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Norm continuity of c_0-semigroups

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Abstract. We show that a positive semigroup T_t on $L_p(\Omega,\nu)$ with generator A and $\|R(\alpha+i\beta)\|\to 0$ as $|\beta|\to\infty$ for some $\alpha\in\mathbb{R}$ is continuous in the operator norm for $t>0$. The proof is based on a criterion for norm continuity in terms of "smoothing properties" of certain convolution operators on general Banach spaces and an extrapolation result for the L_p-scale, which may be of independent interest.

1. Introduction. C_0-semigroups T_t which are continuous in the operator norm of a Banach space for $t>0$ have a number of interesting properties which do not hold for general c_0-semigroups, e.g.

• The spectral mapping theorem holds, i.e.

$$\sigma(T_t)\setminus\{0\}=e^{t\sigma(A)}$$

where A is the generator of T_t ([3], Theorem 2.19).

• As a consequence, the Lyapunov stability theorem holds, i.e. if the right halfplane belongs to the resolvent set of A, then every mild solution $y(t)=T_tx$, $x\in X$, of the Cauchy problem

$$y'(t)=Ay(t),\qquad y(0)=x$$

is exponentially stable.

• The semigroup operators T_t are compact if and only if the resolvent operators $R(\lambda,A)$ are compact (see [9], Chapter 2, Theorem 3.3).

It is therefore important to find natural conditions on the resolvent of A that guarantee that T_t is norm continuous. Since

$$R(\lambda,A)=\int_0^\infty e^{-\lambda t}T_t\,dt,\qquad \lambda>\omega(T_t),$$

it follows from the Riemann–Lebesgue lemma that $\|R(\alpha+i\beta,A)\|\to 0$ as $|\beta|\to\infty$ for $\alpha>\omega(T_t)$ is a necessary condition for norm continuity (see [9], Chapter 2, Theorem 3.6). It was shown first by P. You [12] (see also [5], [7],

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[8]) that in a Hilbert space this necessary condition is also sufficient, but it seems to be an open question whether this is also true in general Banach spaces.

In this paper we give a partial affirmative answer: If T_t is a positive semigroup on $L_p(\Omega,\nu)$, $1<p<\infty$, then the condition $\|R(\alpha+i\beta,A)\|\to 0$ as $|\beta|\to\infty$ for some $\alpha>\omega(T_t)$ implies indeed the norm continuity of T_t (Theorem 3.3 of Section 3).

The proof uses a characterization of norm continuity in terms of a smoothing property of the convolution operator

$$Kf(t)=\int_0^t T_{t-s}(f(s))\,ds \tag{1}$$

(see Def. 2.1) on $L_p([0,\tau],X)$, which we also use to give a simple proof of the Hilbert space result quoted above (Theorem 2.4 and Corollary 2.5 in Section 2).

As a further preparation for the proof of our main result, we prove an extrapolation result for the convolution operator (1) (Theorem 3.1), which allows us to reduce the L_p-case to the L_2-case and which may be of independent interest.

Some further characterizations of norm continuity of a semigroup may be found in [4] and [6].

We use standard notation: X denotes a Banach space and $B(X)$ the bounded linear operators on X with the operator norm. U_X stands for the unit ball of X. For a c_0-semigroup T_t with generator A on X we denote by $s(A)$ the *spectral bound*, $s(A)=\sup\{\operatorname{Re}\lambda:\lambda\in\sigma(A)\}$, and by $\omega(T_t)$ the *growth bound*,

$$\omega(T_t)=\inf\{\omega:\exists C\ \|T_t\|\le Ce^{\omega t}\}.$$

2. Norm continuity and convolution operators. For every c_0-semigroup T_t on a Banach space X we can define a convolution operator $Kf(t)=\int_0^t T_{t-s}(f(s))\,ds$, $t\le\tau$, on $L_p([0,\tau],X)$. We will characterize the norm continuity of T_t by the following "smoothness" property of K:

2.1. DEFINITION. A bounded operator K on $L_p([0,\tau],X)$ satisfies the *Riesz criterion* (R_p), $1\le p<\infty$, if

$$(\mathrm{R}_p)\qquad \int_0^\tau \|(Kf)(t+h)-(Kf)(t)\|^p\,dt\to 0\quad \text{as } h\to 0$$

uniformly for $f\in L_p([0,\tau],X)$ with $\|f\|\le 1$ (and Kf extended periodically from $[0,\tau]$ to $\mathbb{R}$).

2.2. REMARKS (on the Riesz condition (R_p)). (a) If $\dim X=\infty$ condition (R_p) does not imply that K is a compact operator on $L_p([0,\tau],X)$. But the proof of the Fréchet–Kolmogorov theorem on compactness in $L_p([0,\tau])$ (see e.g. [1], 2.26) shows that K has (R_p) if and only if K is "almost smoothing" in the following sense: For all $\varepsilon>0$ there is an $M<\infty$ such that

$$(\mathrm{S}_p)\qquad K(U_{L_p(X)})\subset MU_{W_p^1(X)}+\varepsilon U_{L_p(X)}.$$

(b) The set of operators on $L_p([0,\tau],X)$ with (R_p) is a closed right ideal in $B(L_p([0,\tau],X))$.

(c) If $S_hf(s)=f(s+h)$ denotes the shift by h for $f\in L_p([0,\tau],X)$ (again f is extended periodically), then (R_p) can be reformulated as

$$\|(I-S_h)K\|_{L_p([0,\tau],X)}\to 0\quad \text{as } h\to 0.$$

Hence condition (R_p) "interpolates".

Sketch of proof for (a). It is clear that (S_p) implies (R_p) since

$$|g(t+h)-g(t)|^p\le h^{p/p'}\int_t^{t+h}|g'(s)|^p\,dt.$$

On the other hand, given (R_p) we choose a $\varphi\in C^\infty(0,\tau)$ with $\varphi\ge 0$, $\int\varphi(t)\,dt=1$ and put $\varphi_\varepsilon(t)=(1/\varepsilon)\varphi(t/\varepsilon)$. The operators $K_\varepsilon f=\varphi_\varepsilon*(Kf)$ map $L_p([0,\tau],X)$ into $W_p^1([0,\tau],X)$ and since

$$K_\varepsilon f(t)-Kf(t)=\int_{|u|\le\varepsilon}\varphi_\varepsilon(u)[Kf(t-u)-Kf(t)]\,du$$

we see from (R_p) that, as $\varepsilon\to 0$,

$$\|K_\varepsilon f-Kf\|_{L_p}\le\sup_{|h|\le\varepsilon}\|Kf(\cdot+h)-Kf\|_{L_p}\to 0.\ \blacksquare$$

2.3. EXAMPLE. *Let $[0,\tau]^2\ni(t,s)\mapsto k(t,s)\in B(X)$ be a Bochner measurable, operator-valued kernel with*

$$\int_0^\tau\Big(\int_0^\tau\|k(t,s)\|_{B(X)}^{p'}\,ds\Big)^{p/p'}dt<\infty,\qquad 1<p<\infty,\ \frac{1}{p}+\frac{1}{p'}=1.$$

Then the operator K on $L_p([0,\tau],X)$ given by

$$Kf(t)=\int_0^\tau k(t,s)(f(s))\,ds$$

satisfies the Riesz condition (R_p).

Proof. For $\|f\|_{L_p(\mathbb{R},X)}\le 1$,

$$\int_0^{\tau-h}\|Kf(t+h)-Kf(t)\|^p\,dt\le\int_0^{\tau-h}\Big(\int_0^\tau\|k(t+h,s)-k(t,s)\|^{p'}ds\Big)^{p/p'}dt\to 0$$

as $h \to 0$, and also

$$\int_A \|Kf(t)\|^p\,dt \le \int_A \Big(\int_0^\tau \|k(t,s)\|^{p'}\,ds\Big)^{p/p'}dt \to 0$$

for $A \subset [0,\tau]$ with $m(A) \to 0$. ∎

2.4. THEOREM. *A c_0-semigroup T_t on a Banach space X is continuous in the operator norm for $t > 0$ if and only if for all $\tau > 0$ the operator $K : L_p([0,\tau],X) \to L_p([0,\tau],X)$, $Kf(t) = \int_0^t T_{t-s}(f(s))\,ds$, satisfies the Riesz condition* (R_p) *for some (all) $p \in (1,\infty)$.*

Proof. If T_t is continuous in the operator norm, then we can write K as $Kf(t) = \int_0^\tau k(t,s)f(s)\,ds$, where the kernel $k(t,s) = T_{t-s}$ for $0 \le s \le t \le \tau$ and $k(t,s) = 0$ otherwise satisfies the assumption of Example 2.3. Hence K has (R_p).

To deduce norm continuity from (R_p) put

$$\Delta_h(t)x = (t+h)T_{t+h}x - tT_tx. \tag{1}$$

Observe that for $t+h \le \tau$,

$$\Delta_h(t)x = \int_0^{t+h} T_{t+h-s}(T_sx)\,ds - \int_0^t T_{t-s}(T_sx)\,ds = Kf(t+h) - Kf(t)$$

with $f(s) = T_sx$, $\|f\|_{L_p([0,\tau],X)} \le \tau^{1/p}C\|x\|$ and $C = \sup\{\|T_s\| : 0 \le s \le \tau\}$. Now ($\mathrm{R}_p$) implies that

$$\int_0^{\tau/2} \|\Delta_h(t)x\|^p\,dt \to 0 \quad \text{as } h \to 0^+ \tag{2}$$

uniformly for $x \in U_X$.

To relate this condition to the continuity of T_t, we use

$$\begin{aligned}\int_0^t T_{t-s}(\Delta_h(s)x)\,ds &= \int_0^t [(s+h)T_{t+h}x - sT_tx]\,ds\\ &= \tfrac12(t+h)^2T_{t+h}x - \tfrac12 t^2T_tx - \tfrac12 h^2T_{t+h}x\end{aligned}$$

Hence for $\|x\| \le 1$ and $t \le \tau/2$,

$$\begin{aligned}\|(t+h)^2T_{t+h}x - t^2T_tx\| &\le 2\int_0^t \|T_{t-s}(\Delta_h(s)x)\|\,ds + h^2\|T_{t+h}\|\\ &\le 2C(\tau/2)^{1/p'}\Big(\int_0^{\tau/2} \|\Delta_h(s)x\|^p\,ds\Big)^{1/p} + h^2C.\end{aligned}$$

Now (2) implies that t^2T_t is norm continuous on $[0,\tau/2]$ and the claim follows. ∎

Criterion 2.4 leads to a short proof of the following Hilbert space result originally due to P. You [12] (see also [5], [7], [8]).

2.5. COROLLARY. *Let X be a Hilbert space. Then a c_0-semigroup T_t with generator A is continuous in the operator norm for $t > 0$ if and only if for some $\alpha > s(A)$,*

$$\|R(\alpha + i\beta, A)\| \to 0 \quad \textit{as } |\beta| \to \infty.$$

REMARK. In the following proof we will use the fact that for a Hilbert space X the Fourier transform

$$\widehat{f}(\beta) = \int_{-\infty}^{\infty} e^{-i\beta t} f(t)\,dt$$

multiplied by $1/\sqrt{2\pi}$ defines an isometry on $L_2(\mathbb{R},X)$. Therefore, for a convolution operator

$$Lf(t) = \int_{-\infty}^{\infty} l_{t-s}(f(s))\,ds \quad \text{on } L_2(\mathbb{R},X)$$

where $t \mapsto l_t \in B(X)$ is strongly integrable and $\int_{-\infty}^\infty \|l_t\|\,dt < \infty$ we have $t \mapsto \widehat{l}_t \in B(X)$ and $\widehat{Lf}(\beta) = \widehat{l}_t(\beta)(\widehat{f}(\beta))$ so that

$$\|L\|_{L_2(\mathbb{R},X)} = \sup_{\beta\in\mathbb{R}} \|(l_t)^\wedge(\beta)\|. \tag{3}$$

Proof of 2.5. Without loss of generality we can assume that $\omega(T_t) < 0$. We can also choose $\alpha = 0$, since by the resolvent equation

$$R(\alpha + i\beta, A) = [I + (\alpha' - \alpha)R(\alpha + i\beta, A)]R(\alpha' + i\beta, A)$$

for all $\omega(T_t) < \alpha < \alpha'$ and the term in square brackets is uniformly bounded in $\beta \in \mathbb{R}$.

It was already checked in [9], Chapter 2, Theorem 3.6, that our assumption on the resolvent is necessary.

On the other hand, since $R(i\beta, A) = (T_t)^\wedge(\beta)$, where $T_t = 0$ for $t < 0$, we may think of $Kf(t) = \int_{-\infty}^\infty T_{t-s}(f(s))\,ds$ as a Fourier multiplier on $L_2(\mathbb{R},X)$ with multiplier function $R(i\beta, A)$, i.e.

$$\widehat{Kf}(\beta) = R(i\beta, A)[\widehat{f}(\beta)].$$

To see that the restriction $K_\tau = \chi_{[0,\tau]}K\chi_{[0,\tau]}$ of K to $L_2([0,\tau],X)$ satisfies (R_2) we split K for every $n \in \mathbb{N}$ into two convolution operators

$K = K_n + L_n$ defined by

$$(K_n f)^\wedge(\beta) = \varphi(\beta/n)R(i\beta, A)[\widehat{f}(\beta)],$$
$$(L_n f)^\wedge(\beta) = [1-\varphi(\beta/n)]R(i\beta, A)[\widehat{f}(\beta)],$$

where φ is C^∞ with support in $[-2,2]$, $\varphi \equiv 1$ on $[-1,1]$ and $|\varphi| \le 1$. By (3), our assumptions now imply that $\|L_n\|_{L_2(X)} \to 0$ as $n \to \infty$. The operators K_n are of the form $K_n f(t) = \int_{-\infty}^{\infty} k_n(t,s)f(s)\,ds$ with the kernel

$$k_n(t,s) = [\varphi(\cdot/n)R(i\cdot, A)]^\vee(t-s),$$

that is, Bochner integrable and bounded in $B(X)$.

Therefore, the restriction $K_{\tau,n} = \chi_{[0,\tau]}K_n\chi_{[0,\tau]}$ of K_n to $L_2([0,\tau],X)$ satisfies the assumptions of Example 2.3 with $p=2$ and we can conclude that $K_{\tau,n}$ has property (R_2). Since $\|K_\tau - K_{\tau,n}\| \le \|L_n\| \to 0$ as $n\to\infty$ it follows from 2.2(b) that K_τ has (R_2). Now the claim follows from Theorem 2.4. ■

3. Positive semigroups on $L_p(\Omega,\nu)$. In order to extend the characterization of norm continuous semigroups on Hilbert spaces (Corollary 2.5) to positive semigroups on $L_p(\Omega,\nu)$ we need an *extrapolation result* for positive operators (Theorem 3.1 below).

It is known (Theorem 2.1 of [11]) that for a positive operator $R : L_p(\Omega,\nu) \to L_p(\Omega,\nu)$ with ν σ-finite, there is a density $0 < g \in L_1(\Omega,\nu)$ such that

$$\widetilde{R} = J^{-1}RJ : L_p(\Omega,\widetilde{\nu}) \to L_p(\Omega,\widetilde{\nu}), \tag{1}$$

where $d\widetilde{\nu} = g\,d\nu$ and $Jf = g^{1/p}f$ is an isometry of $L_p(\Omega,\widetilde{\nu})$ onto $L_p(\Omega,\nu)$, extends to a bounded operator on $L_q(\Omega,\widetilde{\nu})$ and

$$\|R\|_{L_q(\widetilde{\nu})} \le 2\|R\|_{L_p(\nu)} \quad \text{for all } 1\le q\le\infty. \tag{2}$$

It seems to be an open problem whether it is possible to extrapolate even a positive c_0-semigroup T_t on $L_p(\Omega,\nu)$ in the same way. The next result shows that one can at least extrapolate the convolution operator

$$Kf(t) = \int_0^t T_{t-s}(f(s))\,ds$$

on $L_p(\mathbb{R}, L_p(\Omega,\nu))$ to the whole $L_q(L_q)$-scale.

3.1. Theorem. *Let T_t be a positive c_0-semigroup on $L_p(\Omega,\nu)$ with $\omega(T_t) < 0$. Then there exists a density $0 < g \in L_1(\Omega,\nu)$ and an isometry $Jf = g^{1/p}f$ of $L_p(\Omega,\widetilde{\nu})$ onto $L_p(\Omega,\nu)$ (where $d\widetilde{\nu} = g\,d\nu$) such that:*

(i) *The operators $J^{-1}R(\lambda,A)J$, with $\operatorname{Re}\lambda \ge 0$, extend to operators on $L_q(\Omega,\widetilde{\nu})$ and there is a constant C with $\|J^{-1}R(\lambda,A)J\|_{L_q(\widetilde{\nu})} \le C$ for all λ with $\operatorname{Re}\lambda\ge 0$ and $1\le q\le\infty$.*

(ii) *The operator $\widetilde{K}f(t) = \int_0^t J^{-1}T_{t-s}J(f(s))\,ds$, $t>0$, extends to a bounded operator on $L_q(\mathbb{R}_+, L_q(\Omega,\widetilde{\nu}))$ and there is a $C<\infty$ with $\|\widetilde{K}\|_{L_q(L_q)} \le C$ for all $1\le q<\infty$.*

(iii) *For all $f\in L_2(\mathbb{R}_+, L_2(\Omega,\widetilde{\nu}))$ we have*

$$(\widetilde{K}f)^\wedge(\beta) = J^{-1}R(i\beta,A)J(\widehat{f}(\beta)).$$

For the proof we need the following result on convolution operators from [10]:

3.2. Lemma. *For a fixed $1\le q<\infty$, let $\mathbb{R}\ni t\mapsto k(t)$ be a function into positive operators on $L_q(\Omega,\nu)$ such that $t\mapsto k(t)f$ is locally Bochner integrable for $f\in L_q(\Omega,\nu)$. Assume that for all $0\le h\in L_q(\Omega,\nu)$ and $0\le g\in L_{q'}(\Omega,\nu)$ we have*

$$\int_{\mathbb{R}} \langle g, k(t)h\rangle\,dt \le C\|g\|_{L_q'}\|h\|_{L_q}.$$

Then the convolution integral

$$Kf(t) = \int_{-\infty}^{\infty} k(t-s)(f(s))\,ds$$

defined for step functions $f:\mathbb{R}\to L_q(\Omega)$ extends to a bounded operator on $L_q(\Omega, L_r(\mathbb{R}))$ for all $1\le r<\infty$ and $\|Kf\|_{L_q(L_r)} \le C\|f\|_{L_q(L_r)}$. ■

Proof of 3.1. We apply the extrapolation result (1) to $R = R(-2\varepsilon, A)$ for some ε with $\omega(T_t) < -2\varepsilon < 0$ where A is the generator of T_t. The operators $J^{-1}T_tJ$ may not extend to $L_q(\Omega,\widetilde{\nu})$ for all q but we can get around this difficulty by using the Yosida approximation of A and T_t:

$$A_\mu = \mu^2R(\mu,A) - \mu, \quad T_{\mu,t} = \exp(tA_\mu), \quad \mu>0.$$

Then $\widetilde{A}_\mu = J^{-1}A_\mu J$ and $\widetilde{T}_{\mu,t} = J^{-1}T_{\mu,t}J$ extend to bounded operators on $L_q(\Omega,\widetilde{\nu})$ for all $1\le q\le\infty$ since $R(\mu,A)\le R$ for $\mu\ge 0$.

It follows from [9], Chapter 1, Lemma 7.2, that

$$\begin{aligned} R(-\varepsilon,\widetilde{A}_\mu) &= (\mu-\varepsilon)^{-1}(\mu I - \widetilde{A})R\Big(\frac{-\mu\varepsilon}{\mu-\varepsilon},\widetilde{A}\Big) \\ &= (\mu-\varepsilon)^{-1}I + \Big(\frac{\mu}{\mu-\varepsilon}\Big)^2 R\Big(\frac{-\mu\varepsilon}{\mu-\varepsilon},\widetilde{A}\Big) \\ &\le I + 4R(-2\varepsilon,\widetilde{A}) \end{aligned} \tag{3}$$

for $\mu\ge\mu_0 := \max(1+\varepsilon, 2\varepsilon)$. Since the semigroups $T_{\mu,t}$ are positive, it follows by (2) that for $\mu\ge\mu_0$, $\lambda\ge-\varepsilon$ and all $1\le q\le\infty$,

$$\|R(\lambda,\widetilde{A}_\mu)\|_{L_q(\widetilde{\nu})} \le 1+4\|R(-2\varepsilon,\widetilde{A})\|_{L_q(\widetilde{\nu})} \le 1+8\|R(-2\varepsilon,A)\|_{L_p(\nu)} =: C. \tag{4}$$

We will also need a uniform growth estimate for $T_{\mu,t}$ in $L_p(\widetilde{\nu})$. Since $\omega(T_t) < -2\varepsilon$ there is a constant D with

$$(\mu+2\varepsilon)^n \|R(\mu,\widetilde{A})^n\|_{L_p(\widetilde{\nu})} \le D \quad \text{for } \mu \ge 0.$$

For $\mu \ge \mu_0$ we then obtain, for all $t \ge 0$,

$$(5) \quad \|T_{\mu,t}\|_{L_p} \le e^{-\mu t} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \mu^{2n} \|R(\mu,\widetilde{A})^n\| \le D e^{-\mu t} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \left(\frac{\mu^2}{\mu+2\varepsilon}\right)^n = D \exp\left(\frac{-2\varepsilon\mu}{\mu+2\varepsilon} t\right) \le D e^{-\varepsilon t}.$$

Now we consider the convolution integrals

$$K_\mu f(t) = \int_{-\infty}^{\infty} k_\mu(t-s)(f(s))\,ds$$

with $k_\mu(t) = \widetilde{T}_{\mu,t}$ for $t \ge 0$ and $k_\mu(t) = 0$ for $t < 0$. For all $0 \le h \in L_q(\Omega,\widetilde{\nu})$, $0 \le g \in L_{q'}(\Omega,\widetilde{\nu})$ and $\mu \ge \mu_0$ we get, by (4),

$$\int_{-\infty}^{\infty} \langle g, k_\mu(t)h\rangle\,dt = \int_0^{\infty} \langle g, \widetilde{T}_{\mu,t}h\rangle\,dt = \langle g, R(0,\widetilde{A}_\mu)h\rangle \le \|R(0,\widetilde{A}_\mu)\|_{L_q(\widetilde{\nu})} \|g\|_{L_{q'}} \|h\|_{L_q} \le C\|g\|_{L_{q'}} \|h\|_{L_q}.$$

Applying Lemma 3.2 to all $L_q(\Omega,\widetilde{\nu})$, $1 \le q < \infty$, with $r = q$ we see that K_μ is a bounded operator on the spaces $L_q(\mathbb{R}, L_q(\Omega,\widetilde{\nu}))$. If we restrict the K_μ's to $L_q(L_q) = L_q(\mathbb{R}_+, L_q(\Omega,\widetilde{\nu}))$ we conclude for all $1 \le q < \infty$ and all $\mu \ge \mu_0$ that

$$(6) \qquad \|K_\mu f\|_{L_q(L_q)} \le C\|f\|_{L_q(L_q)}$$

where $t \mapsto f(t) \in L_\infty(\Omega,\widetilde{\nu}) \subset L_q(\Omega,\widetilde{\nu})$ is a finite step function. Since such step functions are dense in $L_q(L_q)$ it remains to show that

$$(7) \qquad \lim_{\mu\to\infty} K_\mu f = \widetilde{K} f \quad \text{in } L_q(L_q).$$

First we note that (7) holds for $q = p$. Indeed, we have $\lim_{\mu\to\infty} \widetilde{T}_{\mu,s} g = \widetilde{T}_s g$ in $L_p(\Omega,\widetilde{\nu})$ for all $g \in L_p(\Omega,\widetilde{\nu})$ and the uniform growth estimate (5). In particular, $K_\mu f$ converges to $\widetilde{K} f$ in measure on $\mathbb{R} \times \Omega$.

Furthermore, for every function $f(t) = \chi_A(t) g$ with a bounded, measurable set $A \subset [0,a]$ and $0 \le g \in L_q(\Omega,\widetilde{\nu})$ the functions $K_\mu f$, $\mu \ge \mu_0$, are uniformly dominated by a function in $L_q(L_q)$:

$$K_\mu f(t) = \int_0^t \widetilde{T}_{\mu,s}[\chi_A(t-s)g]\,ds = \int_0^\infty e^{-\varepsilon s} \chi_A(t-s)(e^{\varepsilon s} \widetilde{T}_{\mu,s} g)\,ds \le e^{-\varepsilon(t-a)} \int_0^\infty e^{\varepsilon s} \widetilde{T}_{\mu,s} g\,ds = e^{-\varepsilon(t-a)} R(-\varepsilon,\widetilde{A}_\mu) g \le e^{-\varepsilon(t-a)} [I + 4R(-2\varepsilon,\widetilde{A})] g$$

for $\mu \ge \mu_0$ by (3). Lebesgue's convergence theorem and (6) now imply (7) and $\|\widetilde{K} f\| \le C\|f\|$.

Now we prove (iii). For a finite step function $t \mapsto f(t) \in L_p(\Omega,\widetilde{\nu}) \cap L_2(\Omega,\widetilde{\nu})$ we see that $\beta \mapsto (\widetilde{K} f)^\wedge(\beta)$ and $\beta \mapsto J^{-1} R(i\beta, A) J(\widehat{f}(\beta))$ belong to $L_\infty(L_p) \cap L_2(L_2)$ and are equal a.e. by the definition of $\widetilde{K}$. Since such step functions are dense in $L_2(L_2)$, (iii) follows from the boundedness of the Fourier transform on $L_2(L_2)$ and (i). ∎

Finally, we are prepared to prove our main result on norm continuity in $L_p(\Omega,\nu)$.

3.3. Theorem. *Let T_t be a positive semigroup on $L_p(\Omega,\nu)$ for $1 < p < \infty$ with generator A. Then T_t is continuous in the operator norm for $t > 0$ if and only if, for some $\alpha > s(A)$,*

$$(8) \qquad \|R(\alpha + i\beta, A)\| \to 0 \quad \text{as } |\beta| \to \infty.$$

3.4. Remark. As noted in the introduction, 3.3 implies that for a positive semigroup on $L_p(\Omega,\nu)$, (8) is sufficient for $s(A) = \omega(T_t)$ and for the spectral mapping theorem to hold.

Proof of 3.3. We may assume that $\omega(T_t) < 0$ and, by Theorem 3.1, that the operators $R(\lambda, A)$ extend to $L_q(\Omega,\nu)$ for all λ with $\operatorname{Re}\lambda \ge 0$ and all $1 \le q \le \infty$. Furthermore, the operator

$$Kf(t) = \int_0^t T_{t-s}(f(s))\,ds$$

on $L_p(L_p)$ extends to $L_q(L_q)$ for all $1 \le q < \infty$, where

$$L_q(L_q) = L_q(\mathbb{R}_+, L_q(\Omega,\nu)),$$

and the norm of all these operators is bounded by one constant $C < \infty$.

By applying the Riesz interpolation theorem to the resolvent on $L_p(\Omega,\nu)$ and $L_{p'}(\Omega,\nu)$, $1/p + 1/p' = 1$, respectively, we obtain

$$\|R(i\beta, A)\|_{L_2(\Omega,\nu)} \le \|R(i\beta, A)\|^{1/2}_{L_{p'}(\Omega,\nu)} \|R(i\beta, A)\|^{1/2}_{L_p(\Omega,\nu)}.$$

Since $\|R(i\beta, A)\|_{L_{p'}(\Omega,\nu)} \le C$ the assumption gives

$$\|R(i\beta, A)\|_{L_2(\Omega,\nu)} \to 0 \quad \text{as } |\beta| \to \infty.$$

By 3.1 and the proof of the Hilbert space result 2.5 we conclude that K restricted to $L_2([0,1], L_2(\Omega,\nu))$ satisfies condition (R_2). If $S_h f(s) = f(s+h)$

again denotes the (periodic) shift by h on $L_q(L_q) := L_q([0,1], L_q(\Omega,\nu))$ then (R_q) for $X = L_q(\Omega,\nu)$ is equivalent to

$$\|(I - S_h)K\| \to 0 \quad \text{as } h \to 0.$$

Hence, again by interpolation, K satisfies (R_p) on $L_p(L_p)$. Indeed, by [2], Theorem 5.1.2, we have

$$\|(I - S_h)K\|_{L_p(L_p)} \le \|(I - S_h)K\|^{1-\theta}_{L_q(L_q)} \|(I - S_h)K\|^{\theta}_{L_2(L_2)}$$

where in the case $1 < p < 2$ we choose $q = 1$ and $\theta = 2/p'$ and in the case $2 < p < \infty$ we choose some q with $p < q < \infty$ and $\theta = 2(q-p)/(p(q-2))$. Since we have $\|(I - S_h)K\|_{L_q(L_q)} \le 2C$, an appeal to Theorem 2.4 completes the proof. ■

References

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Function spaces and spectra of elliptic operators on a class of hyperbolic manifolds

by

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Abstract. The paper deals with quarkonial decompositions and entropy numbers in weighted function spaces on hyperbolic manifolds. We use these results to develop a spectral theory of related Schrödinger operators in these hyperbolic worlds.

1. INTRODUCTION

A bounded connected domain Ω in $\mathbb{R}^n$ is called a *d-domain* if, roughly speaking, its boundary $\partial\Omega$ is an inner Minkowski d-set. Here $n-1 \le d < n$, with $d = n-1$ in the case of a Lipschitzian boundary, whereas $n-1 < d < n$ indicates fractal distortions. We convert Ω in a non-compact hyperbolic manifold M of bounded geometry and with positive injectivity radius by introducing the Riemannian metric

$$ds^2 = g^2(x)\, dx^2, \quad x \in \Omega, \tag{1.1}$$

where $g(x)$ is a positive C^∞ function in Ω with

$$(\mathrm{dist}(x, \partial\Omega))^{-1} \sim g(x), \quad x \in \Omega, \tag{1.2}$$

where "$\sim$" means that the quotient of the two functions involved can be estimated from above and from below by positive constants which are independent of $x \in \Omega$. Based on [Tri86] and [Tri87] we developed in [Tri92], Ch. 7, a theory of two scales of function spaces $F^s_{pq}(M)$ and $B^s_{pq}(M)$ on (abstract) Riemannian manifolds with bounded geometry and positive injectivity radius. These scales include (fractional) Sobolev spaces, (classical) Besov spaces, Hölder–Zygmund spaces and (inhomogeneous) Hardy spaces. Under the above more special circumstances there is no problem to introduce weighted spaces $F^s_{pq}(M, g^\varkappa)$ with $\varkappa \in \mathbb{R}$. The paper deals with the following topics:

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