STUDIA MATHEMATICA 134 (3) (1999)

# Implicit functions from locally convex spaces to Banach spaces

by

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**Abstract.** We first generalize the classical implicit function theorem of Hildebrandt and Graves to the case where we have a Keller $C_{\Pi}^{k}$-map $f$ defined on an open subset of $E \times F$ and with values in $F$, for $E$ an arbitrary Hausdorff locally convex space and $F$ a Banach space. As an application, we prove that under a certain transversality condition the preimage of a submanifold is a submanifold for a map from a Fréchet manifold to a Banach manifold.

**0. Introduction and preliminaries.** Our main objective is the following

IMPLICIT FUNCTION THEOREM. *Let $E$ and $F$ be locally convex spaces with $F$ Banach. Assuming $k \in \mathbb{N} \cup \{\infty\}$, let $f : E \times F \supseteq \operatorname{dom} f \to F$ be a $C^k$-map with $f(x,y) = z$. If $\partial_2 f(x,y) : F \to F$ is bijective, then there exist open sets $U$ and $V$ in $E$ and $F$, respectively, such that $(x,y) \in U \times V \subseteq \operatorname{dom} f$ and the set $f^{-1}(z) \cap (U \times V)$ is a $C^k$-map $E \supseteq U \to F$.*

Here $\mathbb{N}$ is the set of positive integers and $\operatorname{dom} f$ is the domain set of the function $f$. By Definition 0.8 below, a $C^k$-map always has open domain. Basing on the above implicit function theorem, we then prove as a corollary the following

THEOREM. *Assuming $k \in \mathbb{N} \cup \{\infty\}$, let $M$ and $N$ be $C^k$-manifolds with $M$ modelled on Fréchet and $N$ on Banach spaces. Let $f : M \supseteq \operatorname{dom} f \to N$ be a $C^k$-map. If $S$ is a $C^k$-submanifold of $N$ and for all $(y,x) \in f^{-1}|S$ conditions* (1) *and* (2) *below are satisfied, then $f^{-1}(S)$ is a $C^k$-submanifold of $M$.*

(1) *$\pi \circ Tf_x : T_x M \to T_y N \to T_y N / T_y S$ is surjective.*
(2) *$\operatorname{Ker}(\pi \circ Tf_x)$ is complemented in $T_x M$.*

Here $f^{-1}|S$ is the relation $f^{-1}$ restricted to the set $S$. The $\pi$ above is the quotient map $T_y N \to T_y N/T_y S$. If $T_y N/T_y S$ is finite-dimensional, then

1991 *Mathematics Subject Classification*: 58C20, 58C15, 58B10, 46G05.

the requirement (2) is superfluous. We treat real and complex scalars simultaneously. Consequently, the holomorphic case is also included; cf. Remarks 0.12 below and [6; Lemma 2.6].

Now we explain our notion of $C^k$ differentiability, which for *real scalars* is a weakened adaptation of the $C_k$ of [1], and which coincides with $C^k_\Pi$ of [4] for maps between locally convex spaces.

Our fundamental category of maps is that formed by all separated limit (or convergence) spaces as objects and all continuous functions from *some open subset* of the domain space to the range space. A *limit space* $X$ is a set $S$ endowed with a convergence (in [1] a pseudo-topology) $\Lambda$, $X = (S, \Lambda)$. It is *separated* iff for any filter $\Phi$ on $S$ converging to both $x$ and $y$, we have $x = y$. A set $V \subseteq S$ is *open* iff $V \in \Phi$ for every filter $\Phi$ converging to any $x \in V$. A function is *continuous* iff it maps convergent filter bases in its domain to convergent filter bases in its range. A filter base in a subset of a limit space is convergent iff the filter generated by it on the whole space is convergent. We really obtain a category by

0.1. LEMMA. *Let $X$, $Y$ be convergence spaces and $f : X \supseteq \operatorname{dom} f \to Y$ a continuous function with* $\operatorname{dom} f$ *open in $X$. If $B$ is open in $Y$, then $A = f^{-1}(B)$ is open in $X$.*

Proof. If a filter $\Phi$ in $X$ converges to $x \in A$, then $\{U \cap \operatorname{dom} f : U \in \Phi\}$ is a filter base in $\operatorname{dom} f$, converging to $x$. Thus the filter $\Psi = \{V : \exists U \in \Phi : f(U \cap \operatorname{dom} f) \subseteq V \subseteq Y\}$ converges to $f(x)$ in $Y$. Hence $B \in \Psi$. So, for some $U \in \Phi$, we have $f(U \cap \operatorname{dom} f) \subseteq B$. Thus $U \cap \operatorname{dom} f \subseteq f^{-1}(B) = A$. Hence $A \in \Phi$, as $\operatorname{dom} f \in \Phi$. We are done, because $\Phi$ and $x$ are arbitrary. ■

As an embedded full subcategory, we consider continuous maps between separated real ($\mathbb{K} = \mathbb{R}$) or complex ($\mathbb{K} = \mathbb{C}$) convergence vector spaces. In [1], these are called pseudo-topological vector spaces. With every convergence space we have associated a topology in the above way. Conversely, a topology defines a convergence. A topological vector space can thus be interpreted as a convergence vector space. For the basic facts about convergence vector spaces, see, e.g., [1] or [2].

For a convergence vector space $E$, denote by $\mathcal{N}_0 E$ the neighbourhood filter of zero in the topology of $E$. Put $\mathcal{V}_0 = \mathcal{N}_0\mathbb{K}$. A filter $\Phi$ on a vector space $X$ is called *equable* iff $\Phi = [\mathcal{V}_0\Phi]$. Here $[\mathcal{V}_0\Phi]$ is the filter on $X$ generated by the filter base $\mathcal{V}_0\Phi$ formed by the sets $VB = \{\lambda x : \lambda \in V \text{ and } x \in B\}$, where $V \in \mathcal{V}_0$ and $B \in \Phi$. A convergence vector space $E$ is called *equable* iff for every zero filter (i.e., converging to zero) $\Phi$ there exists a smaller equable zero filter $\Psi$ (i.e., $\Psi \subseteq \Phi$). With every convergence vector space $E$, we associate the equable space $E^{\mathrm{eq}}$ with the same underlying vector space and with zero filters being those filters on $E$ for which there is a smaller equable zero filter. Every topological vector space $E$ is equable, because $\mathcal{N}_0 E$ is equable and contained in every zero filter. For more about equable spaces, see [1]. There $E^{\mathrm{eq}}$ is denoted by $E^{\#}$.



With a pair $E, F$ of convergence vector spaces, we associate the equable space $\mathcal{L}_{\mathrm{eq}}(E,F)$ of continuous linear mappings as follows. Call a filter $\zeta$ on $E$ *quasi bounded* iff $[\mathcal{V}_0\zeta]$ is a zero filter in $E$. Let $M = \mathcal{L}(E,F)$ be the vector space of continuous linear maps $E \to F$ (defined on all of $E$). Equip $M$ with the unique vector convergence whose zero filters are those filters $\Phi$ on $M$ such that for all quasi bounded $\zeta$ in $E$, the filter $[\Phi\zeta]$ on $F$ is a zero filter in $F$. In this way, we obtain the convergence vector space $\mathcal{L}_{\mathrm{qb}}(E,F)$. Now put $\mathcal{L}_{\mathrm{eq}}(E,F) = (\mathcal{L}_{\mathrm{qb}}(E,F))^{\mathrm{eq}}$. In [1] the space $\mathcal{L}_{\mathrm{eq}}(E,F)$ is denoted by $L^{\#}(E;F)$. For locally convex spaces $E, F$, we have $\mathcal{L}_{\mathrm{eq}}(E,F) = \mathcal{L}_\Pi(E,F)$, where the latter space is that considered in [4]. By [4; p. 56, Corollary 0.7.4], we have

0.2. LEMMA. *Let $G$, $F$ be locally convex spaces with $F$ normable. Then a filter $\Phi$ on $\mathcal{L}(G,F)$ converges to zero in $\mathcal{L}_{\mathrm{eq}}(G,F)$ iff*

$$(\Theta) \qquad \exists U \in \mathcal{N}_0 G \ \forall W \in \mathcal{N}_0 F \ \exists L \in \Phi : \quad LU \subseteq W.$$

*Moreover, the convergence of $\mathcal{L}_{\mathrm{eq}}(F,F)$ is given by the norm topology of the normed space $\mathcal{L}(F,F)$ of continuous linear maps $F \to F$.*

For a proof, note that condition $(\Theta)$ above is equivalent to the requirement that the filter $\Phi$ is a zero filter in the space $\mathcal{L}_\Theta(G,F)$ of [4].

Let $E$ be a convergence vector space. By a *differentiable curve* in $E$, we mean a function $c : [0,1] \to E$ which is continuous at the points 0 and 1 and, moreover, for $0 < t < 1$ there exists $c'(t) \in E$ such that the filter base

$$\{\{(s-t)^{-1}(c(s) - c(t)) : 0 \le s \le 1 \text{ and } 0 < |s-t| < \delta\} : \delta > 0\}$$

converges to $c'(t)$ in $E$. By separatedness, $c'(t)$ is unique. The function $c' : ]0,1[\, \ni t \mapsto c'(t) \in E$ is the *derivative* of $c$.

0.3. LEMMA. *Let $E$ be a locally convex space and $U$ a closed convex set in $E$. If $c$ is a differentiable curve in $E$ with $c(0) = 0_E \in U$ and* $\operatorname{rng} c' \subseteq U$, *then* $\operatorname{rng} c \subseteq U$, *in particular*, $c(1) \in U$.

Proof. Without restriction, assume real scalars. If the claim is false, then $c(s) \notin U$ for some $s \in\, ]0,1]$. By Hahn–Banach, for some $\ell \in \mathcal{L}(E,\mathbb{R})$ we have $\ell(U) \subseteq\, ]-\infty, 1[$ and $\ell(c(s)) = 1$. The function $\ell \circ c$ satisfies the requirements of the classical intermediate value theorem. Consequently, for some $t \in\, ]0,s[$ we get $1 \le s^{-1} = s^{-1}\ell(c(s)) = (\ell \circ c)'(t) = \ell(c'(t)) \in \ell(U) \subseteq\, ]-\infty, 1[$, a contradiction. ■

Our principal concept of differentiability for maps $f : E \supseteq \operatorname{dom} f \to F$ between locally convex, or, more generally, between (equable) convergence vector spaces, is FB-differentiability, originally introduced in [1]. As auxiliary concepts we need G- and MK-differentiability. Here the letters refer to

Gateaux, Michal and Keller. First we define the corresponding "smallness" or remainder concepts.

0.4. Definitions. Let $E, F$ be convergence vector spaces and $r : E \to F$ a function (defined on all of $E$). Let X denote any of the symbols G, MK or FB. We say that $r : E \to F$, or the triple $\widetilde{r} = (E, F, r)$, is X-*small* iff the corresponding condition below is satisfied. For MK we make the restriction that $E, F$ have to be topological vector spaces.

(G) For all $h \in E$, the filter base $\{\{t^{-1}r(th) : 0 < |t| < \delta\} : \delta > 0\}$ converges to $0_F$ in $F$, i.e., $\lim_{t\to 0} t^{-1}r(th) = 0_F$.

(MK) $\exists U \in \mathcal{N}_0 E\ \forall W \in \mathcal{N}_0 F\ \exists V \in \mathcal{N}_0 E\ \forall t \in \mathbb{K}\setminus\{0\}, h \in U : th \in V \Rightarrow t^{-1}r(th) \in W$.

(FB) For all quasi bounded $\zeta$ in $E$, the filter base $\{\{t^{-1}r(th) : 0 < |t| < \delta$ and $h \in B\} : \delta > 0$ and $B \in \zeta\}$ converges to $0_F$ in $F$.

For a convergence vector space $G$, if $r : E \to F$ is X-small and $b \in \mathcal{L}(F, G)$, then also $b \circ r : E \to G$ is X-small (with the above restriction for MK). Trivially, FB-smallness implies G-smallness. Using 0.5 below, we get MK $\Rightarrow$ FB $\Rightarrow$ G. So G-smallness is the weakest.

0.5. Proposition. *If $\widetilde{r}$ is* MK-*small, then $\widetilde{r}$ is* FB-*small.*

Proof. Let $\widetilde{r} = (E, F, r)$ be MK-small. (So $E, F$ are topological vector spaces.) For arbitrary quasi bounded $\zeta$ in $E$, assuming 0.4(MK), we have to prove that for all $W \in \mathcal{N}_0 F$, there exist $\delta > 0$ and $B \in \zeta$ such that for all $t \in \mathbb{K}$ and $h \in B$, we have the implication: $0 < |t| < \delta \Rightarrow t^{-1}r(th) \in W$.

Since $\mathcal{N}_0 E \subseteq [\mathcal{V}_0\zeta]$, for $U$ as in 0.4(MK), we first get $\varepsilon B_1 \subseteq U$ for some $\varepsilon > 0$ and $B_1 \in \zeta$. Then, for arbitrary $W \in \mathcal{N}_0 F$, put $\varepsilon W$ in the place of $W$ in 0.4(MK). Again using $\mathcal{N}_0 E \subseteq [\mathcal{V}_0\zeta]$, for the $V$ of 0.4(MK), we find $\delta > 0$ and $B_2 \in \zeta$ such that $tB_2 \subseteq V$ for $|t| < \delta$. Now $B = B_1 \cap B_2 \in \zeta$. Let then $t \in \mathbb{K}$ and $h \in B$ be arbitrary with $0 < |t| < \delta$. Putting $s = \varepsilon^{-1}t$ and $k = \varepsilon h$, we have $k \in U$ and $sk = th \in V$. So we get $t^{-1}r(th) = \varepsilon^{-1}(s^{-1}r(sk)) \in \varepsilon^{-1}(\varepsilon W) = W$. ■

From now on, by a *vector map* we mean a triple $\widetilde{f} = (E, F, f)$ such that $E$, $F$ are convergence vector spaces and $f$ is a function defined on some subset of $E$ and with range included in $F$, i.e., $f \subseteq E \times F$. Instead, we may use the phrase "map $f : E \supseteq \operatorname{dom} f \to F$".

0.6. Definitions. Let $\widetilde{f} = (E, F, f)$ be a vector map and let X denote any of the symbols G, MK, FB. Then a pair $(\ell, r)$ is called an X-*expansion* of $\widetilde{f}$ at $x$ iff $x$ is an interior point of $\operatorname{dom} f$ in the topology of $E$, $\ell \in \mathcal{L}(E, F)$, $r : E \to F$ is X-small, and $f(x + h) = f(x) + \ell(h) + r(h)$ for $x + h \in \operatorname{dom} f$.

If X = MK, we require $E, F$ to be topological vector spaces. By a standard argument, the linear mapping $\ell$ is unique. The map $\widetilde{f}$ is said to be X-*differentiable at* $x$ iff there exists an X-expansion at $x$. A map is X-*differentiable* iff it is X-differentiable at every point in its domain. Then the domain is necessarily open. Mere "differentiability" will from now on refer to FB-differentiability.



The (Gateaux) *derivative function* $f'$ of the map $\widetilde{f}$ is defined to be the set of all pairs $(x, \ell)$ such that for some $r$ the pair $(\ell, r)$ is a G-expansion of $\widetilde{f}$ at $x$. Then the function $f' : \operatorname{dom} f' \to \mathcal{L}(E, F)$ is defined at every point at which $\widetilde{f}$ is G-, FB- or MK-differentiable.

The (Frölicher–Bucher) *derivative* (map) of a vector map $\widetilde{f}$ is the vector map $D\widetilde{f} = (E, \mathcal{L}_{\mathrm{eq}}(E, F), f'|A)$ having as domain set the subset $A$ of $\operatorname{dom} f'$ formed by the points at which $\widetilde{f}$ is FB-differentiable. If $\widetilde{f}$ is differentiable, then $A = \operatorname{dom} f' = \operatorname{dom} f$.

Following the conventional usage, we may write $Df$ instead of $D\widetilde{f}$, and also $f$ instead of $\widetilde{f} = (E, F, f)$.

0.7. Proposition. *Let $E, F$ be locally convex spaces with $F$ normable, and let the map $\widetilde{f} = (E, F, f)$ be* G-*differentiable. If $f'$ is continuous $E \supseteq \operatorname{dom} f \to \mathcal{L}_{\mathrm{eq}}(E, F)$, then $\widetilde{f}$ is* MK-*differentiable.*

Proof. In view of Lemma 0.2, the claim follows from [4; p. 76, Th. 1.2.11]. ■

Let now $C_0$ be the (proper) class (not a set) consisting of all continuous vector maps $(E, F, f)$, where $E, F$ are equable and $\operatorname{dom} f$ is open in $E$. Putting $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\infty + 1 = \infty$, we then construct our differentiability classes $C^k$ for $k \in \mathbb{N}_0 \cup \{\infty\}$ as follows.

0.8. Definition. A vector map $f$ belongs to $C^k$ iff there are maps $f^{(l)}$ in $C_0$ for $l \in \mathbb{N}_0, l < k + 1$, such that $f^{(0)} = f$ and $f^{(i)}$ is differentiable with $f^{(i+1)} = Df^{(i)}$ for $i \in \mathbb{N}_0$, $i < k$.

We have the usual recursivity of continuous differentiability: for any $k \in \mathbb{N}_0 \cup \{\infty\}$, $f \in C^{k+1} \Leftrightarrow (f \in C^1$ and $Df \in C^k) \Leftrightarrow (f$ differentiable and $f, Df \in C^k)$.

Consider maps between equable spaces. By a trivial induction, one sees that constants, defined on an open subset of the domain space, are in $C^\infty$. This implies, by the above recursivity, that continuous linear maps are in $C^\infty$. To prove that continuous bilinear maps are in $C^\infty$, using [1; p. 44, 4.2.3], one only has to show the continuity of the derivative. By [1; Propositions 6.3.3, p. 72 and 2.8.3, p. 24], the bilinear composition map $\operatorname{comp} : \mathcal{L}_{\mathrm{eq}}(E, F) \times \mathcal{L}_{\mathrm{eq}}(F, G) \ni (k, \ell) \mapsto \ell \circ k \in \mathcal{L}_{\mathrm{eq}}(E, G)$ is continuous, hence in $C^\infty$.

For maps $\widetilde{f} = (E, F, f)$ and $\widetilde{g} = (F, G, g)$ in $C^0 = C_0$, we define the composition $\widetilde{f}\widetilde{g}$ as $(E, G, g \circ f)$. Here $\operatorname{dom}(g \circ f) = f^{-1}(\operatorname{dom} g)$ is open in

$E$ by Lemma 0.1. If $\widetilde{f}, \widetilde{g} \in C^1$, then $\widetilde{f}\widetilde{g}$ is differentiable and the first order *chain rule* $(g \circ f)' = \mathrm{comp} \circ [f', g' \circ f]$ holds [1; pp. 38–41]. In general, for maps $\widetilde{f}_i = (E, F_i, f_i)$ where $i = 1, 2$, we define the map $[\widetilde{f}_1, \widetilde{f}_2]$ as $[f_1, f_2] : E \supseteq (\mathrm{dom}\, f_1) \cap (\mathrm{dom}\, f_2) \ni x \mapsto (f_1(x), f_2(x)) \in F_1 \times F_2$ (cf. [1; p. 3, 1.3.2]).

Each class $C^k$ forms a category under the composition $(\widetilde{f}, \widetilde{g}) \mapsto \widetilde{f}\widetilde{g}$ defined above. This follows from a general order $k$ chain rule, Proposition 0.11 below. We also get functors $T : C^{k+1} \to C^k$ by forming tangent maps; for $\widetilde{f} = (E, F, f)$ put $T : \widetilde{f} \mapsto T\widetilde{f} = (E \times E, F \times F, Tf)$, where $Tf : (\mathrm{dom}\, f) \times E \ni (x, u) \mapsto (f(x), f'(x)u)$.

0.9. LEMMA. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $(E, F, f) \in C^k$, and let $G$ be equable. Then the implication $b \in \mathcal{L}(F, G) \Rightarrow (E, G, b \circ f) \in C^k$ holds.*

Proof. By the equivalence $g \in C^\infty \Leftrightarrow \forall k \in \mathbb{N}_0 : g \in C^k$, it suffices to prove the claim for $k \in \mathbb{N}_0$. Use induction. The case $k = 0$ is trivial. Let now $f \in C^{k+1}$ and assume our claim to hold for $k$. Then $f$ is differentiable and $f, Df \in C^k$. Trivially then also $b \circ f$ is differentiable and $(b \circ f)'(x) = b \circ f'(x)$ for $x \in \mathrm{dom}\, f$. Thus $(b \circ f)' = \Gamma \circ f'$, where the partial composition map $\Gamma : \mathcal{L}_{\mathrm{eq}}(E, F) \ni \ell \mapsto b \circ \ell = \mathrm{comp}(\ell, b) \in \mathcal{L}_{\mathrm{eq}}(E, G)$ is linear and continuous. By our induction hypothesis, $b \circ f, D(b \circ f) \in C^k$. Hence $b \circ f \in C^{k+1}$. ■

0.10. PROPOSITION. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $\widetilde{f} = (E, F, f)$ and $\widetilde{g} = (E, G, g)$. Then the implication $\widetilde{f}, \widetilde{g} \in C^k \Rightarrow [\widetilde{f}, \widetilde{g}] \in C^k$ holds.*

Proof. Again use induction. To prove the claim for $k = 0$, one only needs to observe that $\mathrm{dom}[f, g] = (\mathrm{dom}\, f) \cap (\mathrm{dom}\, g)$ is open in $E$. Let $k \in \mathbb{N}_0$, $\widetilde{f}, \widetilde{g} \in C^{k+1}$, and assume our claim to hold for $k$. Then $\widetilde{f}, \widetilde{g}$ are differentiable and $\widetilde{f}, \widetilde{g}, D\widetilde{f}, D\widetilde{g} \in C^k$. By the induction hypothesis, then $[\widetilde{f}, \widetilde{g}], [D\widetilde{f}, D\widetilde{g}] \in C^k$. Now one can verify $[\widetilde{f}, \widetilde{g}]$ to be differentiable and $[f, g]'(x) = [f'(x), g'(x)]$ to hold for all $x \in (\mathrm{dom}\, f) \cap (\mathrm{dom}\, g)$. Thus we may write $[f, g]' = \mathrm{X} \circ [f', g']$, where $\mathrm{X} : \mathcal{L}_{\mathrm{eq}}(E, F) \times \mathcal{L}_{\mathrm{eq}}(E, G) \ni (\ell_1, \ell_2) \mapsto [\ell_1, \ell_2] \in \mathcal{L}_{\mathrm{eq}}(E, F \times G)$ is linear and continuous [1; p. 79, 6.4.13, p. 30, 2.9.1, p. 21, 2.6.4]. By the preceding lemma, $D[\widetilde{f}, \widetilde{g}] \in C^k$, hence $[\widetilde{f}, \widetilde{g}] \in C^{k+1}$. ■

0.11. PROPOSITION. *Let $k \in \mathbb{N}_0 \cup \{\infty\}$, $\widetilde{f} = (E, F, f)$ and $\widetilde{g} = (F, G, g)$. Then the implication $\widetilde{f}, \widetilde{g} \in C^k \Rightarrow \widetilde{f}\widetilde{g} \in C^k$ holds.*

Proof. Proceeding by induction, first note that the case $k = 0$ is trivial. Let then $k \in \mathbb{N}_0$, $\widetilde{f}, \widetilde{g} \in C^{k+1}$, and assume our claim to hold for $k$. Then $\widetilde{f}, \widetilde{g}$ are differentiable, $\widetilde{f}, \widetilde{g} \in C^1$ and $\widetilde{f}, \widetilde{g}, D\widetilde{f}, D\widetilde{g} \in C^k$. Put $\widetilde{h} = [D\widetilde{f}, \widetilde{f}(D\widetilde{g})]\widetilde{c}$, where $\widetilde{c}$ is $\mathrm{comp} : \mathcal{L}_{\mathrm{eq}}(E, F) \times \mathcal{L}_{\mathrm{eq}}(F, G) \to \mathcal{L}_{\mathrm{eq}}(E, G)$. Then $\widetilde{c} \in C^\infty \subseteq C^k$. By our induction hypothesis, and by Proposition 0.10 above, we have $\widetilde{f}\widetilde{g}, \widetilde{h} \in C^k$. By the first order chain rule, $\widetilde{f}\widetilde{g}$ is differentiable and $D(\widetilde{f}\widetilde{g}) = \widetilde{h}$. Then $\widetilde{f}\widetilde{g} \in C^{k+1}$ follows. ■



0.12. REMARKS. By [2; p. 114, Remark], [5; p. 306, Lemma] and [4; p. 110, 2.9.5. Remarks (1)], one can obtain the following results for real scalars. Let $E$, $F$ be locally convex spaces, $U$ open in $E$, and $f : U \to F$ a function. Then $f$ is $C^k$ in our sense iff $f \in C^k_\Pi(U, F)$ in the sense of [4]. If, moreover, $k = \infty$ and $E$ is Fréchet, this is equivalent to $f \in C^\infty_\Lambda(U, F)$ in the sense of [4] for $\Lambda = \mathrm{qb}, \mathrm{c}, \mathrm{b}, \mathrm{pk}, \mathrm{k}, \mathrm{s}$. If also $F$ is Fréchet, all these are equivalent to $f \in \mathcal{L}ip^\infty(U, F)$ in the sense of [2].

One can also prove that if $E, F$ are Fréchet, then $f$ being $k$ times (continuously) differentiable in the sense of [8] is equivalent to $f \in C^k_{\mathrm{c}}(U, F)$ in the sense of [4]. Our concept of $C^k$ is, in general, weaker than that considered in [1]. In this respect, there is some confusion in [4], because Keller seems to think that they are equivalent for maps between locally convex spaces. However, this is not the case even for scalar-valued maps defined on the whole of an infinite-dimensional Hilbert space. Cf. [4; p. 11, (2), p. 74, 1.2.7. Remark, p. 97, 2.6.3. Remarks (2)].

Proceeding by induction, using Lemma 0.2, [1; p. 42, Prop. 4.1.1], and the recursive property stated after 0.8, one can directly prove that for normable spaces $E, F$, we have $(E, F, f) \in C^k$ iff $f$ is a $C^k$-function $E \supseteq \mathrm{dom}\, f \to F$ in the classical sense.

0.13. REMARKS. Denote by $\mathcal{L}is(E, F)$ the set of linear homeomorphisms $E \to F$. By the open mapping theorem, for *Fréchet* spaces $E, F$, we have $\ell \in \mathcal{L}is(E, F)$ iff $\ell$ is a continuous linear isomorphism $E \to F$. Later we need to know that for a Banach space $F$, the map $\mathrm{inv} : \mathcal{L}_{\mathrm{eq}}(F, F) \supseteq \mathcal{L}is(F, F) \ni \ell \mapsto \ell^{-1} \in \mathcal{L}_{\mathrm{eq}}(F, F)$ is in $C^\infty$. This follows from the last part of 0.12 by recalling that in Banach space calculus inv is a $C^\infty$-map. In particular, note that $\mathcal{L}is(F, F)$ is open in $\mathcal{L}_{\mathrm{eq}}(F, F)$.

We also need to know that for $(E_1, F, f), (E_2, G, g) \in C^k$, the map $f \times g : E = E_1 \times E_2 \supseteq (\mathrm{dom}\, f) \times (\mathrm{dom}\, g) \ni (x, y) \mapsto (f(x), g(y)) \in F \times G$ is in $C^k$. This follows from 0.10 and 0.11, because we can write $f \times g = [f \circ p, g \circ q]$, where $p : E \to E_1$ and $q : E \to E_2$ are the natural continuous linear projections.

**1. Implicit Function Theorem.** Let $E, F$ be locally convex spaces. Let the topology of $F$ be given by a Banach norm $x \mapsto \|x\|$. Put $G = E \times F$. Let $\mathrm{B}(x, \delta)$, $\mathrm{B}(\delta)$ and $\overline{\mathrm{B}}(\delta)$ be the open and closed balls in $F$ with radius $\delta$ and centered at $x$ and zero, respectively. The partial derivatives below are derivatives of the corresponding partial maps.

1.1. LEMMA. *Let $\varphi : G \supseteq \mathrm{dom}\, \varphi \to F$ be in $C^1$ with $\varphi$ and $\varphi'$ mapping zero to zero. Then for every $\ell \in \mathcal{L}(E, F)$ there exists a continuous seminorm $p$ in $E$, an open neighbourhood $U$ of zero in $E$, and $\delta > 0$ such that for $V = \mathrm{B}(\delta)$ we have $U \times V \subseteq \mathrm{dom}\, \varphi$ and the set $g = \{(x, y) \in$*

$U \times V : \varphi(x,y) + \ell(x) = y\}$ *is a function with* $\operatorname{dom} g = U$, $\operatorname{rng} g \subseteq V$, *and* $\|g(u) - g(v)\| \le p(u-v)$ *for all* $u, v \in U$.

Proof. Put $\Phi = \{B : \exists A \in \mathcal{N}_0 G : \varphi'[A] \subseteq B \subseteq \mathcal{L}(G,F)\}$. Then, by the assumptions, $\Phi$ is a zero filter in $\mathcal{L}_{\mathrm{eq}}(G,F)$. Applying Lemma 0.2, let $W_u \in \mathcal{N}_0 G$ be the $U$ given by condition $(\Theta)$.

The linear mapping $\ell$ being continuous, we have $\ell(U_1) \subseteq \mathrm{B}(1)$ for some absolutely convex $U_1 \in \mathcal{N}_0 E$. We can assume $U_1 \times \mathrm{B}(\delta_1) \subseteq W_u$ for some $\delta_1 > 0$. Let $p_1$ be the Minkowski functional of $U_1$. Put $p = 6p_1$ to obtain the continuous seminorm searched for. Let $q : G \ni (x,y) \mapsto p_1(x) + \frac{1}{2}\|y\|$.

For $\alpha = \inf\{\frac{1}{2}, \frac{1}{3}\delta_1\}$, let $\mathrm{B}(\alpha)$ be the $W$ in condition $0.2(\Theta)$. Then, for some $U_0 \in \mathcal{N}_0 E$ and some $\delta > 0$, we have $\|\varphi'(z)w\| < \alpha$ for $z \in U_0 \times \mathrm{B}(\delta)$ and $w \in W_u$. By the openness of $\operatorname{dom}\varphi$, we can assume $U_0 \subseteq U_1$ and $U_0 \times \mathrm{B}(\delta) \subseteq \operatorname{dom}\varphi$.

By continuity of $\psi : \operatorname{dom}\varphi \ni (x,y) \mapsto \varphi(x,y) + \ell(x)$ we find an absolutely convex open neighbourhood $U$ of zero in $E$ with $U \subseteq U_0$ such that $\|\psi(x, 0_F)\| < \frac{1}{3}\delta$ for $x \in U$. Write $V = \mathrm{B}(\delta)$ and $W = U \times V$. Then $W \subseteq U_0 \times \mathrm{B}(\delta) \subseteq \operatorname{dom}\varphi$.

Next, we consecutively state and prove claims (1) through (6) below. From these claims, we get our lemma.

(1) $\forall z \in W,\ w \in G : \|\varphi'(z)w\| \le q(w)$. To prove this, let $z \in W$, $w = (u,v) \in G$ and assume first $s = q(w) \ne 0$. Then

$$p_1\left(\frac{\alpha}{s}u\right) = \frac{\alpha}{s}p_1(u) \le \frac{\alpha}{s}(q(w)) = \alpha \le \frac{1}{2}$$

and

$$\left\|\frac{\alpha}{s}v\right\| = \frac{\alpha}{s}\|v\| = \frac{2\alpha}{s}\left(\frac{1}{2}\|v\|\right) \le \frac{2\alpha}{s}(q(w)) = 2\alpha \le \frac{2}{3}\delta_1.$$

Hence $\frac{\alpha}{s}u \in U_1$ and $\frac{\alpha}{s}v \in \mathrm{B}(\delta_1)$. So $\frac{\alpha}{s}w = (\frac{\alpha}{s}u, \frac{\alpha}{s}v) \in U_1 \times \mathrm{B}(\delta_1) \subseteq W_u$. Then

$$\|\varphi'(z)w\| = \frac{s}{\alpha}\left\|\varphi'(z)\left(\frac{\alpha}{s}w\right)\right\| < \frac{s}{\alpha}\alpha = s = q(w).$$

Assuming next $q(w) = 0$, for all $s > 0$ we have $p_1(su) = \|sv\| = 0$, hence $sw \in U_1 \times \{0_F\} \subseteq W_u$. So $s\|\varphi'(z)w\| = \|\varphi'(z)(sw)\| < \alpha$. Thus $\|\varphi'(z)w\| = 0 \le q(w)$.

(2) $\forall w, z \in W : \|\varphi(w) - \varphi(z)\| \le q(w-z)$. To prove this, fix $w, z$ and let $c : [0,1] \ni t \mapsto \varphi(z + t(w-z)) - \varphi(z) \in F$. Then $c$ is a differentiable curve in $F$ with $c(0) = 0_F$ and $c'(t) = \varphi'(z + t(w-z))(w-z)$ for $0 < t < 1$. So, by (1), we have $\|c'(t)\| \le q(w-z)$. Taking then $\overline{\mathrm{B}}(q(w-z))$ as the $U$ in Lemma 0.3, we get $\|\varphi(w) - \varphi(z)\| = \|c(1)\| \le q(w-z)$.



(3) $\forall x \in U,\ y \in \overline{\mathrm{B}}(\frac{2}{3}\delta) : \psi(x,y) \in \overline{\mathrm{B}}(\frac{2}{3}\delta)$. For a proof, by (2) we first obtain $\|\varphi(x,y) - \varphi(x,0_F)\| \le q(0_E, y) = \frac{1}{2}\|y\|$. Now recall that $\|\psi(x,0_F)\| < \frac{1}{3}\delta$, to get

$$\begin{aligned}\|\psi(x,y)\| &\le \|\psi(x,y) - \psi(x,0_F)\| + \|\psi(x,0_F)\| \\ &= \|\varphi(x,y) + \ell(x) - (\varphi(x,0_F) + \ell(x))\| + \|\psi(x,0_F)\| \\ &= \|\varphi(x,y) - \varphi(x,0_F)\| + \|\psi(x,0_F)\| \\ &< \tfrac{1}{2}\|y\| + \tfrac{1}{3}\delta \le \tfrac{1}{2}\cdot\tfrac{2}{3}\delta + \tfrac{1}{3}\delta = \tfrac{2}{3}\delta.\end{aligned}$$

(4) $\forall x \in U, y, z \in V : \|\psi(x,y) - \psi(x,z)\| \le \frac{1}{2}\|y - z\|$. As in (3), we calculate:

$$\begin{aligned}\|\psi(x,y) - \psi(x,z)\| &= \|\varphi(x,y) + \ell(x) - (\varphi(x,z) + \ell(x))\| \\ &= \|\varphi(x,y) - \varphi(x,z)\| \le q(0_E, y-z) = \tfrac{1}{2}\|y-z\|.\end{aligned}$$

By (3) and (4), for a fixed $x \in U$, the function $\{(v,z) : (x,v,z) \in \psi\}$, i.e., $y \mapsto \psi(x,y)$, is contractive $\overline{\mathrm{B}}(\frac{2}{3}\delta) \to \overline{\mathrm{B}}(\frac{2}{3}\delta)$. Thus $(x,y,y) \in \psi$ for a unique $y \in \overline{\mathrm{B}}(\frac{2}{3}\delta) \subseteq \mathrm{B}(\delta) = V$. Then $g = \{(x,y) : (x,y,y) \in \psi\} \cap (U \times V)$ is a function $U \to V$.

(5) $\forall u \in E : \|\ell(u)\| \le 2p_1(u)$. For a proof, assuming first $s = p_1(u) \ne 0$, we have $p_1(\frac{1}{2s}u) = \frac{1}{2s}p_1(u) = \frac{1}{2}$. Hence $\frac{1}{2s}u \in U_1$. Thus $\ell(\frac{1}{2s}u) \in \mathrm{B}(1)$, and we get $\|\ell(u)\| = 2s\|\ell(\frac{1}{2s}u)\| \le 2s = 2p_1(u)$. Assuming now $p_1(u) = 0$, for all $s > 0$ we have $su \in U_1$, hence $s\|\ell(u)\| = \|\ell(su)\| < 1$. Thus $\|\ell(u)\| = 0 \le 2p_1(u)$.

(6) $\forall u, v \in U : \|g(u) - g(v)\| \le p(u-v)$. To prove this, write $x = g(u)$ and $y = g(v)$, so that $x = \psi(u,x) = \varphi(u,x) + \ell(u)$ and $y = \psi(v,y) = \varphi(v,y) + \ell(v)$. By (2) and (5), we obtain

$$\begin{aligned}\|x - y\| &= \|\varphi(u,x) - \varphi(v,y) + \ell(u-v)\| \\ &\le \|\varphi(u,x) - \varphi(v,y)\| + \|\ell(u-v)\| \\ &\le q(u-v, x-y) + 2p_1(u-v) = 3p_1(u-v) + \tfrac{1}{2}\|x-y\|,\end{aligned}$$

from which we get $\|g(u) - g(v)\| = \|x - y\| \le 6p_1(u-v) = p(u-v)$. ■

1.2. Lemma. *Let* $f : G \supseteq \operatorname{dom} f \to F$ *be in* $C^1$ *with* $f(x_0, y_0) = 0_F$ *and* $\partial_2 f(x_0,y_0)$ *bijective* $F \to F$. *Then there exists a continuous seminorm* $p$ *in* $E$, *an open neighbourhood* $U$ *of* $x_0$ *in* $E$, *and* $\delta > 0$ *such that for* $V = \mathrm{B}(y_0, \delta)$ *and* $g = f^{-1}(0_F) \cap (U \times V)$, *we have* $U \times V \subseteq \operatorname{dom} f$ *and* $g$ *is a function* $U \to V$ *satisfying* $\|g(u) - g(v)\| \le p(u-v)$ *for all* $u, v \in U$.

Proof. Put $\ell_1 = \partial_1 f(x_0,y_0)$ and $\ell_2 = \partial_2 f(x_0,y_0)$. Then $\ell_1 \in \mathcal{L}(E,F)$ and $\ell_2 \in \mathcal{L}is(F,F)$, by 0.13. Hence $\ell = -\ell_2^{-1} \circ \ell_1 \in \mathcal{L}(E,F)$. Defining $\psi, \varphi : G \supseteq -(x_0,y_0) + \operatorname{dom} f \to F$ by $\psi(x,y) = y - (\ell_2^{-1} \circ f)(x_0 + x, y_0 + y)$ and $\varphi(x,y) = \psi(x,y) - \ell(x)$, we have $\psi, \varphi \in C^1$ and $\varphi(0_E, 0_F) = 0_F$.

Trivially, $f^{-1}(0_F) = \{(x_0 + x, y_0 + y) : (x, y, y) \in \psi\}$. Moreover, for $(u, v) \in G$ we have

$$\begin{aligned}\varphi'(0_E, 0_F)(u, v) &= \psi'(0_E, 0_F)(u, v) - \ell(u)\\ &= v - \ell_2^{-1}(f'(x_0, y_0)(u, v)) + \ell_2^{-1}(\ell_1 u)\\ &= v - \ell_2^{-1}(\ell_1 u + \ell_2 v) + \ell_2^{-1}(\ell_1 u) = 0_F.\end{aligned}$$

So also $\varphi'$ maps zero to zero.

By Lemma 1.1, there exists a continuous seminorm $p$ in $E$, an open neighbourhood $U_0$ of zero in $E$, and $\delta > 0$ such that the conclusion of 1.1 holds with $U$ replaced by $U_0$, and $g$ replaced by $g_0 = \{(x, y) \in U_0 \times \mathrm{B}(\delta) : \varphi(x, y) + \ell(x) = y\}$. Then for $U = x_0 + U_0$, we get our claim, since $f^{-1}(0_F) \cap (U \times V) = \{(x_0 + x, y_0 + y) : (x, y) \in g_0\}$. ■

1.3. Lemma. *Let $f : G \supseteq \operatorname{dom} f \to F$ be in $C^1$ with $f(x_0, y_0) = 0_F$ and $\partial_2 f(x_0, y_0)$ bijective $F \to F$. Then there exists a continuous seminorm $p$ in $E$, an open neighbourhood $U$ of $x_0$ in $E$, and $\delta > 0$ such that for $V = \mathrm{B}(y_0, \delta)$ we have $U \times V \subseteq \operatorname{dom} f$ and the set $g = f^{-1}(0_F) \cap (U \times V)$ is a continuous function $U \to V$ satisfying*

(1) $\partial_2 f(U \times V) \subseteq \mathcal{L}is(F, F)$,
(2) $(E, F, g)$ *is* MK-*differentiable*,
(3) $g'(x) = -(\partial_2 f(x, g(x)))^{-1} \circ (\partial_1 f(x, g(x)))$ *for all* $x \in U$.

Proof. For $j_2 : F \ni v \mapsto (0_E, v) \in G$, the partial composition map $\Gamma : \mathcal{L}_{\mathrm{eq}}(G, F) \ni \ell \mapsto \ell \circ j_2 \in \mathcal{L}_{\mathrm{eq}}(F, F)$ is continuous. From $\partial_2 f = \Gamma \circ f'$ and the continuity of $f'$, we get the continuity of $\partial_2 f : G \supseteq \operatorname{dom} f \to \mathcal{L}_{\mathrm{eq}}(F, F)$. Applying Lemma 0.1 and recalling 0.13, we see that $A = (\partial_2 f)^{-1}(\mathcal{L}is(F, F))$ is open in $G$ and that $(x_0, y_0) \in A$.

Let $p$, $U$, $\delta$, $V$ be given by Lemma 1.2 applied to $f|A : G \supseteq A \cap (\operatorname{dom} f) \to F$. Then the only nontrivial claims are (2) and (3), which we now prove. For this, let $z_1 = (x_1, y_1) \in g$ be arbitrary. The function $f'$ is continuous $G \supseteq \operatorname{dom} f \to \mathcal{L}_{\mathrm{eq}}(G, F)$. Thus, by Proposition 0.7, $(G, F, f)$ is MK-differentiable. So, for some MK-small $s : G \to F$, we have $f(z_1 + w) = f(z_1) + f'(z_1)w + s(w)$ for $z_1 + w \in \operatorname{dom} f$. Write $\ell_1 = \partial_1 f(z_1)$ and $\ell_2 = \partial_2 f(z_1)$. Then $\ell_2 \in \mathcal{L}is(F, F)$. Consequently, $s_1 = -\ell_2^{-1} \circ s$ is MK-small $G \to F$.

Define $r : E \to F$ by $h \mapsto s_1(h, g(x_1 + h) - g(x_1))$ for $x_1 + h \in U$, and by $h \mapsto 0_F$ otherwise. To show that $\widetilde{r} = (E, F, r)$ is MK-small, let $U_0 \in \mathcal{N}_0 G$ be the $U$ of 0.4(MK) for the map $\widetilde{s}_1 = (G, F, s_1)$. Then, for some $U_1 \in \mathcal{N}_0 E$ and $\delta_1 > 0$, we have $U_1 \times \mathrm{B}(\delta_1) \subseteq U_0$. Putting now $U_1' = U_1 \cap p^{-1}[0, \delta_1[$, we have $U_1' \in \mathcal{N}_0 E$.

To show that $U_1'$ can be taken as the $U$ in 0.4(MK), let $W \in \mathcal{N}_0 F$ be arbitrary, and let $V_0 \in \mathcal{N}_0 G$ be the $V$ of 0.4(MK) for the map $\widetilde{s}_1$. Then, for some $V_2 \in \mathcal{N}_0 E$ and $\delta_2 > 0$, we have $V_2 \times \mathrm{B}(\delta_2) \subseteq V_0$. Putting now $V_1' = V_2 \cap (-x_1 + U) \cap (p^{-1}[0, \delta_2[)$, we get $V_1' \in \mathcal{N}_0 E$.



To show that $V_1'$ can be taken as the $V$ in 0.4(MK), let $t \in \mathbb{K} \setminus \{0\}$ and $h \in U_1'$ be such that $th \in V_1'$. Then $x_1 + th \in U$ and $h \in p^{-1}[0, \delta_1[$. Writing $k = t^{-1}(g(x_1 + th) - g(x_1))$, we have $\|k\| \le |t|^{-1} p(th) = p(h) < \delta_1$. Hence $w = (h, k) \in U_0$. Since $th \in V_1' \subseteq p^{-1}[0, \delta_2[$, we have $\|tk\| = \|g(x_1 + th) - g(x_1)\| \le p(th) < \delta_2$. So $tw = (th, tk) \in V_0$. Hence

$$t^{-1} r(th) = t^{-1} s_1(th, g(x_1 + th) - g(x_1)) = t^{-1} s_1(th, tk) = t^{-1} s_1(tw) \in W.$$

Above we have shown that $\widetilde{r}$ is MK-small. To conclude the proof, let $x_1 + h \in U$, and write $k = g(x_1 + h) - g(x_1)$. Then

$$\begin{aligned}0_F = f(x_1 + h, g(x_1 + h)) &= f(x_1 + h, y_1 + k)\\ &= f(z_1) + f'(z_1)(h, k) + s(h, k) = \ell_1(h) + \ell_2(k) + s(h, k).\end{aligned}$$

Hence

$$\begin{aligned}g(x_1 + h) = g(x_1) + k &= g(x_1) - \ell_2^{-1}(\ell_1(h)) - \ell_2^{-1}(s(h, k))\\ &= g(x_1) - \ell_2^{-1}(\ell_1(h)) + r(h). \ ■\end{aligned}$$

Our Implicit Function Theorem is a trivial corollary of the following

1.4. Proposition. *Assume $k \in \mathbb{N} \cup \{\infty\}$. Let $f : E \times F \supseteq \operatorname{dom} f \to F$ be in $C^k$ with $f(x_0, y_0) = z_0$ and $\partial_2 f(x_0, y_0)$ bijective $F \to F$. Then there exists an open neighbourhood $U$ of $x_0$ in $E$ and $\delta > 0$ such that for $V = \mathrm{B}(y_0, \delta)$ and $g = f^{-1}(z_0) \cap (U \times V)$, the relations* (1)–(4) *below hold.*

(1) $U \times V \subseteq \operatorname{dom} f$,
(2) $(E, F, g) \in C^k$ *and* $\operatorname{dom} g = U$ *and* $\operatorname{rng} g \subseteq V$,
(3) $\partial_2 f(x, g(x)) \in \mathcal{L}is(F, F)$ *for* $x \in U$,
(4) $g'(x) = -(\partial_2 f(x, y))^{-1} \circ (\partial_1 f(x, y))$ *for* $(x, y) \in g$.

Proof. Taking the function $w \mapsto f(w) - z_0$ for $f$ in Lemma 1.3, we find $U$ and $\delta$. Then everything claimed is trivial except $\widetilde{g} = (E, F, g) \in C^k$. Using induction, we prove this by showing that $\widetilde{g} \in C^l$ for $l < k + 1$. For $l = 0$ this is mere continuity of $g$, which is given by Lemma 1.3.

Assume now $l < k$ and $\widetilde{g} \in C^l$. We assert that $\widetilde{g} \in C^{l+1}$. By Proposition 0.5 and Lemma 1.3(2), $\widetilde{g}$ is differentiable. So it suffices to show $D\widetilde{g} \in C^l$. Write $G = E \times F$, $K = \mathcal{L}_{\mathrm{eq}}(G, F)$, $L = \mathcal{L}_{\mathrm{eq}}(E, F)$ and $M = \mathcal{L}_{\mathrm{eq}}(F, F)$. Then $D\widetilde{g} = (E, L, g')$. By (4), and with $j_1 : E \to G$ and $j_2 : F \to G$ the natural injections, the map $D\widetilde{g}$ can be decomposed as

$$\begin{aligned}x &\overset{1}{\mapsto} (x, x) = (x, x') \overset{2}{\mapsto} (x, g(x')) = z \overset{3}{\mapsto} f'(z) = \ell \overset{4}{\mapsto} (\ell, \ell) = (\ell, \ell')\\ &\overset{5}{\mapsto} (\ell \circ j_1, \ell' \circ j_2) = (u, v) \overset{6}{\mapsto} (u, -v^{-1}) = (u, w) \overset{7}{\mapsto} w \circ u = g'(x),\end{aligned}$$

where we have the following maps:

(1) $E \to E \times E$, continuous linear, thus in $C^\infty \subseteq C^l$.

(2) $\mathrm{id}_E \times g : E \times E \supseteq E \times U \to G$, which by the induction hypothesis and by 0.13 is in $C^l$.

(3) $(G, K, f') \in C^{k-1} \subseteq C^l$.

(4) $K \to K \times K$, continuous linear, thus in $C^\infty \subseteq C^l$.

(5) $\Gamma_1 \times \Gamma_2 : K \times K \to L \times M$, where we have the partial composition functions $\Gamma_1 : K \ni \ell \mapsto \ell \circ j_1 \in L$ and $\Gamma_2 : K \ni \ell' \mapsto \ell' \circ j_2 \in M$. These are continuous linear, hence so is the map (5), thus in $C^\infty \subseteq C^l$.

(6) $\mathrm{id}_L \times (-\mathrm{inv}) : L \times M \supseteq L \times \mathcal{L}is(F, F) \to L \times M$, by 0.13 in $C^l$.

(7) $\mathrm{comp} : L \times M \to L$, in $C^\infty \subseteq C^l$, by what we said after 0.8.

By Proposition 0.11, we have $D\widetilde{g} \in C^l$. ∎

**2. Infinite-dimensional manifolds.** To define manifolds, assume we are given a class $C_s$ consisting of some G-differentiable maps $\widetilde{f} = (E, F, f)$. Assume also that $C_s$ becomes a category **S** under the composition introduced after 0.8. We also require the chain rule $(g \circ f)'(x)h = g'(f(x))(f'(x)h)$ to hold for $\widetilde{f}, \widetilde{g} \in C_s$, $x \in f^{-1}(\mathrm{dom}\, g)$, $h \in E$. Let $\mathcal{O}$ be the class of objects of **S**, and for $E, F \in \mathcal{O}$, let $\mathcal{S}(E, F)$ be the set of all functions $f$ such that $(E, F, f) \in C_s$.

We want to build categories whose objects are manifolds. To make this possible, manifolds, as defined below, have to be sets and not proper classes. To this end, we use the following technical trick. We call a space $E \in \mathcal{O}$ *standard*, and write $E \in \mathcal{O}_s$, iff the underlying vector space of $E$ is of the form $\mathbb{K}^{(I)}$ for some cardinal number $I$. The vectors of $\mathbb{K}^{(I)}$ are those of $\mathbb{K}^I$ with only finitely many nonzero coordinates. Then, for every $E \in \mathcal{O}$, the class $\{F : F \in \mathcal{O}_s \text{ and } \mathcal{L}is(E, F) \neq \emptyset\}$ is always a set. We assume this set to be nonempty.

Later we choose $\mathcal{O}$ to be the class of all Fréchet spaces, and with $k \in \mathbb{N} \cup \{\infty\}$ fixed (so $k > 0$) we let $C_s$ be the class $C^k$ restricted to those maps $(E, F, f)$ for which $E, F \in \mathcal{O}$. To enlarge the applicability of what follows, recall Remarks 0.12.

For any class $A$, recall that $\mathrm{dom}\, A = \{x : \exists y : (x, y) \in A\}$, $\mathrm{rng}\, A = \{y : \exists x : (x, y) \in A\}$, and put $\mathrm{Dom}\, A = \bigcup\{\mathrm{dom}\, \phi : \phi \in \mathrm{dom}\, A\}$.

2.1. Definitions. A relation $A$ is called an *atlas* iff $\mathrm{Dom}\, A$ is a set, $\emptyset \notin \mathrm{dom}\, A$, and $(\phi, E), (\psi, F) \in A$ implies that $\phi$ is an injection, $E, F \in \mathcal{O}$ and $\psi \circ \phi^{-1} \in \mathcal{S}(E, F)$.

For an atlas $A$, the members of $\mathrm{rng}\, A$ are its *model spaces*. Any $\alpha = (\phi, E)$ for which also $A \cup \{\alpha\}$ is an atlas and $\mathrm{dom}\, \phi \subseteq \mathrm{Dom}\, A$, is called a *chart* for $A$. If also $x \in \mathrm{dom}\, \phi$, then $\alpha$ is called a *chart at* $x$. The *underlying set* of an atlas $A$ is $\mathrm{Dom}\, A$. An atlas is called *standard* iff its model spaces are standard.



A standard atlas $M$ is called a *manifold* iff we have $(\phi, E) \in M$ whenever $M \cup \{(\phi, E)\}$ is a standard atlas with $\mathrm{dom}\, \phi \subseteq \mathrm{Dom}\, A$. Thus manifolds are maximal standard atlases for some fixed set.

Let $M$ and $N$ be manifolds and consider a function $f \subseteq (\mathrm{Dom}\, M) \times (\mathrm{Dom}\, N)$ (¹). Then $f : M \supseteq \mathrm{dom}\, f \to N$ is called *smooth* (²) iff $\psi \circ f \circ \phi^{-1} \in \mathcal{S}(E, F)$ whenever $(\phi, E) \in M$ and $(\psi, F) \in N$. We also say that the (manifold) map, i.e., the triple $(M, N, f)$ is smooth. Note that we *do not* require $\mathrm{dom}\, f = \mathrm{Dom}\, M$.

Recall that for a topological vector space $G$, its closed topological vector subspaces $E, F$ are called *pairwise complemented*, and each is called a *topological complement* of the other, iff the function $\ell : E \times F \ni (x, y) \mapsto x + y \in G$ is a linear homeomorphism $E \times F \to G$. By the open mapping theorem, for a *Fréchet space* $G$, it suffices for $\ell$ to be bijective.

2.2. Definitions. Let $M$ be a manifold modelled on topological vector spaces and $S \subseteq \mathrm{Dom}\, M$. Then $S$ is called a *submanifold* of $M$ iff for every $x \in S$, there exists a chart $(\phi, G)$ for $M$ at $x$ and pairwise complemented topological vector subspaces $E$, $F$ of $G$ such that $E \cap (\mathrm{rng}\, \phi) = \phi(S)$, and $\phi(S)$ is open in $E$. We also call the pair $(M, S)$ a submanifold.

The manifold (structure) $\widetilde{S}$ of a submanifold (set) $S$ of $M$ is given by the following construction. Let $\mathrm{Prop}[M, N, S]$ mean that $N$ is a manifold with $\mathrm{Dom}\, N = S$, and that for all manifolds $P$ and functions $f : \mathrm{Dom}\, P \to S$ we have the equivalence: $(P, M, f)$ smooth $\Leftrightarrow$ $(P, N, f)$ smooth. Then $\widetilde{S} = \{\alpha : \exists N : \alpha \in N \text{ and } \mathrm{Prop}[M, N, S]\}$ is the unique $N$ satisfying condition $\mathrm{Prop}[M, N, S]$; cf. [7; pp. 24–25].

2.3. Definitions. The *manifold generated* by an atlas $A$ is the set $\{(\phi, E) : A \cup \{(\phi, E)\}$ is an atlas, $\mathrm{dom}\, \phi \subseteq \mathrm{Dom}\, A$ and $E \in \mathcal{O}_s\}$. If $A \cup B$ is an atlas and $\mathrm{Dom}\, A = \mathrm{Dom}\, B$, then $A, B$ are *equivalent* atlases and they generate the same manifold.

The *manifold topology* defined by an atlas $A$ is the topology for $\mathrm{Dom}\, A$ generated by the base $\{\phi^{-1}(\mathrm{dom}\, f) : \exists E, F \in \mathcal{O} : (\phi, E) \in A \text{ and } f \in \mathcal{S}(E, F)\}$.

We call $g$ a *local diffeomorphism* $E \to F$ (*at* $x$) iff $(E, F, g), (F, E, g^{-1}) \in C_s$ (and $x \in \mathrm{dom}\, g$).

Having given our general basic definitions, we now specialize to the case $\mathcal{O} = \{E : E \text{ Fréchet}\}$, $k \in \mathbb{N} \cup \{\infty\}$, $C_s = \{(E, F, f) \in C^k : E, F \in \mathcal{O}\}$.

2.4. Proposition. *Let $G$ be a Fréchet space and $F$ a Banach space. Let $(z_0, y_0) \in f \in \mathcal{S}(G, F)$, and let $\ell_1 = f'(z_0) : G \to F$ be surjective*

(¹) Abusively, $f \subseteq M \times N$.

(²) Or a $C_s$-map; likewise we would say: $C_s$-atlas / (sub)manifold / diffeomorphism.

*with* $\operatorname{Ker} \ell_1$ *complemented in* $G$. *Then there exists a Fréchet space* $E$ *and a local diffeomorphism* $g : E \times F \to G$ *at* $(0_E, y_0)$, *such that* $g(0_E, y_0) = z_0$, $\operatorname{rng} g \subseteq \operatorname{dom} f$, *and* $(f \circ g)(x, y) = y$ *for* $(x, y) \in \operatorname{dom} g$.

Proof. Let $E$ be $\operatorname{Ker} \ell_1$ considered as a topological vector subspace of $G$. Then $E$ is Fréchet. Let $F_1$ be a complement of $E$. Then $F_1$ is also Fréchet. Now $\ell_1 | F_1$ is a continuous linear isomorphism $F_1 \to F$. By the open mapping theorem, also $\ell = (\ell_1 | F_1)^{-1}$ is continuous. (So $F_1$ is normable, hence Banach.) Letting $p : G \to E$ and $p_1 : G \to F_1$ be the projections, we have $\ell \circ \ell_1 = p_1$. Write $x_0 = z_0 - \ell(y_0)$.

Put $E_1 = E \times F$ and $G_1 = E_1 \times F$, and consider the map $\widetilde{f_1} = (G_1, F, f_1)$, where $f_1((x, v), y) = f(x + x_0 + \ell(y)) - v$ for $x + x_0 + \ell(y) \in \operatorname{dom} f$. Then for $w_0 = (0_E, y_0)$, our map $\widetilde{f_1}$ satisfies the requirements of the Implicit Function Theorem with $f_1(w_0, y_0) = 0_F$ and $\partial_2 f_1(w_0, y_0) = f'(z_0) \circ \ell = \mathrm{id}_F$. So we find open neighbourhoods $U_1$ and $V_1$ of $w_0$ and $y_0$ in $E_1$ and $F$, respectively, and a function $g_1 : U_1 \to V_1$ satisfying $(E_1, F, g_1) \in C_s$, $g_1(0_E, y_0) = y_0$, and $f(x + x_0 + \ell(g_1(x, v))) = v$ for $(x, v) \in U_1$. Differentiating with respect to $v$, we get $\partial_2 g_1(0_E, y_0) = \mathrm{id}_F$.

Put $g_0 : U_1 \ni (x, y) \mapsto x + x_0 + \ell(g_1(x, y))$. Then $(E_1, G, g_0) \in C_s$, and we have

$$(*) \qquad g_0(0_E, y_0) = z_0 \quad \text{and} \quad (f \circ g_0)(x, y) = y \text{ for } (x, y) \in U_1.$$

Taking into account the direct sum decomposition of $G$, we can check that $g_0$ is injective. However, we do not know whether $(G, E_1, g_0^{-1}) \in C_s$. To prove that a suitable restriction of $g_0$ is a local diffeomorphism, we make another application of the Implicit Function Theorem.

Let $G_2 = G \times F$, and consider the map $\widetilde{f_2} = (G_2, F, f_2)$, where $f_2(z, y) = g_1(p(z - z_0), y) - \ell_1(z - z_0)$ for $(p(z - z_0), y) \in U_1$. Then $\widetilde{f_2}$ satisfies the requirements of the Implicit Function Theorem with $f_2(z_0, y_0) = y_0$ and $\partial_2 f_2(z_0, y_0) = \mathrm{id}_F$. So we find open neighbourhoods $U_2$ and $V_2$ of $z_0$ and $y_0$ in $G$ and $F$, respectively, and a function $g_2 : U_2 \to V_2$ satisfying $(G, F, g_2) \in C_s$ and $g_1(p(z - z_0), g_2(z)) = y_0 + \ell_1(z - z_0)$ for all $z \in U_2$. We also have $g_2(z_0) = y_0$.

Let $h : U_2 \ni z \mapsto (p(z - z_0), g_2(z)) \in E_1$, and put $g = g_0 | (\operatorname{rng} h)$. Then $(G, E_1, h) \in C_s$ and $\operatorname{rng} h \subseteq \operatorname{dom} g_0$. In order to show $(E_1, G, g) \in C_s$, it suffices to show that $\operatorname{rng} h$ is open in $E_1$. For all $z \in U_2$, we have

$$\begin{aligned} g_0(h(z)) &= g_0(p(z - z_0), g_2(z)) \\ &= p(z - z_0) + x_0 + \ell(g_1(p(z - z_0), g_2(z))) \\ &= p(z) - p(z_0) + z_0 - \ell(y_0) + \ell(y_0) + \ell(\ell_1(z - z_0)) \\ &= p(z) - p(z_0) + z_0 + p_1(z) - p_1(z_0) = z. \end{aligned}$$

Hence $h \subseteq g_0^{-1}$. Then $\operatorname{rng} h = g_0^{-1}(\operatorname{dom} h) = g_0^{-1}(U_2)$ is open in $E_1$, because $g_0$ is continuous. From $h \subseteq g_0^{-1}$ and $g = g_0 | (\operatorname{rng} h)$ it follows that $g^{-1} = h$. From $(*)$ and $(0_E, y_0) = h(z_0) \in \operatorname{rng} h$, we see that $g$ satisfies the remaining assertions. ∎



In Banach space calculus there is a corresponding "dual" theorem to Proposition 2.4 above; cf. [7; p. 16, Corollary 1s] or [3; p. 215, A.9]. Here we cannot prove such a theorem, because we would need an inverse function theorem for maps between general Fréchet spaces.

We now prove our Theorem stated at the beginning. Tangent spaces and tangent maps are defined as in [7; pp. 26–27].

Proof. For given $x \in f^{-1}(S)$, it suffices to find a chart $(\phi, E)$ for $M$ at $x$ such that for some pairwise complemented topological vector subspaces $E'$ and $E''$ of $E$, the set $\phi(f^{-1}(S))$ is included and open in $E'$. To prove this, choose charts $(\phi_1, G) \in M$ at $x$ and $(\psi, F) \in N$ at $y = f(x)$ such that for some pairwise complemented Banach subspaces $F_1$ and $E_2$ of $F$, we have $F_1 \cap (\operatorname{rng} \psi) = \psi(S)$ and the latter set is open in $F_1$. Let $q$ be the projection $F \to E_2$, and put $\varphi = q \circ \psi \circ f \circ \phi_1^{-1}$. Then we have the commutative diagram

$$\begin{tikzcd} T_x M \arrow[rr, "T f_x"] \arrow[d, "\cong"] & & T_y N \arrow[r, "\pi"] \arrow[d, "\cong"'] & T_y N / T_y S \arrow[d, "\cong"'] \\ G \arrow[rr, "(\psi \circ f \circ \phi_1^{-1})'(\phi_1(x))"'] & & F \arrow[r, "q"'] & E_2 \end{tikzcd}$$

where $\cong$ means linear homeomorphism. To apply Proposition 2.4, note that $\varphi'(\phi_1(x)) = q \circ ((\psi \circ f \circ \phi_1^{-1})'(\phi_1(x)))$, and use conditions (1) and (2) of our Theorem. So we find a Fréchet space $E_1$ and a local diffeomorphism $\theta : E = E_1 \times E_2 \to G$ such that $\theta(0_{E_1}, q(\psi(y))) = \phi_1(x)$ and $\varphi(\theta(u, v)) = v$ for $(u, v) \in \operatorname{dom} \theta$. Writing $\phi = (\theta^{-1} \circ \phi_1) | (f^{-1}(\operatorname{dom} \psi))$, we get a chart $(\phi, E)$ for $M$ at $x$. Moreover, we have

$$\begin{aligned} \phi(f^{-1}(S)) &= \theta^{-1}(\phi_1(f^{-1}(S) \cap f^{-1}(\operatorname{dom} \psi))) \\ &= \theta^{-1}(\phi_1(f^{-1}(\psi^{-1}(\psi(S))))) \\ &= \theta^{-1}(\phi_1(f^{-1}(\psi^{-1}(F_1 \cap (\operatorname{rng} \psi))))) \\ &= \theta^{-1}(\phi_1(f^{-1}(\psi^{-1}(q^{-1}(0_F))))) = (\varphi \circ \theta)^{-1}(0_F) \\ &= (E_1 \times \{0_F\}) \cap (\operatorname{dom} \theta), \end{aligned}$$

which is open in the topological vector subspace $E' = E_1 \times \{0_F\}$ of $E$. ∎

2.5. Corollary. *Let* $k \in \mathbb{N} \cup \{\infty\}$, *and let* $f : E \supseteq \operatorname{dom} f \to F$ *be in* $C^k$ *with* $E$ *Fréchet and* $F$ *Banach. Let* $y \in \operatorname{rng} f$, *and let* $\phi$ *be the identity either on* $\operatorname{dom} f$ *or on* $E$. *Then, under conditions* (1) *and* (2) *below,* $f^{-1}(y)$ *is a submanifold of the manifold generated by the atlas* $\{(\phi, E)\}$. *Condition* (2) *is superfluous if* $F$ *is finite-dimensional.*

(1) $f'(x) : E \to F$ *is surjective for all* $x \in f^{-1}(y)$,
(2) $\operatorname{Ker} f'(x)$ *is complemented in* $E$ *for all* $x \in f^{-1}(y)$.

Under the conditions of Corollary 2.5, one can prove that the manifold topology and the induced subspace topology of $f^{-1}(y)$ coincide. One can also see that the submanifold has an equivalent atlas modelled on closed subspaces of $E$ having complements linearly homeomorphic to $F$.

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*Received December 3, 1997*
*Revised version October 19, 1998* (4014)

STUDIA MATHEMATICA 134 (3) (1999)

# Embedding of random vectors into continuous martingales

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**Abstract.** Let $E$ be a real, separable Banach space and denote by $L^0(\Omega, E)$ the space of all $E$-valued random vectors defined on the probability space $\Omega$. The following result is proved. There exists an extension $\widetilde{\Omega}$ of $\Omega$, and a filtration $(\widetilde{\mathfrak{F}}_t)_{t\ge 0}$ on $\widetilde{\Omega}$, such that for every $X \in L^0(\Omega, E)$ there is an $E$-valued, continuous $(\widetilde{\mathfrak{F}}_t)$-martingale $(M_t(X))_{t\ge 0}$ in which $X$ is embedded in the sense that $X = M_\tau(X)$ a.s. for an a.s. finite stopping time $\tau$. For $E = \mathbb{R}$ this gives a Skorokhod embedding for all $X \in L^0(\Omega, \mathbb{R})$, and for general $E$ this leads to a representation of random vectors as stochastic integrals relative to a Brownian motion.

**1. Introduction.** In 1960 Skorokhod [11] proved that for any mean zero, square integrable random variable $X$ there is a Brownian motion $(B_t)_{t\ge 0}$ and a stopping time $\tau$ such that $X$ and $B_\tau$ have the same distribution. There now exist a series of different proofs of this so-called Skorokhod embedding (cf. [1], [2], [5] and [8]). One possibility to get Skorokhod's result is the following. First one constructs a continuous martingale $(M_s)_{0\le s\le 1}$ such that $X = M_1$ a.s. Then by the theorem of Dubins and Schwarz [6] there exists a Brownian motion $(B_t)_{t\ge 0}$ such that $M_t = B_{\tau_t}$, where $\tau_t = [M](t)$, the quadratic variation of $(M_t)$ at time $t$. In particular, one gets $X = M_1 = B_{\tau_1}$ a.s.

The Skorokhod embedding is a result for one-dimensional random variables and in general one cannot expect similar results for random vectors with values in $\mathbb{R}^n$ or—even more general—in a Banach space. For this reason we consider another type of embedding, which is not so restricted to the real line. We embed random vectors into continuous vector-valued martingales with the additional nice property that they are stochastic integrals relative to a Brownian motion.

In §2 we first consider the case of integrable Banach space valued random vectors. If $L^1(\Omega, E)$ denotes the space of all integrable random vectors de-

1991 *Mathematics Subject Classification*: 60G44, 60H05, 60B11.
*Key words and phrases*: Skorokhod embedding, martingale, stochastic integral, Brownian motion.