0}$ such that

$$N_t = \int_0^t \sqrt{\psi(s)}\, dB_s$$

for all $t \geq 0$. Therefore the definition

$$F_t(X) := G_t(X)\sqrt{\psi(t)}$$

for $t \geq 0$ gives a function $F : L^0(\Omega, E) \to \mathcal{P}(\widehat{\Omega}, (\widehat{\mathfrak{F}}_t), E)$ such that for all $x' \in E'$ and all $t \geq 0$,

$$\int_0^t \langle F_s(X), x'\rangle^2\, ds < \infty \quad \text{a.s., and} \tag{4.11}$$

$$\widetilde{M}_t(X) = \int_0^t F_s(X)\, dB_s. \tag{4.12}$$

Finally, we define $\tau : L^0(\Omega, E) \to \zeta_{\mathrm{f}}(\widehat{\mathfrak{F}}_t)$ by

$$\tau(X) := \inf\{t \geq 0 \mid M_t(X) = X\}.$$

Then (1) holds because of (4.11), and (4.12) implies (2). ∎

References

[1] J. Azéma et M. Yor, *Une solution simple au problème de Skorohod*, in: Séminaire de Probabilités XIII, Lecture Notes in Math. 721, Springer, 1979, 90–115 and 625–633.

[2] R. Chacon and J. B. Walsh, *One-dimensional potential embedding*, in: Séminaire de Probabilités X, Lecture Notes in Math. 511, Springer, 1976, 19–23.

[3] E. Dettweiler, *Banach space valued processes with independent increments and stochastic integration*, in: Probability in Banach Spaces IV, Lecture Notes in Math. 990, Springer, 1983, 54–83.

[4] —, *Stochastic integration of Banach space valued functions*, in: L. Arnold and P. Kotelenez (eds.), Stochastic Space-Time Models and Limit Theorems, Reidel, 1985, 53–79.

[5] L. Dubins, *On a theorem of Skorohod*, Ann. Math. Statist. 39 (1968), 2094–2097.
[6] L. Dubins and G. Schwarz, *On continuous martingales*, Proc. Nat. Acad. Sci. U.S.A. 53 (1965), 913–916.
[7] R. M. Dudley, *Wiener functionals as Itô integrals*, Ann. Probab. 5 (1977), 140–141.
[8] —, *Real Analysis and Probability*, Wadsworth & Brooks-Cole, 1989.
[9] J. Hoffmann-Jørgensen, *Probability in Banach spaces*, in: Ecole d'Eté de Probabilités de Saint-Flour VI, Lecture Notes in Math. 598, Springer, 1977, 1–186.
[10] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, 1988.
[11] A. V. Skorohod [A. V. Skorokhod], *Studies in the Theory of Random Processes*, Addison-Wesley, 1965.

Mathematisches Institut der
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen, Germany

Received January 29, 1998
Revised version July 12, 1998 (4043)

STUDIA MATHEMATICA 134 (3) (1999)

Convergence in nonisotropic regions of harmonic functions in $\mathbb{B}^n$

by

CARME CASCANTE and JOAQUIN M. ORTEGA (Barcelona)

To the memory of Joaquin M. Cascante

Abstract. We study the boundedness in $L^p(\mathbb{S}^n)$ of the projections onto spaces of functions with spectrum contained in horizontal strips. We obtain some results concerning convergence along nonisotropic regions of harmonic extensions of functions in $L^p(\mathbb{S}^n)$ with spectrum included in these horizontal strips.

1. Introduction. This work deals with some topics related to the expansion of functions in $L^2(\mathbb{S}^n)$, $\mathbb{S}^n$ the unit sphere in $\mathbb{C}^n$, in terms of harmonic homogeneous polynomials $H(r,s)$ of bidegree (r,s). The projections K_{rs} of $L^2(\mathbb{S}^n)$ onto $H(r,s)$ extend to $L^1(\mathbb{S}^n)$ and permit defining for every $f \in L^1(\mathbb{S}^n)$ the *spectrum* of f, $\operatorname{spec} f = \{(r,s) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : K_{rs} f \neq 0\}$. The orthogonal projection from $\bigoplus_{r,s} H(r,s)$ to $\bigoplus_r H(r,0)$ can be identified with the Cauchy–Szegő projection and it is well known that it can be continuously extended to $L^p(\mathbb{S}^n)$, $p > 1$. What happens if we project to other $\bigoplus_{(r,s)\in\Omega} H(r,s)$? This is a very difficult problem whose answer is not known even for the Fourier expansions when $n = 1$. The first object of this work is to study the boundedness in L^p when Ω is a horizontal strip $\Omega_{0k} = \{(r,s) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \le s \le k\}$.

It is well known that the harmonic extensions of $L^p(\mathbb{S}^n)$ to $\mathbb{B}^n$ have a limit a.e. along nontangential regions and that if the function is in H^p, that is, its spectrum is in $\mathbb{Z}_+ \times \{0\}$, then there is convergence along admissible regions that are tangential in some directions, if $n > 1$. Is there any relation of this fact with the spectrum of the function? The second topic of this work is to study convergence along admissible and other tangential regions of harmonic extensions of functions with spectrum in Ω_{0k}.

The paper is organized as follows: in the second section we show that,

1991 *Mathematics Subject Classification*: 42B20, 32A40.
Key words and phrases: harmonic and holomorphic functions, tangential convergence.
Both authors partially supported by DGICYT Grant PB95-0956-C02-01 and CIRIT Grant 1998-SGR00052.