STUDIA MATHEMATICA 134 (2) (1999)

Spectral localization, power boundedness and invariant subspaces under Ritt's type condition

by

Yu. LYUBICH (Haifa)

Abstract. For a bounded linear operator T in a Banach space the Ritt resolvent condition $\|R_\lambda(T)\| \le C/|\lambda - 1|$ ($|\lambda| > 1$) can be extended (changing the constant C) to any sector $|\arg(\lambda - 1)| \le \pi - \delta$, $\arccos(C^{-1}) < \delta < \pi/2$. This implies the power boundedness of the operator T. A key result is that the spectrum $\sigma(T)$ is contained in a special convex closed domain. A generalized Ritt condition leads to a similar localization result and then to a theorem on invariant subspaces.

The *Ritt resolvent condition* [6] is the inequality

$$\|R_\lambda\| \le \frac{C}{|\lambda - 1|} \quad (|\lambda| > 1;\ C = \text{const} > 0) \tag{1}$$

for the resolvent $R_\lambda \equiv R_\lambda(T) = (T - \lambda I)^{-1}$ of a bounded linear operator T in a Banach space X. Of course, according to (1) the spectrum $\sigma(T)$ lies in the closed unit disk $|\lambda| \le 1$. Our main goal is to precisely localize the spectrum (Theorems 2 and 3). In particular, we extend the inequality of form (1) (with another C) to the sector

$$S_\delta = \{\lambda : |\arg(\lambda - 1)| \le \pi - \delta\}, \qquad 1 \notin S_\delta, \tag{2}$$

with $\arccos(C^{-1}) < \delta < \pi/2$. This sector is the maximal possible for any given C (see Remark 1 to Theorem 2).

By a theorem of O. Nevanlinna ([5], Theorem 4.5.4) any sectorial extension with $0 < \delta < \pi/2$ implies the power boundedness of T,

$$\sup_{n\ge 0} \|T^n\| < \infty, \tag{3}$$

and also the following quantitative version of the well-known Katznelson–Tzafriri Theorem [2]:

$$\|T^n - T^{n+1}\| = O(n^{-1}). \tag{4}$$

Thus, we obtain

1991 *Mathematics Subject Classification*: 47A10, 47A35.

THEOREM 1. *Any operator T satisfying the Ritt condition is power bounded and estimate* (4) *holds.*

This answers a question raised by J. Zemánek at the 10th Matrix Conference (Haifa, January 1998). Earlier it was only known that $\|T^n\| = O(\ln n)$ under the Ritt condition [8]. R. K. Ritt himself obtained $\|T^n\| = o(n)$.

Theorem 1 was independently proven by B. Nagy and J. Zemánek [4], also by a sectorial extension of (1). Since in [4] extension is not maximal possible, the proof turns out to be very short.

Let us start with some known simple facts.

The *resolvent set* $\varrho(T) = \mathbb{C} \setminus \sigma(T)$ is open; moreover, if $\lambda \in \varrho(T)$ then

$$\{\mu : |\mu - \lambda| < \|R_\lambda\|^{-1}\} \subset \varrho(T). \tag{5}$$

This immediately follows from the formula

$$R_\mu = \sum_{k=0}^{\infty} R_\lambda^{k+1}(\mu - \lambda)^k \tag{6}$$

where the series uniformly converges on the above mentioned disk. As a consequence, we have the lower bound

$$\operatorname{dist}(\lambda, \sigma(T)) \geq \|R_\lambda(T)\|^{-1} \quad (\lambda \in \varrho(T)). \tag{7}$$

Therefore if $\mu \in \partial\sigma(T)$ then $\|R_\lambda\| \to \infty$ as $\lambda \to \mu$, $\lambda \in \varrho(T)$.

When applied to (1) the last statement shows that actually

$$\sigma(T) \subset \{\mu : |\mu| < 1\} \cup \{1\} \tag{8}$$

under the Ritt condition and, moreover,

$$\|R_\lambda\| \leq \frac{C}{|\lambda - 1|} \quad (|\lambda| \geq 1,\ \lambda \neq 1). \tag{9}$$

Note that always $C \geq 1$ since $\lambda R_\lambda \to -I$ as $|\lambda| \to \infty$. Setting $q = C^{-1}$, so that $0 < q \leq 1$, we obtain the inequality

$$\operatorname{dist}(\lambda, \sigma(T)) \geq q|\lambda - 1| \quad (|\lambda| \geq 1). \tag{10}$$

PROPOSITION 1. *The Ritt condition implies that*

$$\sigma(T) \subset M \equiv \{\mu : |\mu| \leq 1 \;\&\; |\mu - e^{i\varphi}| \geq q|e^{i\varphi} - 1| \ (|\varphi| \leq \pi)\}. \tag{11}$$

Obviously, $1 \in M$. In the case $q = 1$ the set M is the singleton $\{1\}$, hence $\sigma(T) = \{1\}$. From now on we assume $q < 1$.

THEOREM 2. *The set M is convex and it is contained in the sector*

$$\pi - \arccos q \leq |\arg(\mu - 1)| \leq \pi \tag{12}$$

with vertex $\mu = 1$ (included in the sector).

Thus, the spectrum $\sigma(T)$ is contained in that sector.

Actually, this sector is tangent to M at the point $\mu = 1$, so it is the minimal sector with vertex 1 containing M.

Proof of Theorem 2. Consider the one-parameter family of circles

$$\Gamma_\varphi = \{\mu : |\mu - e^{i\varphi}| = q|e^{i\varphi} - 1|\} \quad (|\varphi| \leq \pi), \tag{13}$$

i.e.

$$\Gamma_\varphi = \{\mu : |\mu e^{-i\varphi} - 1|^2 = 4q^2 \sin^2(\varphi/2)\} \quad (|\varphi| \leq \pi). \tag{14}$$

The family $\{\Gamma_\varphi\}$ has two envelopes, inside and outside the unit disk respectively. We are interested in the inner one which we denote by L. In order to find L we follow the standard prescription to differentiate equation (14) with respect to φ and then consider both equations simultaneously. Thus, we consider the system of equations

$$\begin{cases} |\mu e^{-i\varphi} - 1|^2 = 4q^2 \sin^2(\varphi/2), & (15)\\ \operatorname{Im}(\mu e^{-i\varphi} - 1) = -q^2 \sin\varphi. & (16)\end{cases}$$

Solving this system under the constraint $|\mu| \leq 1$ we find

$$\operatorname{Re}(\mu e^{-i\varphi} - 1) = -2q|\sin(\varphi/2)|\sqrt{1 - q^2\cos^2(\varphi/2)} \tag{17}$$

and then

$$\mu = e^{i\varphi}(1 - 2q|\sin(\varphi/2)|\sqrt{1 - q^2\cos^2(\varphi/2)} - iq^2 \sin\varphi) \quad (|\varphi| \leq \pi). \tag{18}$$

The last curve is, in fact, the envelope L: at every $\varphi \neq 0$ the point μ defined by (18) belongs to the circle Γ_φ and μ is the tangency point of L and Γ_φ. For $\varphi = 0$ we have $\mu = 1$. Here only the one-sided tangents exist.

Obviously, L is a closed curve: $\mu = 2q - 1$ for $\varphi = \pm\pi$. Moreover, L is symmetric with respect to the real axis, $\mu(-\varphi) = \overline{\mu(\varphi)}$, and $L \cap \mathbb{R} = \{\mu(0), \mu(\pi)\} = \{1, 2q-1\}$. We establish that L is a convex curve, the boundary of a convex closed domain N, and then we prove that $M = N$.

Let us simplify (18) using some auxiliary parameters. By elementary algebra, for

$$t = \frac{q}{\sqrt{1 - q^2}}|\sin(\varphi/2)| \tag{19}$$

we obtain

$$\mu = q^2 + (1 - q^2)e^{i\varphi}(\sqrt{1 + t^2} - t)^2 \quad (|\varphi| \leq \pi). \tag{20}$$

Using the hyperbolic substitution

$$t = \operatorname{sh}(\tau/2) \tag{21}$$

we get

$$\mu = q^2 + (1 - q^2)e^{-\tau + i\varphi}. \tag{22}$$

By (19) and (21), τ is a continuous function of φ such that

$$\operatorname{sh}(\tau/2) = \frac{q}{\sqrt{1-q^2}}|\sin(\varphi/2)|. \tag{23}$$

Obviously, the function τ is infinitely smooth everywhere except at $\varphi = 0$. At that point the derivative $\tau'(\varphi)$ has a jump,

$$\tau'(\pm 0) = \pm\frac{q}{\sqrt{1-q^2}}. \tag{24}$$

As we see from (22), our curve L is an affine image of the curve

$$\Lambda = \{\zeta : \zeta = e^{-\tau+i\varphi}\ (|\varphi| \le \pi)\}. \tag{25}$$

It is sufficient to check that Λ is a convex curve. Note that Λ is simple, i.e. has no self-intersections. In this case a sufficient condition for convexity is

$$\det(\zeta', \zeta'') > 0 \quad (\varphi \ne 0), \qquad \det(\zeta'(-0), \zeta'(+0)) > 0. \tag{26}$$

Indeed, the first inequality means that the tangent vector rotates in the positive direction when the angle $\varphi = \arg\zeta$ increases in the intervals $(-\pi, 0)$, $(0, \pi)$; the second inequality means that the jump of the tangent vector at $\varphi = 0$ is also positively oriented.

By differentiation of (25),

$$\zeta' = e^{-\tau+i\varphi}u, \qquad \zeta'' = e^{-\tau+i\varphi}v \tag{27}$$

where

$$u = -\tau' + i, \qquad v = [(\tau')^2 - \tau'' - 1] - 2i\tau'. \tag{28}$$

Since the determinant is orthogonally invariant, we have

$$\det(\zeta', \zeta'') = e^{-2\tau}\det(u, v). \tag{29}$$

In turn,

$$\det(u, v) = \operatorname{Im}(\overline{u}v) = \tau'' + (\tau')^2 + 1. \tag{30}$$

By differentiation of (23),

$$\begin{aligned}\tau''\operatorname{ch}(\tau/2) + \frac{1}{2}(\tau')^2\operatorname{sh}(\tau/2) &= -\frac{1}{2}\frac{q}{\sqrt{1-q^2}}|\sin(\varphi/2)| \\ &= -\frac{1}{2}\operatorname{sh}(\tau/2) \quad (\varphi \ne 0).\end{aligned} \tag{31}$$

Hence we get

$$\tau'' = -\tfrac{1}{2}[(\tau')^2 + 1]\operatorname{th}(\tau/2) \tag{32}$$

and then

$$\det(u, v) = \left[1 - \tfrac{1}{2}\operatorname{th}(\tau/2)\right] \cdot [(\tau')^2 + 1] > 0. \tag{33}$$

Thus, the first part of condition (26) is true. The second part is also true. Indeed,

$$\begin{aligned}\det(\zeta'(-0), \zeta'(+0)) &= \det(u'(-0), u'(+0)) = \tau'(+0) - \tau'(-0) \\ &= \frac{2q}{\sqrt{1-q^2}} > 0.\end{aligned} \tag{34}$$

The convex curve L lies in its tangent sector at the point $\mu = 1$. The sector is just (12). Indeed, L is symmetric, and (22) with (24) imply that

$$\mu'(+0) = (1-q^2)(-\tau'(+0) + i) = \sqrt{1-q^2}\, e^{i\theta} \tag{35}$$

where $\theta = \arccos(-q) = \pi - \arccos q$.

In order to finish the proof of Theorem 2 we consider the convex closed domain N bounded by L and prove that $M = N$. Note that $N \setminus \{1\}$ lies in the open unit disk (since $|\mu| < 1$ by (22) with $\varphi \ne 0$). Let

$$\Delta_\varphi = \{\mu : |\mu| \le 1 \,\&\, |\mu - e^{i\varphi}| < q|e^{i\varphi} - 1|\} \quad (\varphi \ne 0) \tag{36}$$

be the circular lune bounded by the corresponding arcs of the circle Γ_φ and the unit circle $|\mu| = 1$. Since the center $e^{i\varphi}$ of Γ_φ lies outside L and L is a convex curve contacting Γ_φ, the lune Δ_φ also lies outside L. Hence, $\Delta_\varphi \cap N = \emptyset$, so that

$$N \cap \bigcup_\varphi \Delta_\varphi = \emptyset. \tag{37}$$

On the other hand, if $\mu \notin N$ and $|\mu| \le 1$ then $\mu \in \Delta_\varphi$ for some $\varphi \ne 0$, namely, for φ such that Γ_φ contacts L at the point nearest to μ. We conclude that the union of all Δ_φ complements N to the closed unit disk. By definition (11), $N = M$. ∎

REMARK 1. As we see from (15), the distance from the point $\lambda = e^{i\varphi}$ $(\varphi \ne 0)$ to the curve L is $2q|\sin(\varphi/2)| = q|\lambda - 1|$. Therefore if T is a normal operator in a Hilbert space such that $\sigma(T) = L$ then

$$\|R_\lambda\| = \frac{1}{q|\lambda - 1|} = \frac{C}{|\lambda - 1|} \quad (|\lambda| = 1,\ \lambda \ne 1). \tag{38}$$

We see that *the convex set M and hence the sector* (12) *are the smallest containing the spectrum under the Ritt condition with any given C.*

REMARK 2. The outer envelope $\widetilde{L}$ is not convex but consists of two parts (in the upper and lower half-planes) which are convex.

REMARK 3. Using the standard rational parametrization of the unit circle one can rewrite equations (15) and (16) in algebraic form. The classical resultant theory shows that $L \cup \widetilde{L}$ is an algebraic curve of degree 6. However, the transcendental representation (22) turns out to be more convenient for our goal.

Now we can prove

COROLLARY. *Let an operator T satisfy the Ritt condition. Then for any sector S_δ of the form* (2) *with* $\arccos q < \delta < \pi/2$ $(q = C^{-1})$ *the estimate*

$$\|R_\lambda\| < \frac{C(\delta)}{|\lambda - 1|} \quad (\lambda \in S_\delta,\ \lambda \neq 1) \tag{39}$$

holds with

$$C(\delta) = C(1 - C\cos\delta)^{-1}. \tag{40}$$

Proof. By Theorem 2, $S_\delta \setminus \{1\} \subset \varrho(T)$. Take a point $\mu \in S_\delta$ with $|\mu| < 1$. The vector $\mu - 1$ determines a chord in the unit disk. Consider the corresponding arc of the unit circle lying in the sector and let λ be the point on the arc whose orthogonal projection onto the chord is μ. Let θ be the acute angle between the vectors $\mu - 1$ and $\lambda - 1$. Then $\theta < \pi/2 - \delta$, hence $\sin\theta < \cos\delta < q$. Looking at the triangle with vertices $1, \lambda, \mu$ we see that $|\mu - \lambda| = |\lambda - 1|\sin\theta$. The Ritt condition yields

$$\|R_\lambda\| \cdot |\mu - \lambda| \le \frac{C|\mu - \lambda|}{|\lambda - 1|} = C\sin\theta < C\cos\delta < 1. \tag{41}$$

Now we can apply (6) and get

$$\|R_\mu\| \le \frac{\|R_\lambda\|}{1 - \|R_\lambda\| \cdot |\mu - \lambda|} < \frac{\|R_\lambda\|}{1 - C\cos\delta}. \tag{42}$$

Using the Ritt condition again we obtain the estimate

$$\|R_\mu\| < \frac{C}{1 - C\cos\delta} \cdot \frac{1}{|\lambda - 1|} < \frac{C}{1 - C\cos\delta} \cdot \frac{1}{|\mu - 1|}. \tag{43}$$

Here the constant factor is greater than C, so (43) is also valid for $\mu \in S_\delta$ with $|\mu| \ge 1$. ■

REMARK. The case $C = 1$ is included in Theorem 2. (We know that $\sigma(T) = \{1\}$ in this case.) The proof does not need Theorem 2, so this case is the simplest one.

Now we consider a more general condition

$$\|R_\lambda\| \le C/\gamma(\varphi) \quad (\lambda = e^{i\varphi},\ 0 < \varphi < 2\pi) \tag{44}$$

where $C = \text{const} > 0$, the function $\gamma(\varphi)$ is defined and continuous on $[0, 2\pi]$ (this range for φ is now more convenient than $[-\pi, \pi]$), and

$$\gamma(\varphi) > 0 \quad (0 < \varphi < 2\pi), \qquad \gamma(2\pi) = \gamma(0) = 0. \tag{45}$$

We do not assume the symmetry $\gamma(2\pi - \varphi) = \gamma(\varphi)$.

Under (44) the spectrum $\sigma(T)$ may partly lie outside the set $D \cup \{1\}$ where D is the open unit disk. Accordingly, $\sigma(T) = \sigma_-(T) \cup \sigma_+(T)$ where $\sigma_\pm(T)$ are those parts of $\sigma(T)$ which are contained in $D \cup \{1\}$ and in $(\mathbb{C} \setminus \overline{D}) \cup \{1\}$ respectively. Here $\overline{D}$ is the closed unit disk. Obviously, $\sigma_-(T) \cap \sigma_+(T) \subset \{1\}$.

THEOREM 3. *Suppose that condition* (44) *is satisfied for an operator T with a function $\gamma \in C^2[0, 2\pi]$ satisfying* (45). *Furthermore, assume that*

$$C > \pi \max\sqrt{\gamma^2 + (\gamma')^2 + (\gamma'')^2}. \tag{46}$$

Then the boundary of the set

$$M = \{\mu : |\mu - e^{i\varphi}| \ge q\gamma(\varphi)\ (0 \le \varphi \le 2\pi)\}, \qquad q = C^{-1}, \tag{47}$$

is the union of two closed simple curves L_- and L_+ such that $L_- \cap L_+ = \{1\}$, and

$$L_- \subset D \cup \{1\}, \qquad L_+ \subset (\mathbb{C} \setminus \overline{D}) \cup \{1\}. \tag{48}$$

If $M_\pm$ stand for the closed domains bounded by the curves $L_\pm$ respectively, then

$$M = M_- \cup \overline{\mathbb{C} \setminus M_+} \tag{49}$$

and

$$\sigma_-(T) \subset M_-, \qquad \sigma_+(T) \subset \overline{\mathbb{C} \setminus M_+}. \tag{50}$$

In addition, the curve L_- is convex, i.e. M_- is a convex closed domain. The curve L_+ is convex iff $\gamma'(2\pi) = \gamma'(0) = 0$. In the opposite case the intersections of M_+ with the half-planes $\{\mu : \operatorname{Im}\mu \ge 0\}$ and $\{\mu : \operatorname{Im}\mu \le 0\}$ are convex.

Both $L_\pm$ are regular C^1-curves except, maybe, at the point $\mu = 1$ (corresponding to $\varphi = 0$ and $\varphi = 2\pi$) where the tangent angle for L_- is

$$\pi/2 + \arcsin(q\gamma'(0)) \le \varphi \le 3\pi/2 + \arcsin(q\gamma'(2\pi)). \tag{51}$$

The tangent angle for L_+ at $\mu = 1$ is the mirror image of (51) *in the imaginary axis.*

Near this point L_+ stays outside the tangent angle. Actually, the union $L_+ \cup L_-$ is a regular smooth curve with only one self-intersection at $\mu = 1$.

We see that L_+ *is convex iff its smoothness extends to the point* $\mu = 1$, so that both the curves $L_\pm$ turn out to be smooth everywhere. Indeed, $\gamma'(0) \ge 0$, $\gamma'(2\pi) \le 0$ by (45), hence if $\gamma'(2\pi) = \gamma'(0)$ then both these values are zero.

Note that Theorem 2 with restriction (46) for C is contained in the non-smooth case of Theorem 3 (cf. (51) and (12) taking the difference in the choice of branches for φ into account).

For the further proof of Theorem 3 it is convenient to rewrite (46) as

$$q\max\sqrt{\gamma^2 + (\gamma')^2 + (\gamma'')^2} < 1/\pi. \tag{52}$$

The constant $1/\pi$ comes from the following purely geometrical

LEMMA. *Let $\omega(\varphi)$ $(0 \le \varphi \le 2\pi;\ \omega(2\pi) = \omega(0))$ be a complex-valued function satisfying the Lipschitz condition*

$$|\omega(\varphi_1) - \omega(\varphi_2)| \le K|\varphi_1 - \varphi_2| \tag{53}$$

with a constant $K < 2/\pi$. Then the closed curve

$$\Omega:\quad \mu = e^{i\varphi} + \omega(\varphi) \quad (0 \le \varphi \le 2\pi) \tag{54}$$

is simple.

Proof. Suppose that $\mu(\varphi_1) = \mu(\varphi_2)$ for some φ_1 and φ_2 with $0 \le \varphi_1 < \varphi_2 < 2\pi$. Then, according to (54), we obtain

$$|e^{i\varphi_2} - e^{i\varphi_1}| = |\omega(\varphi_2) - \omega(\varphi_1)| < \frac{2}{\pi}(\varphi_2 - \varphi_1). \tag{55}$$

Setting $\alpha = \frac{1}{2}(\varphi_2 - \varphi_1)$, so that $0 < \alpha < \pi$, one can rewrite inequality (55) as

$$\frac{\sin\alpha}{\alpha} < \frac{2}{\pi}. \tag{56}$$

On the other hand,

$$\begin{aligned}|\omega(\varphi_2) - \omega(\varphi_1)| &\le |\omega(2\pi) - \omega(\varphi_2)| + |\omega(\varphi_1) - \omega(0)| \\ &< \frac{2}{\pi}(2\pi - \varphi_2 + \varphi_1),\end{aligned} \tag{57}$$

i.e.

$$\frac{\sin(\pi - \alpha)}{\pi - \alpha} < \frac{2}{\pi}. \tag{58}$$

The function $\sigma(t) = (\sin t)/t$ decreases on $(0,\pi)$ and $\sigma(\pi/2) = 2/\pi$. Hence, if $\sigma(t) < 2/\pi$ then $t > \pi/2$. Therefore we get a contradiction: $\alpha > \pi/2$ from (56) and $\pi - \alpha > \pi/2$ from (58). ∎

REMARK. *The bound $K < 2/\pi$ is the best possible.* Indeed, for $\omega(\varphi) = -2(i/\pi)\arcsin(\sin\varphi)$ the exact Lipschitz constant is just $2/\pi$ but $\mu(3\pi/2) = \mu(\pi/2) = 0$, so the corresponding curve Ω is not simple.

Proof of Theorem 3. As in Theorem 2 we consider the one-parameter family of circles

$$\Gamma_\varphi = \{\mu : |\mu - e^{i\varphi}| = q\gamma(\varphi)\} \quad (0 \le \varphi \le 2\pi) \tag{59}$$

and find its envelope from the system of equations

$$\begin{cases}|\mu e^{-i\varphi} - 1|^2 = q^2\gamma^2, & (60)\\ \mathrm{Im}(\mu e^{-i\varphi} - 1) = -q^2\gamma\gamma' & (61)\end{cases}$$

$(\gamma = \gamma(\varphi),\ \gamma' = \gamma'(\varphi)$ for short). Since

$$q|\gamma'| < 1 \tag{62}$$

by (52), the system has exactly two solutions:

$$\mathrm{Re}(\mu e^{-i\varphi} - 1) = \pm q\gamma\sqrt{1 - q^2(\gamma')^2}, \tag{63}$$

so, respectively,

$$\mu_\pm = e^{i\varphi}(1 \pm q\gamma\sqrt{1 - q^2(\gamma')^2} - iq^2\gamma\gamma') \quad (0 \le \varphi \le 2\pi). \tag{64}$$

We have

$$|\mu_\pm|^2 = 1 + q^2\gamma^2 \pm 2q\gamma\sqrt{1 - q^2(\gamma')^2}, \tag{65}$$

so that $|\mu_+| > 1$ for $0 < \varphi < 2\pi$ trivially but $|\mu_-| < 1$ is also true. Indeed, (52) provides

$$q\sqrt{\gamma^2 + 4(\gamma')^2} < 2/\pi \tag{66}$$

and, a fortiori,

$$q\gamma < 2\sqrt{1 - q^2(\gamma')^2}. \tag{67}$$

The envelope of the family $\{\Gamma_\varphi\}$ is just the union $L_- \cup L_+$ of the closed curves $L_\pm = \{\mu : \mu = \mu_\pm(\varphi)\ (0 \le \varphi \le 2\pi)\}$, and, as we already know, (48) holds, so that

$$L_- \cap L_+ = \{1\}. \tag{68}$$

In order to simplify (64) we introduce an auxiliary function of φ,

$$\theta = \arcsin(q\gamma'). \tag{69}$$

By (62) this definition is correct and $|\theta| < \pi/2$, hence $\cos\theta > 0$. Now we can write

$$\mu_\pm = e^{i\varphi} \pm q\gamma e^{i(\varphi\mp\theta)}. \tag{70}$$

Note that the function θ is of class $C^1[0,2\pi]$, hence $L_\pm$ are of the same class except, maybe, at $\mu = 1$ $(\varphi = 0,\ \varphi = 2\pi)$.

Let us extend the range of the parameter q by adjoining the negative values with the same bound for $|q|$ as (52) for $q > 0$. Actually, now we deal with the one-parameter family of curves

$$L_q:\quad \mu_q = e^{i\varphi} - q\gamma e^{i(\varphi+\theta_q)} \quad (0 \le \varphi \le 2\pi) \tag{71}$$

where θ_q is defined by (69), and our $\mu_\pm$ are, in fact, $\mu_{\pm q}$ where $q > 0$. Therefore it is sufficient to prove that L_q is convex for all q such that

$$|q|\sqrt{\gamma^2 + (\gamma')^2 + (\gamma'')^2} < 1/\pi. \tag{72}$$

From now on we omit the subscript q in (71) for short, so

$$L:\quad \mu = e^{i\varphi} - q\gamma e^{i(\varphi+\theta)} \tag{73}$$

with a fixed q satisfying (72).

To prove that L is simple we apply the Lemma with

$$\omega(\varphi) = -q\gamma e^{i(\varphi+\theta)} \quad (0 \le \varphi \le 2\pi). \tag{74}$$

The function ω is of class $C^1[0,2\pi]$ and

(75) $$\omega'(\varphi) = -qe^{i(\varphi+\theta)}(\gamma' + i(1+\theta')\gamma).$$

Hence,

(76) $$|\omega'(\varphi)| = |q|\sqrt{(1+\theta')^2\gamma^2 + (\gamma')^2}$$

where

(77) $$\theta' = \frac{q\gamma''}{\sqrt{1-q^2(\gamma')^2}}$$

according to (69). Obviously $|\theta'| < 1$ by (52). Therefore

(78) $$|\omega'(\varphi)| \le 2|q|\sqrt{\gamma^2 + (\gamma')^2} < 2/\pi,$$

hence ω satisfies (53) with $K < 2/\pi$ as required.

For further geometrical properties of L we need the derivatives of μ from (73). First of all,

(79) $$\mu' = e^{i(\varphi+\theta)}u$$

where

(80) $$u = i(\cos\theta - q\gamma(1+\theta')),$$

taking into account that $\sin\theta = q\gamma'$ by (69). Note that $\mathrm{Re}(u) = 0$. We prove that $u \neq 0$, which yields that $\mu' \neq 0$, i.e. L has no singularities (except, maybe, at $\mu = 1$).

Now we show that

(81) $$\cos\theta - q\gamma(1+\theta') > 0.$$

Indeed, (81) can be rewritten by (77) and (69) as

(82) $$q\gamma\sqrt{1-q^2(\gamma')^2} + q^2[(\gamma')^2 + \gamma\gamma''] < 1.$$

But this expression is less than

(83) $$|q|\gamma + q^2[(\gamma')^2 + \gamma\gamma''] \le |q|\gamma + q^2[\gamma^2 + (\gamma')^2 + (\gamma'')^2]$$

This, in turn, is less than $\pi^{-1} + \pi^{-2}$ by (52), so (82) is true a fortiori.

By (79) and (80) the right-sided tangent vectors to L at $\mu = 1$ are directed as

(84) $$e^{i[\pi/2+\arcsin(q\gamma'(0))]}, \quad e^{i[\pi/2+\arcsin(q\gamma'(2\pi))]}.$$

Hence, the corresponding tangent angle is just (51) for $q > 0$ and its mirror image in $i\mathbb{R}$ for $q < 0$.

Now let us temporarily assume that $\gamma \in C^3[0,2\pi]$, hence $\theta \in C^2[0,2\pi]$ and then $\mu \in C^2[0,2\pi]$. In this case it follows from (79) and (80) that

(85) $$\mu'' = e^{i(\varphi+\theta)}v$$

where

(86) $$v = i(1+\theta')u + u',$$

so that

(87) $$\mathrm{Re}(v) = -(1+\theta')(\cos\theta - q\gamma(1+\theta')).$$

This yields

(88) $$\det(\mu',\mu'') = \det(u,v) = \mathrm{Im}(\overline{u}v) = (1+\theta')(\cos\theta - q\gamma(1+\theta'))^2$$

and then

(89) $$\det(\mu',\mu'') > 0$$

because of $|\theta'| < 1$ and (81).

We have already got all the convexity statements of Theorem 3 but under the extra assumption $\gamma \in C^3[0,2\pi]$. For $\gamma \in C^2[0,2\pi]$ we can approximate γ by functions $\gamma_n \in C^3[0,2\pi]$ $(n = 1,2,\ldots)$ under the same condition (52). Indeed, (52) determines a ball in $C^2[0,2\pi]$ where the subset of C^3-functions is dense. The boundary conditions $\gamma_n(2\pi) = \gamma_n(0) = 0$ can also be ensured. Therefore the function $\mu \in C^1[0,2\pi]$ given by (73) is the limit of a sequence $\{\mu_n\}$ such that the corresponding domains are convex. The limit domain turns out to be convex as well.

Now the localization (50) of the spectrum can be proven as in Theorem 2. We do not repeat those arguments. ■

With a view to some further applications we now estimate from below the distance function

(90) $$d(\varphi) = \min\{\mathrm{dist}(\mu_-(\varphi), L_+), \mathrm{dist}(\mu_+(\varphi), L_-)\}.$$

Obviously, $d(0) = d(2\pi) = 0$ and $d(\varphi) > 0$ for $0 < \varphi < 2\pi$.

PROPOSITION 2. *Let $\gamma(\varphi)$ satisfy the conditions of Theorem 3 and assume it is convex in some neighborhoods of the points $\varphi = 0$ and $\varphi = 2\pi$. If $\varepsilon > 0$ is sufficiently small then there exists $a = a(\varepsilon) > 0$ such that*

(91) $$d(\varphi) \ge a\gamma(\varphi) \quad (0 < \varphi < \varepsilon,\ 2\pi - \varepsilon < \varphi < 2\pi).$$

Proof. For definiteness we will prove that

(92) $$\mathrm{dist}(\mu_-(\varphi), L_+) \ge a\gamma(\varphi) \quad (0 < \varphi < \varepsilon).$$

Let γ be convex on $[0,\delta]$ $(0 < \delta < \pi/2)$, i.e. $\gamma''(\varphi) \ge 0$ for $0 \le \varphi \le \delta$. Since $\gamma'(0) \ge 0$, we have $\gamma'(\varphi) \ge 0$ for $0 \le \varphi \le \delta$, so γ is non-decreasing on this segment.

Fix φ and set $\mu = \mu_-(\varphi)$ for short. Consider the point $\lambda = \mu_+(\psi)$ such that

(93) $$\mathrm{dist}(\mu, L_+) = |\mu - \lambda|.$$

We get $0 < \psi < \delta$ if $0 < \varphi < \varepsilon$ with sufficiently small ε. Also let $\varepsilon < \delta$ and then $0 < \varepsilon < \pi/2$ a fortiori.

Geometrically, λ is the tangency point of L_+ to the circle Γ_ψ (see (59)). The vector $\mu - \lambda$ is the inner normal to L_+ at λ, therefore it is directed as $e^{i\psi} - \lambda$ ($e^{i\psi}$ is the center of Γ_ψ). The curve L_- has no points inside Γ_ψ while $\mu \in L_-$. Since the radius of Γ_ψ equals $q\gamma(\psi)$, we obtain the inequality

$$|\mu - \lambda| \ge 2q\gamma(\psi). \tag{94}$$

Now we need to properly compare $\gamma(\psi)$ and $\gamma(\varphi)$. If $\psi \ge \varphi$ then $\gamma(\psi) \ge \gamma(\varphi)$ therefore we can assume that $\psi < \varphi$.

According to (79) and (80) the tangent vector to L_+ at the point λ is

$$\tau = ie^{i(\psi-\eta)}v \tag{95}$$

where $v > 0$ (see (81)) and $\eta = \arcsin(q\gamma'(\psi))$ (see (69)), so that

$$0 \le \eta < \arcsin(1/\pi) \tag{96}$$

by (72). The vector $i\tau$ is normal to L_+ at λ, hence

$$\mu - \lambda = -\varrho v e^{i(\psi-\eta)} \tag{97}$$

where $\varrho > 0$ since $i\tau$ is also directed toward the interior of L_+. Now (94) can be written as

$$\varrho v \ge 2q\gamma(\psi). \tag{98}$$

According to (70),

$$\mu = e^{i\varphi} - q\gamma(\varphi)e^{i(\varphi+\theta)}, \quad \lambda = e^{i\psi} + q\gamma(\psi)e^{i(\psi-\eta)} \tag{99}$$

and now (97) takes the form

$$e^{i\varphi} - e^{i\psi} = q\gamma(\varphi)e^{i(\varphi+\theta)} - (\varrho v - q\gamma(\psi))e^{i(\psi-\eta)}. \tag{100}$$

This can be rewritten as the real system

$$\begin{cases} \cos\varphi - \cos\psi = q\gamma(\varphi)\cos(\varphi+\theta) - (\varrho v - q\gamma(\psi))\cos(\psi-\eta), & (101) \\ \sin\varphi - \sin\psi = q\gamma(\varphi)\sin(\varphi+\theta) - (\varrho v - q\gamma(\psi))\sin(\psi-\eta), & (102) \end{cases}$$

which can be considered as a system of linear equations with unknowns $q\gamma(\varphi)$ and $\varrho v - q\gamma(\psi)$. Its determinant is $\sin(2\alpha + \theta + \eta)$ where $\alpha = \frac{1}{2}(\varphi - \psi) > 0$. The determinant is not zero since $0 < 2\alpha + \theta + \eta < \varepsilon + 2\arcsin(1/\pi) < \pi$. Therefore we can solve the system and get the quotient of the unknowns,

$$\frac{\varrho v - q\gamma(\psi)}{q\gamma(\varphi)} = \frac{\cos(\alpha+\theta)}{\cos(\alpha+\eta)}. \tag{103}$$

By convexity of γ we have $\gamma'(\psi) \le \gamma'(\varphi)$ and then $\eta \le \theta$, which implies $\cos(\alpha+\theta) \le \cos(\alpha+\eta)$ since $\alpha + \theta < \varepsilon/2 + \arcsin(1/\pi) < \pi/2$. Now (103) implies $\varrho v - q\gamma(\psi) \le q\gamma(\varphi)$ and (102) yields

$$\sin\varphi - \sin\psi \le 2q\gamma(\varphi). \tag{104}$$

On the other hand,

$$\frac{\sin\varphi - \sin\psi}{\varphi - \psi} = \cos\xi \tag{105}$$

where $\psi < \xi < \varphi$, so that $\cos\xi > \cos\varphi > \cos\varepsilon$. As a result,

$$0 < \varphi - \psi \le \frac{2q\gamma(\varphi)}{\cos\varepsilon}. \tag{106}$$

Using convexity again we obtain

$$\gamma(\psi) \ge \gamma(\varphi) - \gamma'(\varphi)(\varphi - \psi) \ge \left(1 - \frac{2q\gamma'(\varphi)}{\cos\varepsilon}\right)\gamma(\varphi) \tag{107}$$

and, finally,

$$\gamma(\psi) > \left(1 - \frac{2}{\pi\cos\varepsilon}\right)\gamma(\varphi) \tag{108}$$

by (72). Combining (108), (94) and (93) we conclude that (92) is valid with

$$a = 2q\left(1 - \frac{2}{\pi\cos\varepsilon}\right) \tag{109}$$

where ε is determined as before. ∎

Combining our results with the local spectral theory developed by Stampfli [7] (cf. [3]) we can prove the following

THEOREM 4. *Suppose that, in addition to the conditions of Theorem* 3, *the function* $\gamma(\varphi)$ *is convex near* $\varphi = 0$ *and*

$$\int_0^{2\pi} \ln\gamma(\varphi)\,d\varphi > -\infty. \tag{110}$$

Suppose that an operator T *satisfies* (44) *with constraint* (46). *If* $\sigma(T) \setminus \{1\}$ *has points in* D *and outside* $\overline{D}$ *then there exist non-trivial hyperinvariant closed subspaces* $X_\pm$ *such that*

$$\sigma(T|X_\pm) \subset M_\pm. \tag{111}$$

Proof. Theorem 3 remains in force if we replace $q = C^{-1}$ by any κ with $0 < \kappa < q$. In this way we obtain a family of curves $L_\pm(\kappa)$ with the same properties as the initial ones, $L_\pm$. However, all these curves intersect the set M (given by (47)) only at the point 1, so that the localization (50) can be sharpened:

$$\sigma_-(T) \setminus \{1\} \subset \operatorname{int} M_-(\kappa), \quad \sigma_+(T) \setminus \{1\} \subset \mathbb{C} \setminus M_+(\kappa) \tag{112}$$

where $M_\pm(\kappa)$ are the closed domains bounded by $L_\pm(\kappa)$ respectively.

For any point $\mu = \mu(\varphi) \in L_\pm(\kappa)$ we have

$$|\mu - \lambda| = \kappa\gamma(\varphi) \quad (\lambda = e^{i\varphi}). \tag{113}$$

We can estimate $\|R_\mu\|$ like (41), (42), namely,

$$\|R_\lambda\| \cdot |\mu - \lambda| \le \frac{C|\mu-\lambda|}{\gamma(\varphi)} = C\kappa < 1. \tag{114}$$

This yields

$$\|R_\mu\| \le \frac{\|R_\lambda\|}{1 - \|R_\lambda\| \cdot |\mu-\lambda|} \le \frac{C}{1-C\kappa} \cdot \frac{1}{\gamma(\varphi)}. \tag{115}$$

Therefore we could suppose from the very beginning that

$$\sigma_-(T) \setminus \{1\} \subset \operatorname{int} M_-, \quad \sigma_+(T) \setminus \{1\} \subset \mathbb{C} \setminus M_+. \tag{116}$$

Following [7] (Theorem 1 with subsequent remarks and Theorem 1′) we only need to construct two functions $f_\pm(\lambda)$ in $H^\infty(\overline{\mathbb{C} \setminus M_+})$ and $H^\infty(M_-)$ respectively, both with no zeros and such that

$$\sup\{\|f_\pm(\lambda) R_\lambda(T)\| : \lambda \in L_\pm\} < \infty \tag{117}$$

and, finally,

$$\int_{L_-}\int_{L_+} \frac{|f_\pm(\lambda)|}{|\lambda-\mu|}\,|d\lambda|\,|d\mu| < \infty. \tag{118}$$

For definiteness we do it for $f_-(\lambda)$. Our construction is a counterpart of a classical one which is well known for $H^\infty(\overline{D})$, where $\overline{D}$ is the closed unit disk (see, e.g., [1], Chapter 4).

Denote by $G(\lambda,\mu)$ the Green function of the Dirichlet problem for M_- and introduce

$$g(\lambda) = \int_0^{2\pi} G(\lambda, \mu_-(\varphi)) \ln \gamma(\varphi)\, ds(\varphi) \quad (\lambda \in \operatorname{int} M_-) \tag{119}$$

where $s(\varphi)$ is the arc length of the curve L_-. The function $g(\lambda)$ is harmonic and continuously extends to $L_- \setminus \{1\}$. Its boundary function is just $\ln\gamma(\varphi)$. Consider the conjugate harmonic function $h(\lambda)$ and set

$$f_-(\lambda) = e^{g(\lambda)+ih(\lambda)} \quad (\lambda \in \operatorname{int} M_-). \tag{120}$$

($h(\lambda)$ is well defined (up to an additive constant) since $\operatorname{int} M_-$ is simply connected.)

The function $f_-(\lambda)$ is analytic in $\operatorname{int} M_-$, it has no zeros and

$$|f_-(\lambda)| = e^{g(\lambda)} \le \int_0^{2\pi} G(\lambda, \mu_-(\varphi))\gamma(\varphi)\, ds(\varphi) \tag{121}$$

by the arithmetic-geometric means inequality. The function on the right-hand side of (121) is also harmonic in $\operatorname{int} M_-$ and continuously extends to $\gamma(\varphi)$ on the boundary. Thus, this function is bounded and we see that $f_- \in H^\infty(M_-)$.

The boundary function for $|f_-(\lambda)|$ is just $\gamma(\varphi)$. Hence,

$$\|f_-(\lambda) R_\lambda(T)\| \le C \quad (\lambda \in L_-), \tag{122}$$

i.e. (117) is valid. Further,

$$\int_{L_-}\int_{L_+} \frac{|f_-(\lambda)|}{|\lambda-\mu|}\,|d\lambda|\,|d\mu| \le \int_{L_+} |d\mu| \int_0^{2\pi} \frac{\gamma(\varphi)}{\operatorname{dist}(\mu_-(\varphi), L_+)}\, ds(\varphi) < \infty \tag{123}$$

by (92), i.e. (118) is valid as well. ∎

References

[1] K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, 1962.

[2] Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. 68 (1986), 313–328.

[3] Yu. Lyubich and V. Matsaev, *Operators with separable spectrum*, in: Amer. Math. Soc. Transl. (2) 47 (1965), 89–129.

[4] B. Nagy and J. Zemánek, *A resolvent condition implying power boundedness*, Studia Math. 134 (1999), 143–151.

[5] O. Nevanlinna, *Convergence of Iterations for Linear Equations*, Birkhäuser, 1993.

[6] R. K. Ritt, *A condition that* $\lim_{n\to\infty} n^{-1}T^n = 0$, Proc. Amer. Math. Soc. 4 (1953), 898–899.

[7] J. G. Stampfli, *A local spectral theory for operators IV; Invariant subspaces*, Indiana Univ. Math. J. 22 (1972), 159–167.

[8] E. Tadmor, *The resolvent condition and uniform power boundedness*, Linear Algebra Appl. 80 (1986), 250–252.

Department of Mathematics
Technion
32000 Haifa, Israel
E-mail: lyubich@techunix.technion.ac.il

Received March 16, 1998
Revised version December 17, 1998 (4065)