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On coerciveness in Besov spaces for abstract parabolic equations of higher order

by

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Abstract. We are concerned with a relation between parabolicity and coerciveness in Besov spaces for a higher order linear evolution equation in a Banach space. As proved in a preceding work, a higher order linear evolution equation enjoys coerciveness in Besov spaces under a certain parabolicity condition adopted and studied by several authors. We show that for a higher order linear evolution equation coerciveness in Besov spaces forces the parabolicity of the equation. We thus conclude that parabolicity and coerciveness in Besov spaces are equivalent.

1. Introduction. Dubinskiĭ [7] classified linear operator differential equations of higher order by means of spectral properties of operator pencils. He introduced the notion of a parabolic equation of higher order. In a Banach space X_0 consider a linear evolution equation of the form

$$\text{(1.1)} \qquad \begin{cases} \sum_{j=0}^{n} A_j {D_t}^{n-j} u(t) = f(t), \quad 0 < t < T, \\ {D_t}^{n-j} u(0) = x_j, \quad j = 1, \ldots, n, \end{cases}$$

where A_j, $j = 1, \ldots, n$, are continuous linear operators from Banach spaces X_j to X_0, respectively, and A_0 is the identity operator in X_0. The X_j are assumed to be continuously embedded in X_{j-1} for $j = 1, \ldots, n$. Dubinskiĭ's parabolicity condition is a spectral condition on the operator pencil $\sum_{j=0}^{n} \lambda^{n-j} A_j$. When $n = 1$, this condition reduces to the usual one on the resolvent of the operator A_1.

For a parabolic equation of order n several solvability results were given by Dubinskiĭ [7] and Obrecht [16]–[18] (see also Tanabe [22], [23]). There are also many results on sufficient conditions for (1.1) to be a parabolic equation (see Favini and Obrecht [8], Favini and Tanabe [9] and the references therein).

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In the preceding paper [26] under a condition of Dubinskiĭ's type we solved the equation (1.1) in an intersection of X_j-valued Besov spaces with suitable exponents, $j = 0, 1, \ldots, n$. The result is that the map

$$P : u \mapsto \Big(\sum_{j=0}^{n} A_j {D_t}^{n-j} u(t), {D_t}^{n-1} u(0), \ldots, u(0)\Big)$$

is an isomorphism between the space of solutions given above and the space of data with certain compatibility relations. We call this property of (1.1) *coerciveness* in Besov spaces. For $n = 1$, a similar property of a parabolic equation has already been obtained in [6], [15], [19], etc. Moreover, in [25] we showed that the condition for parabolicity of a first order equation follows from coerciveness in Besov spaces. In this paper we prove that for a higher order equation, coerciveness in Besov spaces implies the parabolicity condition assumed in [26]. This together with the result of [26], given below in Section 5, shows that parabolicity in the sense of Dubinskiĭ and coerciveness in Besov spaces are equivalent for a linear evolution equation of any order.

To deduce parabolicity from coerciveness in Besov spaces we consider a system of first order equations for the unknown functions $({D_t}^{n-1}u, \ldots, u)$ in the product space $X_0 \times \ldots \times X_{n-1}$ and construct from a solution of (1.1) a semigroup of bounded linear operators in $X_0 \times \ldots \times X_{n-1}$ relating to the system. Spectral properties of the operator pencil $\sum_{j=0}^{n} \lambda^{n-j} A_j$ are then derived from the Laplace transform of the semigroup. Note that, as in [26], Brézis–Fraenkel's condition on the spaces X_j, $j = 0, 1, \ldots, n$, as assumed in [18], is not necessary. In [26] we solved the equation (1.1) by reduction to the same system as we considered above. For another method of solving (1.1) with the use of fundamental solutions, we refer the reader to [16]–[18], [22] and [23].

2. Notation and preliminaries. $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. $\mathbb{Z}_+$ is the set of nonnegative integers. Let E and F be Banach spaces. $\mathcal{L}(E,F)$ is the space of bounded linear operators from E to F with uniform operator norm $\|\cdot\|_{\mathcal{L}(E,F)}$. We write simply $\mathcal{L}(E,E) = \mathcal{L}(E)$.

For $1 \le p \le \infty$, $0 < T \le \infty$ and $l \in \mathbb{Z}_+ \cup \{\infty\}$, we consider the following E-valued function spaces. $\mathcal{D}'(0,T;E)$ is the space of distributions on $(0,T)$. The derivatives of $f \in \mathcal{D}'(0,T;E)$ are denoted by ${D_t}^l f$. $L^p(0,T;E)$ and $L^p_*(0,T;E)$ are the L^p spaces with respect to the Lebesgue measure dt and the measure $t^{-1}dt$ on $(0,T)$, respectively. For an interval $I = (0,T)$, $(0,T]$, $[0,T)$ or $[0,T]$, $C^l(I;E)$ is the space of l times continuously differentiable functions on I. $C^l_B(I;E)$ is the subspace of $C^l(I;E)$ which consists of the functions whose derivatives of order up to l belong to $L^\infty(0,T;E)$. In the notation above we omit the symbol E when $E = \mathbb{R}$ or $\mathbb{C}$.

We now define Besov spaces on intervals with or without boundary conditions. Assume $1 \le p \le \infty$, $1 \le q \le \infty$, $0 < \theta < \infty$, $0 < T \le \infty$. Set $m = [\theta] + 1$, where $[\theta]$ is the largest integer which does not exceed θ. For a strongly measurable function f on $(0,T)$ with values in E put

$$[f]_{B^\theta_{p,q}(0,T;E)} = \Big| h^{-\theta} \Big| \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(\cdot + kh) \Big|_{L^p(0,T-mh;E)} \Big|_{L^q_*(0,T/m)}.$$

The subspaces $B^\theta_{p,q}(0,T;E)$ and $\dot{B}^\theta_{p,q}(0,T;E)$ of $L^p(0,T;E)$ are defined as follows:

(1) $f \in L^p(0,T;E)$ belongs to $B^\theta_{p,q}(0,T;E)$ if

$$[f]_{B^\theta_{p,q}(0,T;E)} < \infty.$$

(2) $f \in L^p(0,T;E)$ belongs to $\dot{B}^\theta_{p,q}(0,T;E)$ if

$$f \in B^\theta_{p,q}(0,T;E) \quad \text{and} \quad |h^{-\theta} |f|_{L^p(0,h;E)}|_{L^q_*(0,T)} < \infty.$$

The spaces $B^\theta_{p,q}(0,T;E)$ and $\dot{B}^\theta_{p,q}(0,T;E)$, called the *Besov spaces*, are equipped with the norms

$$|f|_{B^\theta_{p,q}(0,T;E)} = |f|_{L^p(0,T;E)} + [f]_{B^\theta_{p,q}(0,T;E)},$$
$$|f|_{\dot{B}^\theta_{p,q}(0,T;E)} = |h^{-\theta}|f|_{L^p(0,h;E)}|_{L^q_*(0,T)} + |f|_{B^\theta_{p,q}(0,T;E)},$$

respectively and are Banach spaces. In this paper we make use of the following properties of the Besov spaces. For the proofs see, for instance, Triebel [24]. In the following proposition $B(\theta,p,q)$ stands for $B^\theta_{p,q}(0,T;E)$ or $\dot{B}^\theta_{p,q}(0,T;E)$.

PROPOSITION 2.1. *Assume* $1 \le p, p' \le \infty$, $1 \le q, q' \le \infty$, $0 < \theta, \theta' < \infty$, $0 < T \le \infty$. *Let* $\theta = l + \sigma$ *with* $l \in \mathbb{Z}_+$ *and* $\sigma \in (0,1]$. *Then* (1)–(6) *below hold.*

(1) $B(\theta,p,q) \subset B(\theta',p,q')$ *when* $\theta > \theta'$.

(2) $B(\theta,p,q) \subset B(\theta - p^{-1} + p'^{-1}, p', q')$ *when* $p' \ge p$, $q' \ge q$, $\theta > p^{-1} - p'^{-1}$.

(3) *For* $\alpha \in C^m_B((0,T))$ *and* $f \in B(\theta,p,q)$ *we have* $\alpha f \in B(\theta,p,q)$.

(4) *For* $f \in B(\theta,p,q)$, *if* $T < \infty$ *then* $\int_0^t f(s)\,ds \in B(1+\theta,p,q)$.

(5) *For* $f \in B(\theta,p,q)$ *we have* ${D_t}^k f \in B(\theta - k,p,q)$, $k = 0, \ldots, l$.

(6) $\dot{B}^\theta_{p,q}(0,T;E) = \{f \in B^\theta_{p,q}(0,T;E) : {D_t}^k f(0) = 0$ *for* $k = 0, \ldots, l-1$, ${D_t}^l f \in \dot{B}^\sigma_{p,q}(0,T;E)\}$. *When* $\sigma - p^{-1}$ *is not an integer, we have*

$$\dot{B}^\sigma_{p,q}(0,T;E) = \begin{cases} B^\sigma_{p,q}(0,T;E), & \sigma - p^{-1} < 0, \\ \{f \in B^\sigma_{p,q}(0,T;E) : f(0) = 0\}, & \sigma - p^{-1} > 0. \end{cases}$$

3. Results. Let X_j, $j=0,1,\ldots,n$, be Banach spaces with norms $|\cdot|_j$. X_j is assumed to be continuously embedded in X_{j-1} for $j=1,\ldots,n$. Let A_j, $j=1,\ldots,n$, be continuous linear operators from X_j to X_0. Put $P(D_t)=\sum_{j=0}^n A_j D_t{}^{n-j}$, where A_0 is the identity operator in X_0.

Assume that $1\le p\le\infty$, $1\le q\le\infty$, $0<\theta<\infty$, $0<T<\infty$. For $u\in\bigcap_{j=0}^n \dot B_{p,q}^{n-j+\theta}(0,T;X_j)$, by Proposition 2.1(5) we have $P(D_t)u\in\dot B_{p,q}^{\theta}(0,T;X_0)$. Hence we may define a linear operator by

$$\dot P:\bigcap_{j=0}^n \dot B_{p,q}^{n-j+\theta}(0,T;X_j)\to \dot B_{p,q}^{\theta}(0,T;X_0),\qquad \dot Pu=P(D_t)u.$$

Further assume $\theta-p^{-1}\notin\mathbb{Z}_+$. By Proposition 2.1(5) the range of the linear operator $u\mapsto(P(D_t)u,D_t{}^{n-1}u(0),\ldots,u(0))$ acting on the space $\bigcap_{j=0}^n B_{p,q}^{n-j+\theta}(0,T;X_j)$ is included in $B_{p,q}^{\theta}(0,T;X_0)\times X_0\times\ldots\times X_{n-1}$. Taking compatibility relations for $(P(D_t)u,D_t{}^{n-1}u(0),\ldots,u(0))$ into account, we get a subspace of the product space. For $t>0$ and $x\in X_j$, $j=0,1,\ldots,n$, put

$$L_j(t,x)=\inf_{\phi\in X_n}\Big(\sum_{k=0}^{j} t^{k-j}|x-\phi|_k+\sum_{k=j+1}^{n} t^{k-j}|\phi|_k\Big)$$

(if $j=n$, the second sum is meant to be zero). Define the subspace Y_j of X_j by

$$Y_j=\{x\in X_j:\sup_{0<t<\infty}L_j(t,x)<\infty\}. \tag{3.1}$$

The space Y_j equipped with the norm $|x|'_j=|x|_j+\sup_{0<t<\infty}L_j(t,x)$ is a Banach space. Several properties of Y_j are collected in [26, Section 3]. Here we recall that $Y_0=X_0$, $Y_n=X_n$ and $Y_j\subset Y_{j-1}$, $j=1,\ldots,n$. We denote by $\mathcal{D}_0$ the linear space which consists of the elements $(f,x_1,\ldots,x_n)$ of $B_{p,q}^{\theta}(0,T;X_0)\times X_0\times\ldots\times X_{n-1}$ satisfying the conditions (1) and (2) below. Set $N=[\theta-p^{-1}]+1$. Put $y_{0j}=x_j$, $j=1,\ldots,n$.

(1) If $k<N$, then $y_{kj}\in X_j$, $j=1,\ldots,n$. In this case put

$$y_{k+1,j}=\begin{cases} D_t{}^k f(0)-\sum_{l=1}^{n}A_l y_{kl}, & j=1,\\ y_{k,j-1}, & j=2,\ldots,n.\end{cases}$$

(2) $y_{Nj}\in(Y_{j-1},Y_j)_{\theta-p^{-1}+1-N,q}$, $j=1,\ldots,n$.

Here $(Y_{j-1},Y_j)_{\eta,q}$, $0<\eta<1$, $1\le q\le\infty$, is the real interpolation space between Y_{j-1} and Y_j. See, for instance, [1], [3] and [24]. We denote by $|\cdot|_{(Y_{j-1},Y_j)_{\eta,q}}$ the norm in $(Y_{j-1},Y_j)_{\eta,q}$. Since by definition we have

$$y_{kj}=\begin{cases} y_{N,j-k+N}, & 1\le j\le n-N+k,\\ y_{k-j+n,n}, & n-N+k<j\le n,\end{cases} \tag{3.2}$$

the space $\mathcal{D}_0$ equipped with the norm

$$|f|_{B_{p,q}^{\theta}(0,T;X_0)}+\sum_{k=0}^{N-1}|y_{kn}|_n+\sum_{j=1}^{n}|y_{Nj}|_{(Y_{j-1},Y_j)_{\theta-p^{-1}+1-N,q}}$$

is a Banach space. The space $\mathcal{D}_0$ describes certain compatibility relations for $(P(D_t)u,D_t{}^{n-1}u(0),\ldots,u(0))$ in the following sense.

LEMMA 3.1. *For $u\in\bigcap_{j=0}^n B_{p,q}^{n-j+\theta}(0,T;X_j)$ we have*

$$(P(D_t)u,D_t{}^{n-1}u(0),\ldots,u(0))\in\mathcal{D}_0.$$

Proof. See [26, Section 6]. ∎

Thus we may define a linear operator by

$$P:\bigcap_{j=0}^n B_{p,q}^{n-j+\theta}(0,T;X_j)\to\mathcal{D}_0,\qquad Pu=(P(D_t)u,D_t{}^{n-1}u(0),\ldots,u(0)).$$

We are now in a position to state our main result.

THEOREM. *The following five statements are equivalent.*

(1) *$\dot P$ is bijective for any (p,q,θ,T).*
(2) *$\dot P$ is bijective for some (p,q,θ,T).*
(3) *P is bijective for any (p,q,θ,T) with $\theta-p^{-1}\notin\mathbb{Z}_+$.*
(4) *P is bijective for some (p,q,θ,T) with $\theta-p^{-1}\notin\mathbb{Z}_+$.*
(5) *The linear operators $P_n(\lambda)$, $\lambda\in\mathbb{C}$, in X_0 given by*

$$\mathcal{D}(P_n(\lambda))=X_n,\qquad P_n(\lambda)x=\sum_{j=0}^{n}\lambda^{n-j}A_jx,$$

are bijective when $\lambda-\omega\in\Sigma\equiv\{\lambda\in\mathbb{C}:|\arg\lambda|\le\psi\}$ for some constants $\omega\in\mathbb{R}$ and $\psi\in(\pi/2,\pi)$. For $j=1,\ldots,n$, we have

$$\sup_{\lambda\in\omega+\Sigma}\|(\lambda-\omega)^{n-j+1}P_n(\lambda)^{-1}\|_{\mathcal{L}(X_0,X_{j-1})}<\infty. \tag{3.3}$$

REMARK. The condition (5) on the operator pencil $P_n(\lambda)$ is essentially the same as the parabolicity condition due to Dubinskiĭ [7]. For $n=1$, the assertion of the theorem has already been proved in [25].

We prove the theorem as follows. Obviously, (1) implies (2), and (3) implies (4). Assume $\theta-p^{-1}\notin\mathbb{Z}_+$. For $f\in\dot B_{p,q}^{\theta}(0,T;X_0)$ we have $(f,0,\ldots,0)\in\mathcal{D}_0$. Moreover, if $u\in\bigcap_{j=0}^n B_{p,q}^{n-j+\theta}(0,T;X_j)$ satisfies $Pu=(f,0,\ldots,0)$, then $u\in\bigcap_{j=0}^n\dot B_{p,q}^{n-j+\theta}(0,T;X_j)$. Hence (2) follows from (4). Therefore, if we prove that (2) implies (5), and that (5) implies (1) and (3), we obtain the theorem.

To show that (2) implies (5) we first construct from a solution of $\dot{P}u = f$ a semigroup of bounded linear operators in $X_0 \times \ldots \times X_{n-1}$ relating to the system of equations

$$(3.4)\qquad \begin{cases} D_t u_1 + \sum_{j=1}^{n} A_j u_j = 0, \\ D_t u_j - u_{j-1} = 0, \quad j = 2, \ldots, n, \\ u_j(0) = x_j, \quad j = 1, \ldots, n. \end{cases}$$

The condition (5) is then deduced from the behavior of the semigroup by means of the Laplace transformation. A detailed proof will be given in the next section. In [26] we have proved that (5) implies (3). Similarly we can prove that (5) implies (1). In Section 5 we give an outline of those proofs.

4. (2) implies (5). Throughout this section E and F denote the product spaces $X_0 \times \ldots \times X_{n-1}$ and $X_1 \times \ldots \times X_n$, respectively. We start by constructing a semigroup of bounded linear operators in E relating to (3.4).

The following lemma is a consequence of a simple computation. Put

$$P_j(D_t) = \sum_{k=0}^{j} A_k D_t^{j-k}, \qquad j = 0, 1, \ldots, n.$$

LEMMA 4.1. *For $u_j \in \mathcal{D}'(0, T; X_j)$ and $f_j \in \mathcal{D}'(0, T; X_{j-1})$, $j = 1, \ldots, n$, the following two relations (4.1_1) and (4.1_2) are equivalent*:

$$(4.1_1)\qquad \begin{cases} P_n(D_t) u_n = \sum_{j=1}^{n} P_{j-1}(D_t) f_j, \\ u_{j-1} = D_t u_j - f_j, \quad j = 2, \ldots, n; \end{cases}$$

$$(4.1_2)\qquad \begin{cases} D_t u_1 + \sum_{j=1}^{n} A_j u_j = f_1, \\ D_t u_j - u_{j-1} = f_j, \quad j = 2, \ldots, n. \end{cases}$$

Suppose that for a set of numbers (p, q, θ, T) the operator $\dot{P}$ is bijective. Put $m = [\theta] + n$. Define the linear operators

$$U_{-m-1,j} : E \to \bigcap_{l=0}^{j-1} \dot{B}_{p,q}^{j-l+\theta}(0, T; X_l), \qquad j = 1, \ldots, n,$$

by

$$(U_{-m-1,j} x)(t) = \frac{t^{m+1}}{(m+1)!} x_j, \qquad x = (x_1, \ldots, x_n) \in E.$$

Then, for integers $k \ge -m$, define the linear operators

$$U_{kj} : E \to \bigcap_{l=0}^{j} \dot{B}_{p,q}^{j-l+\theta}(0, T; X_l), \qquad j = 1, \ldots, n,$$

successively by solving the equations

$$(4.2)\qquad \begin{cases} \dot{P} U_{kn} x = \sum_{j=1}^{n} P_{j-1}(D_t) D_t U_{k-1,j} x, \\ U_{k,j-1} x = D_t U_{kj} x - D_t U_{k-1,j} x, \quad j = 2, \ldots, n. \end{cases}$$

By Lemma 4.1 the equation (4.2) is equivalent to

$$(4.3_k)\qquad \begin{cases} D_t U_{k1} x + \sum_{j=1}^{n} A_j U_{kj} x = D_t U_{k-1,1} x, \\ D_t U_{kj} x - U_{k,j-1} x = D_t U_{k-1,j} x, \quad j = 2, \ldots, n. \end{cases}$$

For $k \ge -m$ and $x \in E$ put

$$(4.4)\qquad v_{kj} = t U_{k-1,j} x + \int_0^t \{(k-2) U_{k-1,j} x - (k+m) U_{kj} x\}\, ds, \qquad j = 1, \ldots, n.$$

By definition we have $v_{-m,j} = 0$, $j = 1, \ldots, n$. For $k \ge -m$ we see that $v_{kj} \in \bigcap_{l=0}^{j} \dot{B}_{p,q}^{j-l+\theta}(0, T; X_l)$, $j = 1, \ldots, n$. For $k \ge -m+1$ from (4.3_k) we can easily derive the equation

$$\begin{cases} D_t v_{k1} + \sum_{j=1}^{n} A_j v_{kj} = D_t v_{k-1,1}, \\ D_t v_{kj} - v_{k,j-1} = D_t v_{k-1,j}, \quad j = 2, \ldots, n. \end{cases}$$

By Lemma 4.1 this implies that

$$\begin{cases} \dot{P} v_{kn} = \sum_{j=1}^{n} P_{j-1}(D_t) D_t v_{k-1,j}, \\ v_{k,j-1} = D_t v_{kj} - D_t v_{k-1,j}, \quad j = 2, \ldots, n. \end{cases}$$

Recall that $v_{-m,j} = 0$, $j = 1, \ldots, n$. Since $\dot{P}$ is injective, by induction on $k \ge -m$ we get $v_{kj} = 0$, $j = 1, \ldots, n$. On the other hand, differentiating (4.4) in t, we see that

$$D_t v_{kj} = (t D_t + k - 1) U_{k-1,j} x - (k+m) U_{kj} x.$$

This implies the following lemma.

LEMMA 4.2. *For $k \ge -m+1$ and $x \in E$ we have*

$$(4.5)\qquad U_{kj} x = \frac{1}{k+m} (t D_t + k - 1) U_{k-1,j} x, \qquad j = 1, \ldots, n.$$

Using (4.5), by induction on k we obtain the following equations in $\mathcal{D}'(0,T;X_j)$, $j=1,\dots,n$:

$$(4.6)\quad D_t^{1-k}U_{kj}x = \frac{m!}{(k+m)!}t^k D_t U_{0j}x, \qquad -m \le k \le 1\ (0!=1),$$

$$(4.7)\quad U_{kj}x = \frac{m!}{(k+m)!}D_t^{k-1}(t^k D_t U_{0j}x), \quad 1 \le k < \infty.$$

The equations (4.6) show that the mappings

$$x \mapsto (k+m)!t^{-k-m}D_t^{1-k}U_{kj}x, \quad -m \le k \le 1,$$

give the same linear operators from E to $\mathcal{D}'(0,T;X_j)$, which we denote by $S_j x$. The derivatives of $S_j x$ are expressed as

$$(4.8)\qquad t^{l+m}D_t^{l}S_j x = (l+m)!\sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}D_t U_{kj}x, \quad l \in \mathbb{Z}_+.$$

These are derived from the equalities

$$\frac{t^{l+m}}{(l+m)!}D_t^{l} f = \sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}D_t^{k}\left\{\frac{t^{k+m}}{(k+m)!}f\right\}, \quad f \in \mathcal{D}'(0,T;X_j),$$

and from (4.7). By (4.5) and (4.8) we get $S_j x \in C^\infty((0,T];X_j)$. Hence we may define the linear operators $S_j(t): E \to X_j$ by

$$S_j(t)x = (S_j x)(t), \quad x \in E.$$

We denote by $S(t)$ the linear operator $x \mapsto (S_1(t)x,\dots,S_n(t)x)$ from E to F. We now prove that the mapping $t \mapsto S(t)$ extends to a semigroup of bounded linear operators in E.

Define $B \in \mathcal{L}(F,E)$ by

$$Bx = \Big(\sum_{j=1}^{n}A_j x_j, -x_1,\dots,-x_{n-1}\Big), \quad x=(x_1,\dots,x_n)\in F.$$

Differentiating (4.3_{-m}) $m+1$ times in t, we obtain

$$(4.9)\qquad D_t S(t)x + BS(t)x = 0, \quad 0 < t \le T.$$

PROPOSITION 4.1. (1) *$x \in E$ belongs to F if and only if there exists $y \in E$ such that*

$$(4.10)\qquad D_t S(t)x + S(t)y = 0, \quad 0 < t \le T.$$

(2) *For $x \in E$, if $S(t)x = 0$, $0 < t \le T$, then $x = 0$.*

(3) *For $x \in F$ and $y \in E$ the relation* (4.10) *is equivalent to $y = Bx$.*

Proof. For $x,y \in E$ consider the relation given by (4.10). First let us rewrite it as follows. For $k \ge -m$ put

$$w_{kj} = U_{k-1,j}x - U_{kj}x - \int_0^t U_{kj}y\,ds, \quad j=1,\dots,n.$$

For $k \ge -m+1$ we have $w_{kj} \in \bigcap_{l=0}^{j}\dot{B}_{p,q}^{j-l+\theta}(0,T;X_l)$, $j=1,\dots,n$. Differentiating w_{1j} in t, by (4.3_1) we have

$$\begin{cases} D_t w_{11} = \sum_{j=1}^{n}A_j U_{1j}x - U_{11}y, \\ D_t w_{1j} = -U_{1,j-1}x - U_{1j}y, \quad j=2,\dots,n. \end{cases}$$

By the definition of S_j we have

$$U_{1j}x = \frac{t^{m+1}}{(m+1)!}S_j x, \quad U_{1j}y = \frac{t^{m+1}}{(m+1)!}S_j y, \quad j=1,\dots,n.$$

Hence, using (4.9), we obtain

$$D_t w_{1j} = -\frac{t^{m+1}}{(m+1)!}(D_t S_j x + S_j y), \quad j=1,\dots,n.$$

Therefore (4.10) is equivalent to $w_{1j}=0$, $j=1,\dots,n$. On the other hand, using (4.3_k) and Lemma 4.1 as before, for $k \ge -m+1$ we obtain

$$\begin{cases} \dot{P}w_{kn} = \sum_{j=1}^{n}P_{j-1}(D_t)D_t w_{k-1,j}, \\ w_{k,j-1} = D_t w_{kj} - D_t w_{k-1,j}, \quad j=2,\dots,n. \end{cases}$$

By the injectivity of $\dot{P}$ we see that $w_{1j}=0$, $j=1,\dots,n$, and hence (4.10), is equivalent to $w_{-m,j}=0$, $j=1,\dots,n$.

Let us prove part (1) of the proposition. It suffices to show that $x \in E$ belongs to F if and only if there exists $y \in E$ such that $w_{-m,j}=0$, $j=1,\dots,n$. Suppose that $x \in F$. Then for $y \in E$ we have $w_{-m,j} \in \bigcap_{l=0}^{j}\dot{B}_{p,q}^{j-l+\theta}(0,T;X_l)$, $j=1,\dots,n$. It is easy to see that

$$\begin{cases} \dot{P}w_{-m,n} = \sum_{j=1}^{n}P_{j-1}(D_t)U_{-m-1,j}(Bx-y), \\ w_{-m,j-1} = D_t w_{-m,j} - U_{-m-1,j}(Bx-y), \quad j=2,\dots,n. \end{cases}$$

Hence, taking $y = Bx$, by the injectivity of $\dot{P}$ we obtain $w_{-m,j}=0$, $j=1,\dots,n$. Conversely, suppose that $w_{-m,j}=0$, $j=1,\dots,n$, for some $y \in E$. Then, by the definition of $w_{-m,j}$ we have $U_{-m-1,j}x = U_{-m,j}x + \int_0^t U_{-m,j}y\,ds$, $j=1,\dots,n$. Since the right-hand side is an X_j-valued function, by the definition of $U_{-m-1,j}x$ we obtain $x \in F$.

Next, let us prove (2). Suppose that $S(t)x=0$. Since (4.10) holds with $y=0$, we have $w_{-m,j}=0$, $j=1,\ldots,n$, with $y=0$. Therefore, by the definition of $w_{-m,j}$ we have $U_{-m-1,j}x=U_{-m,j}x$, $j=1,\ldots,n$. Differentiating this equation $m+1$ times in t and recalling the definition of $S(t)$, we obtain $x=S(t)x=0$.

Finally, we prove (3). As in the proof of (1) we have $D_tS(t)x+S(t)Bx=0$, $0<t\le T$. Therefore, if (4.10) holds, then (2) implies that $y=Bx$. ∎

LEMMA 4.3. *There exist constants K_0 and K_1 such that for $x\in E$, $l\in\mathbb{Z}_+$ and $0<t\le T$ we have*

$$\sum_{j=1}^{n}\sum_{i=1}^{j} t^{l+i-j}|D_t{}^lS_j(t)x|_{i-1}\le K_1K_0{}^l l!(T/t)^{m-\theta+p^{-1}}\sum_{j=1}^{n}\sum_{i=1}^{j}T^{i-j}|x_j|_{i-1}.$$

In particular, the function $S:(0,T]\to\mathcal{L}(E)$, $t\mapsto S(t)$, is analytic.

Proof. Since $\dot P$ is bicontinuous, there exists a constant C_0 such that for $u\in\bigcap_{j=0}^{n}\dot B^{n-j+\theta}_{p,q}(0,T;X_j)$ we have

$$\sum_{i=1}^{n+1}|u|_{\dot B^{n-i+\theta+1}_{p,q}(0,T;X_{i-1})}\le C_0|\dot Pu|_{\dot B^{\theta}_{p,q}(0,T;X_0)}.$$

Taking $u=U_{kn}x$, $k\ge -m$, by Proposition 2.1(5) we get

$$\sum_{i=1}^{n}|D_tU_{kn}x|_{\dot B^{n-i+\theta}_{p,q}(0,T;X_{i-1})}\le C_1\sum_{j=1}^{n}\sum_{i=1}^{j}|D_tU_{k-1,j}x|_{\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})}$$

with a constant C_1 independent of k. Since by (4.2) we have

$$U_{kj}x=D_t{}^{n-j}U_{kn}x-\sum_{l=j+1}^{n}D_t{}^{l-j}U_{k-1,l}x,\quad j=1,\ldots,n-1,$$

by Proposition 2.1(5) we get

$$\begin{aligned}\sum_{i=1}^{j}|D_tU_{kj}x|_{\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})}&\le C_2\sum_{i=1}^{j}|D_tU_{kn}x|_{\dot B^{n-i+\theta}_{p,q}(0,T;X_{i-1})}\\&\quad+C_2\sum_{l=j+1}^{n}\sum_{i=1}^{j}|D_tU_{k-1,l}x|_{\dot B^{l-i+\theta}_{p,q}(0,T;X_{i-1})},\quad j=1,\ldots,n-1,\end{aligned}$$

with a constant C_2 independent of k. Hence we obtain

$$\sum_{j=1}^{n}\sum_{i=1}^{j}|D_tU_{kj}x|_{\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})}\le C_3\sum_{j=1}^{n}\sum_{i=1}^{j}|D_tU_{k-1,j}x|_{\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})}$$

with a constant C_3 independent of k. By a simple computation we obtain

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{i=1}^{j}|D_tU_{kj}x|_{\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})}\\&\qquad\le C_3{}^{k+m+1}\sum_{j=1}^{n}\sum_{i=1}^{j}|D_tU_{-m-1,j}x|_{\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})}\\&\qquad\le C_4C_3{}^{k+m+1}T^{m-\theta+p^{-1}}\sum_{j=1}^{n}\sum_{i=1}^{j}T^{i-j}|x_j|_{i-1}\end{aligned}$$

with a constant C_4 independent of k. For $f\in\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})$ and $0<t\le T$ we have $|f|_{\dot B^{j-i+\theta}_{p,q}(0,t;X_{i-1})}\le|f|_{\dot B^{j-i+\theta}_{p,q}(0,T;X_{i-1})}$. Therefore it follows from (4.8) that

$$\begin{aligned}&\sum_{j=1}^{n}\sum_{i=1}^{j}|s^{l+m}D_s{}^lS_jx|_{\dot B^{j-i+\theta}_{p,q}(0,t;X_{i-1})}\\&\qquad\le(l+m)!\sum_{k=0}^{l}\binom{l}{k}\sum_{j=1}^{n}\sum_{i=1}^{j}|D_sU_{kj}x|_{\dot B^{j-i+\theta}_{p,q}(0,t;X_{i-1})}\\&\qquad\le C_4C_3{}^{m+1}T^{m-\theta+p^{-1}}(l+m)!(1+C_3)^l\sum_{j=1}^{n}\sum_{i=1}^{j}T^{i-j}|x_j|_{i-1}.\end{aligned}$$

By (4.5) and (4.8) we get $t^{l+m+1}D_t{}^lS_j(t)x\in\dot B^{j-i+\theta+1}_{p,q}(0,T;X_{i-1})$. Hence $t^{l+m+J}D_t{}^lS_j(t)x=\int_0^tD_s(s^{l+m+J}D_s{}^lS_jx)\,ds$, $J=1,2,\ldots$ Applying Hölder's inequality to the integrals, $J=2,1$, we obtain

$$\begin{aligned}&t^{l+m+2}|D_t{}^lS_j(t)x|_{i-1}\\&\quad\le C_5t^{j-i+\theta+2-p^{-1}}\{|h^{-j+i-\theta-1+p^{-1}}h^{l+m+2}D_h{}^{l+1}S_jx|_{L^q_*(0,t;X_{i-1})}\\&\qquad+(l+m+2)|h^{-j+i-\theta-1+p^{-1}}h^{l+m+1}D_h{}^lS_jx|_{L^q_*(0,t;X_{i-1})}\}\\&\quad\le C_5t^{j-i+\theta+2-p^{-1}}\{|h^{-(j-i+\theta)}|s^{l+m+2}D_s{}^{l+2}S_jx|_{L^p(0,h;X_{i-1})}|_{L^q_*(0,t)}\\&\qquad+2(l+m+2)|h^{-(j-i+\theta)}|s^{l+m+1}D_s{}^{l+1}S_jx|_{L^p(0,h;X_{i-1})}|_{L^q_*(0,t)}\\&\qquad+(l+m+2)(l+m+1)|h^{-(j-i+\theta)}|s^{l+m}D_s{}^lS_jx|_{L^p(0,h;X_{i-1})}|_{L^q_*(0,t)}\}\end{aligned}$$

with a constant C_5 independent of l. Combining the estimates above, we arrive at the conclusion. ∎

LEMMA 4.4. *For* $0 < t_1 < T$ *and* $\widetilde{u} \in C^0((0,t_1);E) \cap \mathcal{D}'(0,t_1;F)$, *if* $D_t\widetilde{u} + B\widetilde{u} = 0$ *in* $\mathcal{D}'(0,t_1;E)$ *and* $\lim_{t\to 0}\widetilde{u}(t) = 0$ *in* E, *then* $\widetilde{u} = 0$ *on* $(0,t_1)$.

Proof. Take $\phi \in C^\infty((0,t_1))$ with $\operatorname{supp}\phi \subset (0,t_1)$. For $\tau \in (t_1,T]$ and $\lambda \in \mathbb{C}\setminus[\tau - t_1,\tau]$, from the equation $D_t\widetilde{u} + B\widetilde{u} = 0$ we have

$$\int_0^{t_1} \frac{\phi'(t)\widetilde{u}(t)}{\lambda - (\tau - t)}\,dt = -x'(\lambda) + Bx(\lambda), \qquad x(\lambda) = \int_0^{t_1} \frac{\phi(t)\widetilde{u}(t)}{\lambda - (\tau - t)}\,dt. \tag{4.11}$$

Notice that by assumption we have $x(\lambda) \in F$. By Lemma 4.3 the function $S : (0,T] \to \mathcal{L}(E)$, $t \mapsto S(t)$, is analytic. Hence S has an analytic continuation to a complex neighborhood of the interval $(0,T]$. The continuation is denoted by S. In the complex neighborhood, consider a rectifiable Jordan curve γ enclosing the interval $[\tau - t_1,\tau]$. Applying $S(\lambda)$ to (4.11) for $\lambda \in \gamma$ and then integrating both sides over γ, by Cauchy's integral formula we have

$$\int_0^{t_1} \phi'(t)S(\tau - t)\widetilde{u}(t)\,dt = \frac{1}{2\pi\sqrt{-1}}\oint_\gamma (S'(\lambda)x(\lambda) + S(\lambda)Bx(\lambda))\,d\lambda.$$

Proposition 4.1(3) implies that $S'(\lambda)x(\lambda) + S(\lambda)Bx(\lambda) = 0$ for $\lambda \in \gamma$. Hence $D_t(S(\tau - t)\widetilde{u}(t)) = 0$ in $\mathcal{D}'(0,t_1;E)$. Since $\lim_{t\to 0}\widetilde{u}(t) = 0$ in E, we obtain $S(\tau - t)\widetilde{u}(t) = 0$ on $(0,t_1)$. Again by the analyticity of S this implies that $S(s)\widetilde{u}(t) = 0$, $0 < s \le T$, for each $t \in (0,t_1)$. By Proposition 4.1(2) we conclude that $\widetilde{u} = 0$ on $(0,t_1)$. ∎

Let us study the behavior of $S_j(t)$ as $t \to 0$. Choose $\varphi \in C^\infty([0,\infty))$ such that φ is 1 on $[0,1]$ and 0 on $[2,\infty)$. For $\tau \in (0,T]$ and $x \in E$, define $u_{-m-1,j} \in \bigcap_{l=0}^{j-1} \dot{B}^{j-l+\theta}_{p,q}(0,T;X_l)$, $j = 1,\dots,n$, by

$$u_{-m-1,j}(t) = \varphi(t/\tau)(U_{-m-1,j}x)(t), \qquad 0 < t < T.$$

Then for $k \ge -m$ define $u_{kj} \in \bigcap_{l=0}^{j} \dot{B}^{j-l+\theta}_{p,q}(0,T;X_l)$, $j = 1,\dots,n$, successively by solving the equations

$$\begin{cases} \dot{P}u_{kn} = \sum_{j=1}^{n} P_{j-1}(D_t)D_t u_{k-1,j}, \\ u_{k,j-1} = D_t u_{kj} - D_t u_{k-1,j}, \quad j = 2,\dots,n. \end{cases}$$

Using Lemmas 4.1 and 4.4, by induction on $k \ge -m-1$ we have $u_{kj} = U_{kj}x$, $j = 1,\dots,n$, on $(0,\tau)$. Hence by (4.8) we get

$$t^{l+m}{D_t}^l S_j x = (l+m)!\sum_{k=0}^{l}(-1)^{l-k}\binom{l}{k}D_t u_{kj}, \quad l \in \mathbb{Z}_+, \quad \text{on } (0,\tau).$$

Using this in place of (4.8), by an argument similar to that of the proof of Lemma 4.3, for $x \in E$, $l \in \mathbb{Z}_+$ and $0 < t \le \tau$ we obtain

$$\sum_{j=1}^{n}\sum_{i=1}^{j} t^{l+i-j}|{D_t}^l S_j(t)x|_{i-1} \le K_1{K_0}^l l!(\tau/t)^{m-\theta+p^{-1}} \sum_{j=1}^{n}\sum_{i=1}^{j}\tau^{i-j}|x_j|_{i-1}$$

with constants K_0 and K_1 independent of l and τ. Taking $t = \tau$, we obtain the following estimate of S.

PROPOSITION 4.2. *There exist constants* K_0 *and* K_1 *such that for* $x \in E$, $l \in \mathbb{Z}_+$ *and* $0 < t \le T$ *we have*

$$\sum_{j=1}^{n}\sum_{i=1}^{j} t^{l+i-j}|{D_t}^l S_j(t)x|_{i-1} \le K_1{K_0}^l l! \sum_{j=1}^{n}\sum_{i=1}^{j} t^{i-j}|x_j|_{i-1}. \tag{4.12}$$

For $t_1, t_2 > 0$ with $t_1 + t_2 \le T$ and $x \in E$ define $\widetilde{u} \in C^\infty([0,t_1];F)$ by

$$\widetilde{u}(t) = \frac{1}{m!}\int_0^t (t-s)^m S(s+t_2)x\,ds, \qquad 0 \le t \le t_1.$$

Using the equation (4.9), we get

$$\begin{cases} D_t\widetilde{u}(t) + B\widetilde{u}(t) = \dfrac{t^m}{m!}S(t_2)x, \quad 0 \le t \le t_1, \\ \widetilde{u}(0) = 0. \end{cases}$$

By (4.3$_{-m}$) the function $(U_{-m,1}S(t_2)x,\dots,U_{-m,n}S(t_2)x)$ satisfies the same equation on $(0,t_1)$ as $\widetilde{u}$ does. Hence $\widetilde{u} = (U_{-m,1}S(t_2)x,\dots,U_{-m,n}S(t_2)x)$ holds on $(0,t_1)$ by Lemma 4.4. Differentiating this $m+1$ times in t, we obtain the semigroup property of S.

PROPOSITION 4.3. *For* $t_1, t_2 > 0$ *with* $t_1 + t_2 \le T$, *we have* $S(t_1 + t_2) = S(t_1)S(t_2)$.

By (4.12) and the semigroup property of S the $\mathcal{L}(E)$-valued function S on $(0,T]$ has an analytic continuation to a sectorial region $\Sigma' = \{t \in \mathbb{C} : |\arg t| \le \phi\}$ with a constant $\phi \in (0,\pi/2)$. The continuation, also denoted by S, enjoys the semigroup property and satisfies the growth condition

$$\limsup_{\substack{t\in\Sigma' \\ t\to\infty}} |t|^{-1}\log\|S(t)\|_{\mathcal{L}(E)} < \infty. \tag{4.13}$$

The operator norm $\|S(t)\|_{\mathcal{L}(E)}$ is not necessarily bounded as t tends to 0 in Σ'. However, if all the components but the first one of $x \in E$ are 0, by (4.12) the norm $|S(t)x|_E$ remains bounded as t tends to 0 in Σ'. By the semigroup property of S there exists a real number ω such that

$$\sup_{\substack{t\in\Sigma' \\ |x_1|_0\le 1}} |t^{j-n}e^{-\omega t}S_n(t)(x_1,0,\dots,0)|_{j-1} < \infty, \qquad j = 1,\dots,n. \tag{4.14}$$

For $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$, define $R(\lambda) \in \mathcal{L}(X_0, X_{j-1})$, $j = 1, \ldots, n$, by

$$R(\lambda)x_1 = \int_0^\infty e^{-\lambda t} S_n(t)(x_1, 0, \ldots, 0)\, dt, \quad x_1 \in X_0.$$

By (4.14) the $\mathcal{L}(X_0, X_{j-1})$-valued function R has an analytic continuation to a sector $\omega + \Sigma$, where

$$\Sigma = \{\lambda \in \mathbb{C} : |\arg \lambda| \le \psi\}$$

with a constant $\psi \in (\pi/2, \pi)$. Moreover, for $j = 1, \ldots, n$, we have

$$\sup_{\lambda \in \omega+\Sigma} \|(\lambda - \omega)^{n-j+1} R(\lambda)\|_{\mathcal{L}(X_0, X_{j-1})} < \infty.$$

To show the property (5) of the theorem for $P_n(\lambda)$ it suffices to prove that $R(\lambda)$ gives the inverse of $P_n(\lambda)$.

LEMMA 4.5. *If $\Re\lambda$ is large enough, then the operator $\lambda + B$ is injective.*

Proof. Suppose that $x \in F$ and $(\lambda + B)x = 0$. For $t, s > 0$ by Proposition 4.1(3) we have $\partial_t(e^{-\lambda t} S(t+s)x) = 0$. This implies that $S(s)e^{-\lambda t}S(t)x = S(s)x$. By Proposition 4.1(2) we get $e^{-\lambda t}S(t)x = x$. Hence, if $\Re\lambda$ is large enough, the growth condition (4.13) gives $x = 0$. ∎

PROPOSITION 4.4. *If $\Re\lambda$ is large enough, then the operator $P_n(\lambda)$ is injective.*

Proof. For $x_n \in X_n$, if $P_n(\lambda)x_n = 0$, then $(\lambda + B)(\lambda^{n-1}x_n, \ldots, x_n) = 0$. Hence the assertion follows from Lemma 4.5. ∎

LEMMA 4.6. *For $x_1 \in X_0$, if $\Re\lambda$ is large enough, then we have $\int_0^\infty e^{-\lambda t} S(t)(x_1, 0, \ldots, 0)\, dt \in F$ and*

$$(\lambda + B) \int_0^\infty e^{-\lambda t} S(t)(x_1, 0, \ldots, 0)\, dt = (x_1, 0, \ldots, 0).$$

Proof. Since $D_s S(s)S(t) = \partial_t S(t + s)$, integrating by parts gives

$$D_s S(s) \int_0^\infty e^{-\lambda t} S(t)(x_1, 0, \ldots, 0)\, dt$$

$$= -S(s)\Big((x_1, 0, \ldots, 0) - \lambda \int_0^\infty e^{-\lambda t} S(t)(x_1, 0, \ldots, 0)\, dt\Big).$$

By Proposition 4.1 this leads to the conclusion. ∎

The following proposition is an immediate consequence of Lemma 4.6.

PROPOSITION 4.5. *For $x_1 \in X_0$, if $\Re\lambda$ is large enough, then $R(\lambda)x_1 \in X_n$ and $P_n(\lambda)R(\lambda)x_1 = x_1$.*

Propositions 4.4 and 4.5 show that when $\Re\lambda$ is large enough, $P_n(\lambda)$ is bijective and $P_n(\lambda)^{-1} = R(\lambda)$. We now prove that this holds for all $\lambda \in \omega + \Sigma$. If $P_n(\lambda)^{-1} = R(\lambda)$ and $P_n(\mu)^{-1} = R(\mu)$, then

$$R(\lambda) - R(\mu) = R(\mu)\Big\{\sum_{j=0}^{n-1} (\mu^{n-j} - \lambda^{n-j}) A_j\Big\} R(\lambda) \quad \text{in } \mathcal{L}(X_0).$$

Since R is an $\mathcal{L}(X_0, X_{n-1})$-valued analytic function, the equality also holds for $\lambda, \mu \in \omega + \Sigma$. This implies that R is an $\mathcal{L}(X_0, X_n)$-valued analytic function in $\omega + \Sigma$. This in turn proves that $P_n(\lambda)R(\lambda) = I$ in $\mathcal{L}(X_0)$ and $R(\lambda)P_n(\lambda) = I$ in $\mathcal{L}(X_n)$ for $\lambda \in \omega + \Sigma$, as required.

5. (5) implies (1) and (3). Suppose that the statement (5) of the theorem is true. We first make a reduction of the equation (1.1). Consider the following transformation of the unknown function of (1.1):

$$u(t) \mapsto u^\varrho(t) \equiv e^{-\varrho t} u(t), \quad \varrho \in \mathbb{C}.$$

The new unknown function u^ϱ should satisfy

$$\text{(5.1)} \qquad \begin{cases} \sum_{j=0}^{n} A_j^\varrho D_t^{\,n-j} u^\varrho(t) = e^{-\varrho t} f(t), & 0 < t < T, \\ D_t^{n-j} u^\varrho(0) = x_j^\varrho, & j = 1, \ldots, n, \end{cases}$$

with

$$A_j^\varrho = \sum_{l=0}^{j} \varrho^{j-l} \binom{n-l}{n-j} A_l \in \mathcal{L}(X_j, X_0), \quad j = 0, 1, \ldots, n,$$

$$x_j^\varrho = \sum_{l=j}^{n} (-\varrho)^{l-j} \binom{n-j}{n-l} x_l, \quad j = 1, \ldots, n.$$

Notice that $\sum_{j=0}^{n} \lambda^{n-j} A_j^\varrho = P_n(\varrho + \lambda)$. For $\varrho > \omega$, by the condition (5) of the theorem, $|P_n(\varrho)x|_0$ gives a norm on X_n equivalent to $|x|_n$. Hence, replacing the equation (1.1) by (5.1) if necessary, we may assume that the condition (5) of the theorem is satisfied with $\omega = 0$ and that (3.3) holds for $j = 1, \ldots, n, n+1$. The correspondence between the data of (1.1) and those of (5.1) is as follows. For $f \in \dot{B}^\theta_{p,q}(0, T; X_0)$ we have $e^{-\varrho t} f \in \dot{B}^\theta_{p,q}(0, T; X_0)$. When $\theta - p^{-1} \notin \mathbb{Z}_+$, define the subspace $\mathcal{D}_\varrho$ of $B^\theta_{p,q}(0, T; X_0) \times X_0 \times \ldots \times X_{n-1}$ for (5.1) in the same manner as we defined $\mathcal{D}_0$ for (1.1) in Section 3. We then have the following lemma.

LEMMA 5.1. *For $(f, x_1, \ldots, x_n) \in B^\theta_{p,q}(0, T; X_0) \times X_0 \times \ldots \times X_{n-1}$, if $(f, x_1, \ldots, x_n) \in \mathcal{D}_0$, then $(e^{-\varrho t} f, x_1^\varrho, \ldots, x_n^\varrho) \in \mathcal{D}_\varrho$.*

Proof. For $(f, x_1, \ldots, x_n) \in \mathcal{D}_0$ let $\{y_{kj} : j = 1, \ldots, n,\ k = 0, \ldots, N\}$ be the sequence given in Section 3. Define $\{y^{\varrho}_{kj} : j = 1, \ldots, n,\ k = 0, \ldots, N\}$ by

$$y^{\varrho}_{kj} = \sum_{l=j}^{n} (-\varrho)^{l-j} \binom{n-j}{n-l} \sum_{m=0}^{k} (-\varrho)^{k-m} \binom{k}{m} y_{ml}.$$

We have $y^{\varrho}_{0j} = x^{\varrho}_j$, $j = 1, \ldots, n$. We see that $y^{\varrho}_{kj} \in X_j$, $j = 1, \ldots, n$, for $k < N$. Moreover, $y^{\varrho}_{Nj} \in (Y_{j-1}, Y_j)_{\theta - p^{-1}+1-N,q}$, $j = 1, \ldots, n$, since

$$y_{ml} \in (Y_{l-m+N-1}, Y_{l-m+N})_{\theta-p^{-1}+1-N,q} \subset (Y_{j-1}, Y_j)_{\theta-p^{-1}+1-N,q}$$

for $j \le l \le n - N + m$, and $y_{ml} \in Y_n$ for $l > n - N + m$ by (3.2). For $k < N$ by direct calculations we obtain the equations

$$y^{\varrho}_{k+1,1} + \sum_{l=1}^{n} A^{\varrho}_l y^{\varrho}_{kl} = {D_t}^k (e^{-\varrho t} f)(0), \quad y^{\varrho}_{k+1,j} = y^{\varrho}_{k,j-1}, \quad j = 2, \ldots, n.$$

This proves that $(e^{-\varrho t} f, x^{\varrho}_1, \ldots, x^{\varrho}_n) \in \mathcal{D}_{\varrho}$. ■

Having added the above assumptions, we now introduce the unknown functions $u_j = {D_t}^{n-j} u$, $j = 1, \ldots, n$, and write (1.1) as a system of equations

$$\begin{cases} D_t u_1(t) + \sum_{j=1}^{n} A_j u_j(t) = f(t), & 0 < t < T, \\ D_t u_j(t) - u_{j-1}(t) = 0, & j = 2, \ldots, n,\ 0 < t < T, \\ u_j(0) = x_j, & j = 1, \ldots, n. \end{cases}$$

With the notations

$$\widetilde{u} = {}^t(u_1, \ldots, u_n), \quad \widetilde{x} = {}^t(x_1, \ldots, x_n), \quad \widetilde{f} = {}^t(f, 0, \ldots, 0),$$

the system is written as a single equation of first order

$$\begin{cases} D_t \widetilde{u} + B\widetilde{u} = \widetilde{f}, & 0 < t < T, \\ \widetilde{u}(0) = \widetilde{x}, \end{cases} \tag{5.2}$$

where B is a matrix of operators given by

$$B = \begin{bmatrix} A_1 & A_2 & \ldots & A_{n-1} & A_n \\ -I & 0 & \ldots & 0 & 0 \\ 0 & -I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & & -I & 0 \end{bmatrix}$$

The matrix B is regarded as a linear operator in $E = X_0 \times \ldots \times X_{n-1}$ with domain $\mathcal{D}(B) = F = X_1 \times \ldots \times X_n$.

The injectivity of the operators $\dot{P}$ and P is a consequence of the following proposition on the uniqueness of rather weak solutions to (5.2).

PROPOSITION 5.1. *For $\widetilde{u} \in C^0((0,T); E) \cap \mathcal{D}'(0,T;F)$, if $D_t\widetilde{u} + B\widetilde{u} = 0$ in $\mathcal{D}'(0,T;E)$ and $\lim_{t\to 0} \widetilde{u}(t) = 0$ in E, then $\widetilde{u} = 0$ on $(0,T)$.*

Proof. By the condition (5) of the theorem the operator $\lambda + B$ is bijective for $\lambda \in \Sigma$. The inverse $(\lambda + B)^{-1}$ is expressed as a matrix of operators with (i,j) component given by

$$\begin{cases} \lambda^{n-i} P_n(\lambda)^{-1} P_{j-1}(\lambda) - \lambda^{j-i-1} I, & i < j, \\ \lambda^{n-i} P_n(\lambda)^{-1} P_{j-1}(\lambda), & i \ge j, \end{cases} \tag{5.3}$$

where $P_k(\lambda) = \sum_{l=0}^{k} \lambda^{k-l} A_l$, $k = 0, 1, \ldots, n$. Let γ be a contour running in Σ from $e^{-\sqrt{-1}\psi}\infty$ to $e^{\sqrt{-1}\psi}\infty$. The integral

$$S(t) = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma} e^{\lambda t} (\lambda + B)^{-1}\, d\lambda, \quad t > 0,$$

gives an $\mathcal{L}(E)$-valued analytic function on $(0,\infty)$. Take $\phi \in C^\infty((0,T))$ with $\operatorname{supp}\phi \subset (0,T)$. For $\tau \ge T$, from the equation $D_t\widetilde{u} + B\widetilde{u} = 0$ we have

$$\int_0^T \phi'(t) e^{\lambda(\tau-t)} \widetilde{u}(t)\, dt = (\lambda + B) \int_0^T e^{\lambda(\tau - t)} \phi(t) \widetilde{u}(t)\, dt.$$

Acting with $(\lambda + B)^{-1}$ on both sides for $\lambda \in \gamma$ and then integrating over γ, we get $\int_0^T \phi'(t) S(\tau - t) \widetilde{u}(t)\, dt = 0$. Hence, as in the proof of Lemma 4.4 we obtain $S(s)\widetilde{u}(t) = 0$, $s > 0$, for each $t \in (0,T)$. The function S has the property of Proposition 4.1(2) (see [26, Lemma 7.1]). Therefore $\widetilde{u} = 0$ on $(0,T)$. ■

In order to prove the surjectivity of $\dot{P}$ and P we regard (5.2) as an equation in a certain subspace of E. Let Y_j, $j = 0, 1, \ldots, n$, be the subspaces of X_j given by (3.1). Put

$$E' = Y_0 \times \ldots \times Y_{n-1}, \quad F' = Y_1 \times \ldots \times Y_n$$

and define the linear operator B' in E' by

$$\mathcal{D}(B') = F', \quad B'x = Bx.$$

Consider the following equation in E':

$$\begin{cases} D_t \widetilde{u} + B'\widetilde{u} = \widetilde{f}, & 0 < t < T, \\ \widetilde{u}(0) = \widetilde{x}. \end{cases} \tag{5.4}$$

For $f \in \dot{B}^{\theta}_{p,q}(0,T;X_0)$, if $\widetilde{u} \in \dot{B}^{1+\theta}_{p,q}(0,T;E') \cap \dot{B}^{\theta}_{p,q}(0,T;F')$ is a solution of (5.4) with $\widetilde{x} = 0$, then the nth component of $\widetilde{u} = {}^t(u_1, \ldots, u_n)$ gives a solution of $\dot{P}u = f$. Similarly, for $(f, x_1, \ldots, x_n) \in \mathcal{D}_0$, from a solution $\widetilde{u} \in B^{1+\theta}_{p,q}(0,T;E') \cap B^{\theta}_{p,q}(0,T;F')$ of (5.4) a solution of $Pu = (f, x_1, \ldots, x_n)$ is obtained as above.

In [25, Section 5] we showed that when $n = 1$, the condition (5) of the theorem implies the conditions (1) and (3). We now show that the result applies to (5.4). First observe the correspondence between the data of (1.1) and those of (5.4). Clearly, $f \in \dot{B}^{\theta}_{p,q}(0,T;X_0)$ implies that $\widetilde{f} \in \dot{B}^{\theta}_{p,q}(0,T;E')$. When $\theta - p^{-1} \notin \mathbb{Z}_+$, define the subspace $\widetilde{\mathcal{D}}_0$ of $B^{\theta}_{p,q}(0,T;E') \times E'$ for (5.4) in the same manner as we defined $\mathcal{D}_0$ for (1.1) in Section 3. Then from $(f, x_1, \ldots, x_n) \in \mathcal{D}_0$ we obtain $\widetilde{f} \in B^{\theta}_{p,q}(0,T;E')$, $\widetilde{x} \in E'$ and $(\widetilde{f}, \widetilde{x}) \in \widetilde{\mathcal{D}}_0$.

The following proposition shows that for the equation (5.4) the condition (5) of the theorem is satisfied with $n = 1$. This completes our proof of the surjectivity of $\dot{P}$ and P.

PROPOSITION 5.2. *For $\lambda \in \Sigma$ the operator $\lambda + B'$ is bijective. We have*

(5.5) $$\sup_{\lambda \in \Sigma} \|\lambda(\lambda + B')^{-1}\|_{\mathcal{L}(E')} < \infty.$$

Proof. The assertion corresponds to that of [26, Lemma 5.6]. We sketch the proof of [26]. The following two lemmas are of importance.

LEMMA 5.2. *For $t > 0$ and $x \in Y_j$, $j = 0, 1, \ldots, n$, put*

$$L'_j(t,x) = \inf_{\phi \in Y_n} \Big(\sum_{k=0}^{j} t^{k-j} |x - \phi|'_k + \sum_{k=j+1}^{n} t^{k-j} |\phi|'_k \Big)$$

(if $j = n$, the last sum is zero). Define the subspace Z_j of Y_j by

$$Z_j = \{x \in Y_j : \sup_{0<t<\infty} L'_j(t,x) < \infty\}$$

with norm $|x|''_j = |x|'_j + \sup_{0<t<\infty} L'_j(t,x)$. Then $Z_j = Y_j$, $j = 0, 1, \ldots, n$, with equivalent norms.

Proof. See [26, Proposition 3.1]. ■

LEMMA 5.3. *We have the following estimates analogous to* (3.3):

(5.6) $$\sup_{\lambda \in \Sigma} \|\lambda^{n-j+1} P_n(\lambda)^{-1}\|_{\mathcal{L}(Y_0, Y_{j-1})} < \infty, \quad j = 1, \ldots, n, n+1.$$

Proof. See [26, Lemma 5.3]. ■

For $\lambda \in \Sigma$ the operator $\lambda + B'$ is bijective. The inverse $(\lambda + B')^{-1}$ is given by a formula similar to (5.3). For $\mu \in \mathbb{C}$ and $x_j \in Y_j$, $j = 0, \ldots, n-1$, put

$$L'(\mu, x_0, \ldots, x_{n-1}) = \inf_{\phi \in Y_n} \sum_{k=0}^{n} \Big| \sum_{j=k}^{n-1} \mu^{k-j} x_j - \mu^{k-n}\phi \Big|'_k$$

(if $k = n$, the inner sum is zero). As shown in [26, Lemma 5.5], it follows from (5.6) that for $\lambda \in \Sigma$ we have

$$|B'(\lambda + B')^{-1}\, {}^t(x_0, \ldots, x_{n-1})|_{E'} \sim L'(\lambda^{-1}, x_0, \ldots, x_{n-1})$$

($|\cdot|_{E'}$ is the norm on the product space E'). This means that the left-hand side is bounded from above by the right-hand side multiplied by a constant independent of x_j, $j = 0, \ldots, n-1$, and of $\lambda \in \Sigma$, and vice versa. It is easy to see that for all x_j, $j = 0, \ldots, n-1$, and $\lambda \in \Sigma$ we have

$$L'(\lambda^{-1}, x_0, \ldots, x_{n-1}) \le \sum_{j=0}^{n-1} L'_j(|\lambda|^{-1}, x_j).$$

Hence, by Lemma 5.2 the norm $|B'(\lambda + B')^{-1}\, {}^t(x_0, \ldots, x_{n-1})|_{E'}$ is bounded from above by $\sum_{j=0}^{n-1} |x_j|'_j$ multiplied by a constant independent of $\lambda \in \Sigma$. We thus obtain $\sup_{\lambda \in \Sigma} \|B'(\lambda + B')^{-1}\|_{\mathcal{L}(E')} < \infty$. This proves (5.5). ■

In contrast to the assertion of Proposition 5.2 the boundedness of the operator $\lambda(\lambda + B)^{-1}$ in $\mathcal{L}(E)$ is conditional: it remains bounded in $\mathcal{L}(E)$ as λ tends to ∞ in the sector Σ if and only if $E' = E$ with equivalent norms (see [26, Remark 5.1]). It seems that in applications this is a strong restriction on the choice of the spaces X_j, $j = 1, \ldots, n-1$.

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