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A note on the hyperreflexivity constant for certain reflexive algebras

by

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Abstract. Using results on the reflexive algebra with two invariant subspaces, we calculate the hyperreflexivity constant for this algebra when the Hilbert space is two-dimensional. Then by the continuity of the angle for two subspaces, there exists a non-CSL hyperreflexive algebra with hyperreflexivity constant C for every $C > 1$. This result leads to a kind of continuity for the hyperreflexivity constant.

In the study of non-selfadjoint operator algebras, the property of hyperreflexivity introduced by Arveson [3] is very important in the class of reflexive algebras. In this paper we study the algebra which is the set of all bounded operators on a Hilbert space H which leave invariant two closed subspaces L, M of H with $L \cap M = 0$, $\overline{L + M} = H$. In symbols,

$$\mathfrak{A} = \{A \in B(H) : AL \subseteq L,\ AM \subseteq M\}.$$

This algebra is the simplest example of a reflexive algebra which is not a CSL algebra (a reflexive algebra whose invariant projection lattice is not commutative), provided the two subspaces are not orthogonal. Results related to this algebra can be found in [1], [2], [4], [5] and [6].

Papadakis [6] and Katavolos *et al.* [4] showed that this algebra is hyperreflexive if and only if $L+M$ is closed. By the calculation of the hyperreflexivity constant for this algebra when the Hilbert space is two-dimensional, we get a result on non-CSL hyperreflexive algebras.

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A weakly closed unital subalgebra $\mathfrak{A}$ of $B(H)$ is called *hyperreflexive* if there is a positive constant k such that

$$d(B, \mathfrak{A}) \le k \sup\{\|P^{\perp} BP\| : P \in \operatorname{Lat} \mathfrak{A}\}$$

for all $B \in B(H)$. The infimum K of such constants k is called the *hyperreflexivity constant* of $\mathfrak{A}$. Now if $T \in {}^{\perp}\mathfrak{A}$ take $k(T)$ to be the infimum of all

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sums $\sum_{n=1}^{\infty}\|T_n\|_1$ where each $T_n \in {}^{\perp}\mathfrak{A}$ is of rank one and $T = \sum_{n=1}^{\infty} T_n$. Arveson (Th. 7.4 of [3]) proved that

$$K = \sup\{k(T) : T \in {}^{\perp}\mathfrak{A},\ \|T\|_1 \le 1\}.$$

Recall that the *preannihilator* ${}^{\perp}\mathfrak{A}$ of $\mathfrak{A}$ is

$${}^{\perp}\mathfrak{A} = \{T \in C_1 : \operatorname{tr}(T^*A) = 0 \text{ for all } A \in \mathfrak{A}\}.$$

In our case, Papadakis [6] and Katavolos *et al.* [4] showed that the algebra $\mathfrak{A}$ is hyperreflexive if and only if $L + M = H$. Hence for H finite-dimensional this algebra is always hyperreflexive.

Let $H = \mathbb{C}^2$ and take two closed subspaces L, M of $\mathbb{C}^2$. Then we may assume that for some θ,

$$L = \{(x,0) : x \in \mathbb{C}\}, \qquad M = \{(y, y\tan\theta) : y \in \mathbb{C}\}.$$

By our hypothesis (L and M are not orthogonal), we may suppose that $0 < \theta < \pi/2$.

In this case,

$$\mathfrak{A} = \left\{ \begin{pmatrix} a & b \\ 0 & a + b\tan\theta \end{pmatrix} : a, b \in \mathbb{C} \right\},$$
$${}^{\perp}\mathfrak{A} = \left\{ \begin{pmatrix} -t & -t\tan\theta \\ s & t \end{pmatrix} : s, t \in \mathbb{C} \right\}.$$

For any $T \in {}^{\perp}\mathfrak{A}$, it is easy to see that

$$\|T\|_1 = \sqrt{(\tan^2\theta + 2)|t|^2 + |s|^2 + 2|t|\cdot|t - s\tan\theta|}.$$

A rank one operator of ${}^{\perp}\mathfrak{A}$ can only be of the following two types:

$$\begin{pmatrix} -t & -t\tan\theta \\ \dfrac{t}{\tan\theta} & t \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}.$$

Their trace norms are

$$\left\| \begin{pmatrix} -t & -t\tan\theta \\ \dfrac{t}{\tan\theta} & t \end{pmatrix} \right\|_1 = \left(\tan\theta + \frac{1}{\tan\theta}\right)|t|,$$
$$\left\| \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} \right\|_1 = |s|.$$

Hence for every $T \in {}^{\perp}\mathfrak{A}$, the following decomposition minimizes the sum of trace norms of rank one summands:

$$\begin{pmatrix} -t & -t\tan\theta \\ s & t \end{pmatrix} = \begin{pmatrix} -t & -t\tan\theta \\ \dfrac{t}{\tan\theta} & t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ s - \dfrac{t}{\tan\theta} & 0 \end{pmatrix}.$$

Therefore the hyperreflexivity constant K is

$$K = \sup_D \left\{ \left(\tan\theta + \frac{1}{\tan\theta} \right)|t| + \left| s - \frac{1}{\tan\theta} t \right| \right\},$$

where $D = \{(s,t) \in \mathbb{C}^2 : (\tan^2\theta + 2)|t|^2 + |s|^2 + 2|t| \cdot |t - s\tan\theta| \le 1\}$.

For all $(s,t) \in D$,

$$\begin{aligned}
&\left\{ \left(\tan\theta + \frac{1}{\tan\theta} \right)|t| + \left| s - \frac{t}{\tan\theta} \right| \right\}^2 \\
&\quad = \frac{1}{\sin^2\theta} \left\{ \frac{1}{\cos^2\theta}|t|^2 + 2|t||t - s\tan\theta| + \cos^2\theta|t - s\tan\theta|^2 \right\} \\
&\quad \le \frac{1}{\sin^2\theta} \left\{ \frac{1}{\cos^2\theta}|t|^2 + (1 - (\tan^2\theta + 2)|t|^2 - |s|^2) \right. \\
&\qquad\qquad \left. + \cos^2\theta(|t| + \tan\theta|s|)^2 \right\} \\
&\quad = \frac{1}{\sin^2\theta}\{1 - (\sin\theta|t| - \cos\theta|s|)^2\} \le \frac{1}{\sin^2\theta}.
\end{aligned}$$

So $K \le 1/\sin\theta$.

Now, put

$$t = \frac{1}{2\sqrt{\tan^2\theta + 1}} e^{i\alpha}, \quad s = \frac{\tan\theta}{2\sqrt{\tan^2\theta + 1}} e^{i(\alpha+\pi)} \quad (\alpha \in \mathbb{R}).$$

By an easy calculation, we can show that $(s,t) \in D$. In this case,

$$\begin{aligned}
&\left(\tan\theta + \frac{1}{\tan\theta} \right)|t| + \left| s - \frac{t}{\tan\theta} \right| \\
&\quad = \frac{\tan^2\theta + 1}{\tan\theta} \frac{1}{2\sqrt{\tan^2\theta + 1}} + \frac{1}{\tan\theta} \frac{\tan^2\theta + 1}{2\sqrt{\tan^2\theta + 1}} \\
&\quad = \frac{\sqrt{\tan^2\theta + 1}}{\tan\theta} = \frac{1}{\sin\theta}.
\end{aligned}$$

Hence $K = 1/\sin\theta$.

From this calculation, we deduce the following result.

THEOREM. *For all $C > 1$, there exists a non-CSL hyperreflexive algebra whith hyperreflexivity constant C.*

We cannot expect the existence of a non-CSL hyperreflexive algebra whose hyperreflexivity constant is 1, because every hyperreflexive algebra with hyperreflexivity constant 1 is CSL.

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