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The real-analytic solutions of the Abel functional equation

by

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Abstract. For the Abel equation on a real-analytic manifold a dynamical criterion of solvability in real-analytic functions is proved.

1. Introduction. In this paper we continue to investigate the Abel equation (A.e.)

$$\varphi(Fx) - \varphi(x) = 1, \quad x \in X, \tag{1.1}$$

from a dynamical point of view (see [1]). In the general case φ is an unknown real-valued continuous function on a topological space X and $F : X \to X$ is a continuous mapping. The following condition is necessary for solvability of the A.e. ([1], Corollary 4.2).

(CW) *All compact subsets of X are wandering.*

This means that for every compact set $K \subset X$ there exists $\nu \ge 1$ such that

$$F^n(K) \cap F^m(K) = \emptyset \quad (n - m \ge \nu). \tag{1.2}$$

This condition is also sufficient if the space X is locally compact and countable at infinity (l.c.c.i.) and the mapping F is injective ([1], Corollary 1.6). The injectivity is substantial ([1], Example 1.7).

The condition (CW) is also sufficient in a smooth situation. Namely, if X is a C^∞-manifold countable at infinity and F is a C^k-diffeomorphism with $0 < k \le \infty$ then (CW) implies the existence of a C^k-solution φ (*the C^k-solvability*). As we said this is also true for $k = 0$. Note that if F is a homeomorphism then (1.2) is equivalent to $F^n(K) \cap K = \emptyset$ $(n \ge \nu)$.

Now we establish a similar criterion for $k = \omega$, i.e. in the real-analytic situation. For short we will say "analytic" or "C^ω".

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Let us recall that a locally compact topological space is called *countable at infinity* if it is a countable union of compact subsets.

MAIN THEOREM. *Let X be a C^ω-manifold countable at infinity and let $F : X \to X$ be a C^ω-diffeomorphism. Then* (CW) *implies C^ω-solvability of the A.e.*

COROLLARY. *For the above described X and F the C^0-solvability implies the C^ω-solvability.*

Our proof of the Main Theorem is based on a reduction of the solvability problem to a conjugacy problem.

Given a C^ω-diffeomorphism $F : X \to X$, one can consider the one-parameter family

$$H_s(x,t) = (Fx, t+s) \quad (x \in X;\ s \in \mathbb{R}) \tag{1.3}$$

of C^ω-diffeomorphisms $\widetilde{X} \to \widetilde{X}$ where $\widetilde{X} = X \times \mathbb{R}$.

Suppose that the A.e. is C^ω-solvable, and φ is its C^ω-solution. The formula

$$\Phi(x,t) = (x, t + \varphi(x)) \quad (x \in X,\ t \in \mathbb{R}) \tag{1.4}$$

yields a C^ω-diffeomorphism $\Phi : \widetilde{X} \to \widetilde{X}$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{H_0} & \widetilde{X} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \widetilde{X} & \xrightarrow[H_1]{} & \widetilde{X} \end{array} \tag{1.5}$$

is commutative. The analytic diffeomorphisms H_0 and H_1 turn out to be analytically conjugate.

Conversely, let (1.5) be commutative with a C^ω-diffeomorphism $\Phi(x,t) = (\Psi(x,t), \varphi(x,t))$. Then for any fixed t the function $x \mapsto \varphi(x,t)$ is a C^ω-solution of the A.e.

Obviously, the same situation takes place in C^k, $0 \le k \le \infty$.

2. Proof of the Main Theorem. For the reader's convenience we recall a well known dynamical construction (see [2], p. 37).

Let $H : Y \to Y$ be an analytic diffeomorphism of an analytic manifold Y, $\dim Y = d$. For any point $y \in Y$ its *orbit* is $O(y) = \{H^n y : n \in \mathbb{Z}\}$. Any two different orbits are disjoint, so that the orbits are classes of an equivalence relation. One can consider the corresponding quotient space Y/H provided with the standard quotient topology: a set $V \subset Y/H$ is open if and only if its preimage $U = \{y \in Y : O(y) \in V\}$ is open in Y. Under condition (CW), the space Y/H is Hausdorff. To show this, consider two distinct orbits, say, $O(y_1) \neq O(y_2)$. If W_1 and W_2 are neighborhoods of y_1 and y_2 respectively, such that

$$H^n(W_1) \cap H^m(W_2) = \emptyset \quad (n, m \in \mathbb{Z}) \tag{2.1}$$

then the neighborhoods

$$V_k = \{O(y) : y \in W_k\} \quad (k = 1,2)$$

separate the orbits in Y/H. Such W_1 and W_2 do exist, since otherwise there exist some sequences $\{l_i\}_{i=1}^\infty$ in $\mathbb{Z}$ and $\{v_i\}_{i=1}^\infty$ in Y such that

$$\lim_{i\to\infty} v_i = y_1, \quad \lim_{i\to\infty} H^{l_i} v_i = y_2, \quad |l_i| \to \infty,$$

and (1.2) is violated for K consisting of y_1, y_2, all v_i and all $H^{l_i} v_i$.

Now let $\{U_\alpha, \Phi_\alpha\}$ be an atlas giving the analytic structure on Y. This means that $\{U_\alpha\}$ is an open covering of Y and $\Phi_\alpha : U_\alpha \to \mathbb{R}^d$ are homeomorphisms onto their images such that all $\Phi_\beta \Phi_\alpha^{-1}$ are C^ω-diffeomorphisms between $\Phi_\alpha(U_\alpha \cap U_\beta)$ and $\Phi_\beta(U_\alpha \cap U_\beta)$.

For any $y \in Y$ choose $\alpha(y)$ such that $y \in U_{\alpha(y)}$ and then a neighborhood $W_y \ni y$ such that $W_y \subset U_{\alpha(y)}$ and $H^n(W_y) \cap W_y = \emptyset$ for all integer n. (The last property can be proved like (2.1)). Then the natural mapping $Y \to Y/H$ restricted to W_y is a homeomorphism Ω_y between W_y and its image. We obtain the atlas $\{\Omega_y(W_y),\ \Phi_{\alpha(y)}\Omega_y^{-1}\}$ giving an analytic structure on Y/H, the *canonical quotient analytic structure.*

Coming back to the proof of the Main Theorem we start with the case of a connected manifold X. Following our preliminary plan it is sufficient to obtain (1.5) with a C^ω-diffeomorphism Φ. However, under condition (CW) we already have (1.5) with a C^∞-diffeomorphism Φ. Indeed, as we know (see [1]), (CW) implies the C^∞-solvability of the A.e.

Now let us introduce the quotient spaces $Y_0 = \widetilde{X}/H_0$ and $Y_1 = \widetilde{X}/H_1$ where $\widetilde{X} = X \times \mathbb{R}$ (see (1.5)). They are analytic manifolds since all H_s satisfy (CW). Since H_0 and H_1 are C^∞-conjugate, the manifolds Y_0 and Y_1 are C^∞-diffeomorphic in the following natural way:

$$\Phi(O_0(x,t)) = O_1(\Phi(x,t))$$

where $O_s(x,t)$ is the H_s-orbit of the point (x,t).

It is a well-known fact that an analytic structure on a C^∞-manifold is unique (see [3], Chapter 2, §5). Hence, there exists an analytic diffeomorphism $\Psi : Y_0 \to Y_1$. Let $p_0 : \widetilde{X} \to Y_0$ and $p_1 : \widetilde{X} \to Y_0$ be the natural projections. Since $\operatorname{Im} \Psi p_0 = \operatorname{Im} p_1 = Y_1$, for any points $x', x'' \in \widetilde{X}$ there exists a unique continuous *lifting* $\widetilde{\Psi} : \widetilde{X} \to \widetilde{X}$, $\widetilde{\Psi}(x') = x''$, providing the commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\Psi}} & \widetilde{X} \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_1} \\ Y_0 & \xrightarrow{\Psi} & Y_1 \end{array}$$

(see [4], Theorem 16.3). By the same argument there exists a unique lifting $\widetilde{\Psi}_1 : \widetilde{X} \to \widetilde{X}$ for Ψ^{-1} such that $\widetilde{\Psi}_1(x'') = x'$. Since $\widetilde{\Psi}\widetilde{\Psi}_1$ and $\widetilde{\Psi}_1\widetilde{\Psi}$ are liftings for id: $\widetilde{X} \to \widetilde{X}$ with $\widetilde{\Psi}\widetilde{\Psi}_1(x'') = x''$ and $\widetilde{\Psi}_1\widetilde{\Psi}(x') = x'$, it follows from the uniqueness that $\widetilde{\Psi}\widetilde{\Psi}_1 = \widetilde{\Psi}_1\widetilde{\Psi} = \mathrm{id}$. Hence, $\widetilde{\Psi} : \widetilde{X} \to \widetilde{X}$ is a homeomorphism. Moreover, since p_0, p_1 are local analytic diffeomorphisms and Ψ is an analytic diffeomorphism, $\widetilde{\Psi}$ is also an analytic diffeomorphism.

Coming back to the previous commutative diagram, we see that

$$\Psi(O_0(x,t)) = O_1(\widetilde{\Psi}(x,t))$$

for any point $(x,t) \in \widetilde{X}$. Hence $\widetilde{\Psi}(z) \in O_1(\widetilde{\Psi}(x,t))$ for any $z \in O_0(x,t)$. In particular,

$$\widetilde{\Psi}(H_0(x,t)) \in O_1(\widetilde{\Psi}(x,t)).$$

The latter means that

$$\widetilde{\Psi}(H_0(x,t)) = H_1^n(\widetilde{\Psi}(x,t)) \quad ((x,t) \in \widetilde{X})$$

for some integer $n = n(x,t)$. Since the space $\widetilde{X} = X \times \mathbb{R}$ is connected, $n(x,t) = \text{const}$. Obviously, $n \neq 0$, otherwise $H_0 = \mathrm{id}$. Hence, H_0 is analytically conjugate to the diffeomorphism

$$H_1^n(x,t) = (F^n x, t+n)$$

by the diffeomorphism $\widetilde{\Psi}$. So, $\widetilde{\Psi}$ satisfies the equation

$$\widetilde{\Psi}(Fx,t) = \widetilde{\Psi}(x,t) + n.$$

It remains to note that the function $\varphi(x) = n^{-1}\widetilde{\Psi}(x,0)$ is a solution of (1.1).

Now let X be a disconnected manifold. It is sufficient to construct a solution of the A.e. on the orbit $\bigcup_{n\in\mathbb{Z}} F^n(U)$ of a connected component U.

Note that $F^n(U)$ is also a connected component for any $n \in \mathbb{Z}$. We have the following alternative: either a) $F^n(U) \neq U$ $(n \in \mathbb{Z})$, or (b) $F^n(U) = U$ for some $n \in \mathbb{Z}$.

In case (a) one can set $\varphi(x) = n$ for $x \in F^n(U)$, $n \in \mathbb{Z}$.

In case (b) let $p = \min\{n > 0 : F^n(U) = U\}$. As we have proved, the equation

$$\varphi(F^p x) - \varphi(x) = 1 \quad (x \in U)$$

has an analytic solution φ. Then the function

$$h(x) = p\varphi(F^{-j}x) + j \quad (x \in F^j(U),\ j = 0, \ldots, p-1)$$

is a solution of the A.e. on the union $\bigcup_n F^n(U)$. ∎

5. Some examples

1. *One-dimensional case.* We single out this case since the wandering property on the line can be reduced to a very simple one. Namely, it is clear that *all compact subsets are wandering with respect to a diffeomorphism* $F : \mathbb{R} \to \mathbb{R}$ *if and only if it has no fixed points, i.e.* $\mathrm{Fix}(F) = \emptyset$.

THEOREM 3.1. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a real-analytic fixed point free diffeomorphism. Then* F *is analytically conjugate to the shift* $x \mapsto x + 1$.

Proof. The quotient space $\mathbb{R}/F$ is a real-analytic manifold. Being a compact connected one-dimensional manifold, it is homeomorphic to the standard circle $\mathbb{T}$ (see [2], p. 138). Hence, $\mathbb{R}/F$ is analytically diffeomorphic to $\mathbb{T}$ provided with the standard analytic structure because the latter is unique (see [2], p. 245). Let $\widetilde{\Psi} : \mathbb{R} \to \mathbb{R}$ be the analytic lifting of an analytic diffeomorphism $\Psi : \mathbb{R} \setminus F \to \mathbb{T}$. As in Section 2 we get

$$\widetilde{\Psi}(Fx) = \widetilde{\Psi}(x) + n$$

where n is an integer, $n \neq 0$. Then the analytic diffeomorphism $\Phi(x) = n^{-1}\widetilde{\Psi}(x)$ conjugates F with the shift. ∎

2. *Some examples on the plane.* The simplest example on $\mathbb{R}^2$ with the above mentioned wandering property is the shift

$$F_e(x) = x + e \quad (x, e \in \mathbb{R}^2).$$

Obviously, any additive function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with $\varphi(e) = 1$ satisfies the A.e. for F_e.

Theorem 3.1 cannot be extended to the two-dimensional case.

EXAMPLE 3.2. Let $X = \mathbb{C}\setminus\mathbb{R}_-$, where $\mathbb{R}_- = \{\xi \in \mathbb{R} : \xi \leq 0\}$. Then X is analytically diffeomorphic to the plane $\mathbb{R}^2 \equiv \mathbb{C}$. Indeed, X is diffeomorphic to the open right half-plane H by $y = \sqrt{x}$, H is diffeomorphic to the unit disk D by $z = (1-y)/(1+y)$, and finally D is diffeomorphic to $\mathbb{C}$ by $w = z/(1-|z|)$. Denote by $T : X \to \mathbb{R}^2$ the composition of the previous analytic diffeomorphisms. Consider

$$G(x) = (\lambda\xi, -\lambda\eta) \quad (x = (\xi, \eta) \in X)$$

with some $\lambda \in (0,1)$. The mapping $F = TGT^{-1}$ is an analytic diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$. Moreover, F is fixed point free since so is G. However, F is not conjugate to any shift F_l since it does not preserve orientation, $\det F'(w) < 0$.

REMARK 3.3. The A.e. corresponding to the mapping F has an analytic solution

$$\varphi(w) = \frac{\ln \|T^{-1}w\|}{\ln \lambda} \quad (w \in \mathbb{R}^2).$$

Hence, F has (CW), a property much stronger than the absence of fixed points.

3. *Shifts on Lie groups*. Let X be a Lie group. Consider the right shift $F_g(x) = xg$ $(x \in X)$ with some $g \in X$. In this case (CW) is fulfilled if and only if the sequence $\{g^n\}_{n=-\infty}^{\infty}$ has no limit points (see [1]). Hence, we have

THEOREM 3.4. *The A.e. corresponding to the shift F_g has an analytic solution if and only if the sequence $\{g^n\}_{n=-\infty}^{\infty}$ has no limit points.*

In particular, consider $X = \mathrm{GL}(p, \mathbb{C})$. Following [1] we call a matrix $g \in \mathrm{GL}(p, \mathbb{C})$ *quasiunitary* if g is similar to a unitary matrix. Obviously, g is quasiunitary if and only if the spectrum g lies on the unit circle and g is diagonalizable. In other words, g is not quasiunitary if and only if the following alternative holds: either (a) there is λ with $|\lambda| \neq 1$ and $e \in \mathbb{C}^p$ such that $ge = \lambda e$, or (b) all eigenvalues of g are of modulus 1 and there is λ with $|\lambda| = 1$ and $e_1, e_2 \in \mathbb{C}^p$ such that $ge_1 = \lambda e_1$ and $ge_2 = \lambda e_2 + e_1$.

LEMMA 3.5. *A matrix $g \in \mathrm{GL}(p, \mathbb{C})$ is quasiunitary if and only if the sequence $\{g^n\}_{n=-\infty}^{\infty}$ has no limit points in $\mathrm{GL}(p, \mathbb{C})$.*

Proof. If g is unitary then, obviously, $\{g^n\}_{n=-\infty}^{\infty}$ has a limit point in the space of $p \times p$ matrices and all limit points are unitary. Hence, if g is quasiunitary then there exists a limit point of $\{g^n\}_{n=-\infty}^{\infty}$ and all of them belong to $\mathrm{GL}(p, \mathbb{C})$.

Let g be not quasiunitary. Then in case (a) of the above mentioned alternative we have $g^n e = \lambda^n e$. The only limit point of $\{g^n e\}_{n=-\infty}^{\infty}$ is zero. Hence, $\{g^n\}_{n=-\infty}^{\infty}$ has no limit points in $\mathrm{GL}(p, \mathbb{C})$.

In case (b), $g^n e_2 = \lambda^n e_2 + n\lambda^{n-1} e_1$, therefore $g^n e_2 \to \infty$ as $|n| \to \infty$. Hence, $\{g^n\}_{n=-\infty}^{\infty}$ has no limit points again. ∎

Combining Theorem 3.4 and Lemma 3.5 we obtain

COROLLARY 3.6. *The A.e. corresponding to the shift F_g on the group $\mathrm{GL}(p, \mathbb{C})$ has an analytic solution if and only if the matrix g is not quasiunitary.*

In fact, this analytic solution can be explicitly presented in the following form:

$$\varphi(x) = \frac{1}{\ln|\det g|} \ln|\det x|$$

if $|\det g| \neq 1$.

If $|\det g| = 1$ then

$$\varphi(x) = \frac{1}{\ln|\lambda|} \ln\|xe\|$$

in case (a) and

$$\varphi(x) = \frac{1}{\|xe_1\|^2} \mathrm{Re} \frac{(xe_1, xe_2)}{\lambda} \quad (x \in \mathrm{GL}(p, \mathbb{R}))$$

in case (b). Here $(\ ,\ )$ is an inner product in $\mathbb{C}^p$ and $\|\cdot\|$ is the corresponding Euclidean norm.

References

[1] G. Belitskii and Yu. Lyubich, *The Abel equation and total solvability of linear functional equations*, Studia Math. 127 (1998), 81–97.

[2] D. Fuks and V. Rokhlin, *The First Course in Topology*, Nauka, Moscow, 1977 (in Russian).

[3] M. W. Hirsch, *Differential Topology*, Springer, New York, 1976.

[4] S.-T. Hu, *Homotopy Theory*, Academic Press, New York, 1959.

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