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Order bounded composition operators on the Hardy spaces and the Nevanlinna class

by

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Abstract. We study the order boundedness of composition operators induced by holomorphic self-maps of the open unit disc D. We consider these operators first on the Hardy spaces H^p $(0 < p < \infty)$ and then on the Nevanlinna class $\mathcal{N}$. Given a non-negative increasing function h on $[0, \infty[$, a composition operator is said to be (X, L_h)-order bounded (we write (X, L_h)-ob) with $X = H^p$ or $X = \mathcal{N}$ if its composition with the map $f \mapsto f^*$, where f^* denotes the radial limit of f, is order bounded from X into L_h. We give a complete characterization and a family of examples in both cases. On the other hand, we show that the $(\mathcal{N}, \log^+ L)$-ob composition operators are exactly those which are Hilbert–Schmidt on H^2. We also prove that the $(\mathcal{N}, L^q)$-ob composition operators are exactly those which are compact from $\mathcal{N}$ into H^q.

1. Introduction. Throughout this paper, we denote by D the open unit disc in the complex plane, by $H(D)$ the space of holomorphic functions on D and by $H(D, D)$ the subset of $H(D)$ consisting of all self-maps of D.

Let φ be in $H(D, D)$. On appropriate subspaces of $H(D)$, the composition operator C_φ is defined by

$$C_\varphi f := f \circ \varphi.$$

We recall that the *Hardy space* H^p $(0 < p < \infty)$ is the subspace of $H(D)$ consisting of all functions satisfying

$$\|f\|_p := \left(\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} < \infty.$$

We also recall that the *Nevanlinna class* $\mathcal{N}$ is the subalgebra of $H(D)$ consisting of all functions such that

$$\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta < \infty.$$

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If $f \in \mathcal{N}$, the radial limit

$$f^*(e^{i\theta}) = \lim_{\substack{r\to 1 \\ r<1}} f(re^{i\theta})$$

exists almost everywhere on the unit circle ∂D (see [2]).

The *Smirnov class* $\mathcal{N}^+$ is the subspace of $\mathcal{N}$ consisting of all functions f such that

$$\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f^*(e^{i\theta})|\,d\theta.$$

The *class* F^+ is the subspace of $H(D)$ consisting of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $|a_n| \le c_\varepsilon e^{\varepsilon\sqrt{n}}$ for all $\varepsilon > 0$.

The following proper inclusions are well known:

$$H^p \subset H^q \subset \mathcal{N}^+ \subset \mathcal{N} \quad \text{for all } 0 < q < p < \infty.$$

Let $h : [0,\infty[\to [0,\infty[$ be an increasing function and (X,d) be a metric additive topological group contained in $H(D)$ such that every $f \in X$ has a radial limit f^* almost everywhere on the unit circle and that C_φ is a self-map of X. The operator C_φ is said to be (X, L_h)-*order bounded*, written (X, L_h)-ob, if its composition with the map $j : f \mapsto f^*$ is order bounded from X into L_h where L_h denotes the set of all measurable functions f on ∂D such that

$$\int_0^{2\pi} h(|f(e^{i\theta})|)\,d\theta < \infty.$$

This amounts to saying that the operator $\widetilde{C}_\varphi := j \circ C_\varphi$ sends every bounded subset of X onto an order bounded subset of L_h.

It is well known that C_φ is a continuous self-map of $\mathcal{N}$ or H^p (this follows from the Littlewood subordination principle: see [2], [8] and [10]) and a lot of work has been devoted to operators C_φ "better than continuous": either compact, or order bounded for some h, or sending the initial space into a smaller subspace. For example, J. H. Shapiro [11] has characterized those $C_\varphi : H^2 \to H^2$ which are compact and, recently, J. S. Choa and H. O. Kim [1] have shown that they are the same as those $C_\varphi : \mathcal{N} \to \mathcal{N}$ which are compact. H. Jarchow and H. Hunziker [5] have shown that the C_φ which are (H^2, L^2)-ob are exactly those which are Hilbert–Schmidt on H^2. J. W. Roberts and M. Stoll [9] have characterized those C_φ which send F^+ into H^q for some and hence all $q > 0$. All these characterizations are given in terms of the behavior of the "analytic moment" sequence

$$\|\varphi^n\|_1 = \frac{1}{2\pi}\int_0^{2\pi} |\varphi^*(e^{i\theta})|^n\,d\theta$$

whose smallness is a quantitative way to express that $|\varphi^*(e^{i\theta})|$ is most of the time far from 1. These results naturally lead to the following questions.

1) Does the coincidence of compact maps $C_\varphi : H^2 \to H^2$ and $C_\varphi : \mathcal{N} \to \mathcal{N}$ still hold if we replace compactness by order boundedness? In Section 4, we give an affirmative answer to this question.

2) The (H^p, L^q)-ob C_φ's were characterized in [5]. Can one characterize the $(H^p, \log^+ L)$-ob and $(\mathcal{N}, L^q)$-ob ones? In both cases, we give a complete characterization (see Theorems 3.1 and 4.4).

3) Does the $(\mathcal{N}, L^q)$-order boundedness improve the compactness of C_φ as for example the (H^2, L^2)-order boundedness does? Rather surprisingly, we shall see that the answer is negative: the C_φ's which are $(\mathcal{N}, L^q)$-ob are exactly those which send $\mathcal{N}$ into H^q compactly. Compared to Roberts–Stoll's result, this latter fact (namely that sending $\mathcal{N}$ into H^q compactly implies $(\mathcal{N}, L^q)$-order boundedness) seems to be due to the huge size of $\mathcal{N}$ with respect to F^+: sending compactly $\mathcal{N}$ into H^q is so restrictive that it forces the $(\mathcal{N}, L^q)$-order boundedness.

The paper is organized as follows. In Section 2, we recall some facts on the class $\mathcal{N}$, the notion of order boundedness and some results on "moment" sequences, taken from [6], which provide a convenient tool to establish the existence of functions $\varphi \in H(D,D)$ relative to prescribed properties of the operators C_φ.

Section 3 is devoted to the study of (H^p, L_h)-ob composition operators and to families of examples.

In Section 4, we deal with the operators C_φ which start from $\mathcal{N}$: either $(\mathcal{N}, L^q)$-ob or compact from $\mathcal{N}$ into H^q. Our main results are Theorems 4.4 and 4.7, where we show that the operators we obtain are among those obtained by J. W. Roberts and M. Stoll [9]; that is, those such that

$$\|\varphi^n\|_1 = O(e^{-\lambda\sqrt{n}}) \quad \text{for some } \lambda > 0.$$

2. Preliminaries

2.1. *The Nevanlinna class*. We recall that $f \in \mathcal{N}$ if $f \in H(D)$ and if

$$\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta < \infty.$$

It follows from the inequalities

$$\log^+ x \le \log(1+x) \le 1 + \log^+ x \quad (x \ge 0)$$

that $f \in \mathcal{N}$ if and only if

$$\|f\|_{\mathcal{N}} := \sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} \log(1+|f(re^{i\theta})|)\,d\theta < \infty.$$

This pseudo-norm allows us to define the following translation invariant metric d:

$$d(f,g)=\|f-g\|_{\mathcal{N}} \quad \text{for all } f,g\in\mathcal{N}.$$

Endowed with this metric and the induced topology (stronger than that of uniform convergence on compact subsets of D), $\mathcal{N}$ becomes a complete metric space, but surprisingly not a topological vector space: there are functions f in $\mathcal{N}$ such that $d(\varepsilon f,0)$ does not tend to zero as ε tends to zero (see [13]). For other properties of $(\mathcal{N},d)$, see [2] and [4].

However, the Smirnov class $(\mathcal{N}^+,d)$ is a topological vector space but not a locally convex vector space (see [16]). The class F^+, equipped with the family of seminorms

$$\|f\|_c := \sum_{n=0}^{\infty}|a_n|e^{-c\sqrt{n}} \quad (c>0),$$

is a locally convex vector space containing $\mathcal{N}^+$ as a dense subspace (see [18]).

LEMMA 2.1.1. (1) *Let $v: D\to[0,\infty[$ be a continuous subharmonic function and $z\in D$. Then*

$$v(z)\le\frac{1+|z|}{1-|z|}\sup_{0\le R<1}\Big(\frac{1}{2\pi}\int_0^{2\pi}v(Re^{it})\,dt\Big).$$

(2) *Let $f\in\mathcal{N}$ and $z\in D$. Then*

$$|f(z)|\le\exp\Big(\frac{2\|f\|_{\mathcal{N}}}{1-|z|}\Big)-1.$$

(3) *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{N}$. Then $|a_n|\le ae^{b\sqrt{n}}$ for some $a,b>0$.*

Proof. (1) Let $0<r<1$. The function $v_r: z\mapsto v(rz)$ is continuous on $\overline{D}$, subharmonic in D and therefore majorized by its Poisson integral in this disc. In particular, we have

$$v_r(z)\le\frac{1}{2\pi}\int_0^{2\pi}v_r(e^{it})P_z(e^{it})\,dt,$$

where P_z denotes the Poisson kernel at $z\in D$:

$$P_z(e^{it})=\frac{1-|z|^2}{|e^{it}-z|^2}\le\frac{1+|z|}{1-|z|}.$$

It follows that

$$v(rz)\le\frac{1+|z|}{1-|z|}\frac{1}{2\pi}\int_0^{2\pi}v(re^{it})\,dt\le\frac{1+|z|}{1-|z|}\sup_{0\le R<1}\frac{1}{2\pi}\int_0^{2\pi}v(Re^{it})\,dt.$$

Letting r tend to 1 gives the desired inequality.

(2) Apply (1) to the positive, continuous and subharmonic function $v(z)=\log(1+|f(z)|)$ to obtain

$$\log(1+|f(z)|)\le\frac{2}{1-|z|}\|f\|_{\mathcal{N}},$$

from which the result follows.

(3) Set $\lambda=\|f\|_{\mathcal{N}}$. (2) and Cauchy's inequalities give, for all $0<r<1$,

$$|a_n|\le\exp\Big(\frac{2\lambda}{1-r}+n\log\frac{1}{r}\Big)\le\exp\Big(\frac{2\lambda}{1-r}+n\frac{1-r}{r}\Big).$$

Optimizing in r $(1-r=\sqrt{2\lambda/n})$ gives

$$|a_n|\le\exp(2\sqrt{2\lambda n}+O(1)),$$

which is the desired result with $b=2\sqrt{2\lambda}$. ■

As is well known, (3) can be replaced by $|a_n|\le c_\varepsilon e^{\varepsilon\sqrt{n}}$ for all $\varepsilon>0$ if $f\in\mathcal{N}^+$ (see [17]), and so $\mathcal{N}^+\subset F^+$. But a reverse inclusion $F^+\subset\mathcal{N}$ does not hold as confirmed by the following proposition (see [3]).

PROPOSITION 2.1.2. *If $\sum_{n=1}^{\infty}|a_n|^2=\infty$, then, for almost all choices of signs, $\sum_{n=1}^{\infty}\pm a_nz^n$ does not belong to $\mathcal{N}$.*

For example, there exist signs such that if $f(z)=\sum_{n=1}^{\infty}\pm(1/\sqrt{n})z^n$, then $f\notin\mathcal{N}$. Of course, $f\in F^+$.

2.2. *Order bounded maps.* Let h be a non-negative increasing function on $[0,\infty[$. We denote by L_h the set of all measurable functions f on ∂D such that

$$\int_0^{2\pi}h(|f(e^{i\theta})|)\,d\theta<\infty.$$

We consider a topological additive group X endowed with a metric d. We recall that a subset E of X is *bounded* if there exists a finite constant s such that $d(x,0)\le s$ for all $x\in E$. A map $T: X\to L_h$ is said to be *order bounded* if the image under T of every bounded set is order bounded. That is, the maximal function

$$M(T,s):=\sup_{x\in\overline{B}_X(0,s)}|Tx|$$

belongs to L_h for all $s>0$. Here $\overline{B}_X(0,s)$ denotes the closed ball in X centred at 0 with radius s.

In the case of composition operators we take $X=H^p$ $(0<p<\infty)$ or $X=\mathcal{N}$ and C_φ is a self-map of X for every $\varphi\in H(D,D)$. On the other hand, we take $h(x)=\log^+x:=\max(\log x,0)$ or $h(x)=x^q$ $(0<q<\infty)$. We shall allways restrict ourselves to those cases, for which L_h is a vector space.

Given a function $\varphi \in H(D,D)$ such that $|\varphi^*(e^{i\theta})| < 1$ almost everywhere, we shall say C_φ is (X, L_h)-*order bounded* (ob) if the operator $\widetilde{C}_\varphi := j \circ C_\varphi : X \to L_h$ is order bounded. According to this definition, the (X, L_h)-ob composition operators are closely related to the point evaluations induced by the points of $D \cap \varphi^*(\partial D)$.

For $X = H^p$, there are two cases: first, for $1 \le p < \infty$, the space H^p endowed with the norm $\|\cdot\|_p$ (defined in Sec. 1) is a Banach space. So the metric we shall consider is $d(f,g) := \|f-g\|_p$. Then, for $0 < p < 1$, $\|\cdot\|_p$ fails to be a norm and $d(f,g) := \|f-g\|_p^p$ defines a metric for which H^p becomes a complete space. In both cases the homogeneity of the metric d implies that, for all $s > 0$,

$$M(\widetilde{C}_\varphi, s) = \begin{cases} sM(\widetilde{C}_\varphi, 1) & \text{if } 1 \le p < \infty, \\ s^{1/p} M(\widetilde{C}_\varphi, 1) & \text{if } 0 < p < 1, \end{cases}$$

and then C_φ is (H^p, L_h)-ob if and only if

$$M_{\widetilde{C}_\varphi} := M(\widetilde{C}_\varphi, 1) \in L_h.$$

The following theorem about point evaluations on H^p is well known (see [19]).

THEOREM 2.2.1. *For all $0 < p < \infty$ and $z \in D$, we have*

$$\sup_{f \in \overline{B}_{H^p}(0,1)} |f(z)| = (1 - |z|^2)^{-1/p}.$$

2.3. *Moment sequences.* We denote by Δ the difference operator defined on the space of sequences $F = (F(n))_{n \in \mathbb{N}}$ by

$$\Delta F(n) := F(n) - F(n+1).$$

Its iterates are defined by

$$\Delta^0 F = F, \quad \Delta^{n+1} F = \Delta(\Delta^n F) \quad \text{for all } n \in \mathbb{N}.$$

The following binomial formula clearly holds:

$$\Delta^n F(k) = \sum_{j=0}^{n} \binom{n}{j} (-1)^j F(j+k) \quad \text{for all } k, n \in \mathbb{N}.$$

A version of the Hausdorff moment theorem (see [15], p. 9) suitable for our purposes can be stated as follows.

THEOREM 2.3.1. *Let F be a sequence of real numbers. There is a Borel measurable function $f : [0,1] \to [0,1]$ such that*

$$F(n) = \int_0^1 f(t)^n \, dt \quad \textit{for all } n \in \mathbb{N}$$

if and only if

$$F(0) = 1 \quad \textit{and} \quad \Delta^n F(k) \ge 0 \quad \textit{for all } k, n \in \mathbb{N}.$$

From now on, every sequence of real numbers satisfying the conditions of Theorem 2.3.1 will be called a *moment sequence.* For example, for any $\varphi \in H(D,D)$ the sequence $(\|\varphi^n\|_1)_{n \in \mathbb{N}}$ which coincides with the sequence $(\|\varphi^{*n}\|_{L^1})_{n \in \mathbb{N}}$ (see [2]) is a moment sequence. More precisely, owing to the analyticity of φ, we shall call this sequence an *analytic moment sequence.*

The condition $\Delta^n F(k) \ge 0$ is not always easy to check; we can sometimes use the following proposition in which $F^{(n)}$ denotes the nth derivative of F.

PROPOSITION 2.3.2. *Suppose that $F : [0, \infty[\to \mathbb{R}$ is a C^∞-function such that $F(0) = 1$ and $\operatorname{sign} F^{(n)} = (-1)^n$ for each $n \in \mathbb{N}$. Then $(F(n))_{n \in \mathbb{N}}$ is a moment sequence.*

This proposition is a consequence of Theorem 2.3.1 and of the following formula which one can prove by induction:

$$\Delta^n F(k) = (-1)^n \underbrace{\int_0^1 \cdots \int_0^1}_{(n \text{ times})} F^{(n)}(k + t_1 + \ldots + t_n) \, dt_1 \ldots dt_n.$$

The analytic moment sequences were characterized among moment sequences (see [6]) by the condition

$$\sum_{n=1}^{\infty} \frac{1}{n} \Delta^n F(0) < \infty.$$

In general, this condition is difficult to check. However, an appeal to the following theorem (see [6]) enables us to avoid this problem. It provides an analytic moment sequence close to a given moment sequence.

THEOREM 2.3.3. *Given any moment sequence $(F(n))_{n \in \mathbb{N}}$, there is a function $\varphi \in H(D,D)$ such that*

$$|F(n) - \|\varphi^n\|_1| \le 1/2^n \quad \textit{for each } n \in \mathbb{N}.$$

This theorem has the following corollary.

COROLLARY 2.3.4. *If $(F(n))_{n \in \mathbb{N}}$ is a moment sequence such that*

$$\lim_{n \to \infty} F(n) = 0 \quad \textit{and} \quad 2^n F(n) \ge M > 1, \quad \textit{as } n \to \infty,$$

then there is a function $\varphi \in H(D,D)$ such that $\|\varphi^n\|_1 \sim F(n)$.

Note that in the conclusion of the last corollary we necessarily have

$$|\varphi^*(e^{i\theta})| < 1 \quad \text{almost everywhere.}$$

In the rest of this paper φ denotes any function of $H(D,D)$ satisfying this condition.

3. (H^p, L_h)-ob composition operators

THEOREM 3.1. *The following are equivalent.*

(1) C_φ *is* (H^p, L_h)*-ob.*

(2) $$\begin{cases} \sum_{n=1}^{\infty} n^{q/p-1}\|\varphi^n\|_1 < \infty & \text{if } h(x) = x^q. \\ \sum_{n=1}^{\infty}\|\varphi^n\|_1/n < \infty & \text{if } h(x) = \log^+ x. \end{cases}$$

Proof. Since $|\varphi^*(e^{i\theta})| < 1$ almost everywhere, it follows from Theorem 2.2.1 that

$$M_{\widetilde{C}_\varphi}(e^{i\theta}) = \sup_{f\in \overline{B}_{H^p}(0,1)} |(f\circ\varphi)^*(e^{i\theta})| = \sup_{f\in \overline{B}_{H^p}(0,1)} |f(\varphi^*(e^{i\theta}))| = (1-|\varphi^*(e^{i\theta})|^2)^{-1/p}.$$

If $h(x) = x^q$, the result is shown in [5]. In the case $h(x) = \log^+ x$, we have

$$h(M_{\widetilde{C}_\varphi}(e^{i\theta})) = -\frac{1}{p}\log(1-|\varphi^*(e^{i\theta})|^2).$$

This equality, together with the estimates $1 \le 1 + |\varphi^*| \le 2$, implies that $M_{\widetilde{C}_\varphi} \in L_h$ if and only if $\log(1-|\varphi^*|) \in L^1$. Using the Taylor series of the function $x \mapsto -\log(1-x)$, we get

$$-\log(1-|\varphi^*(e^{i\theta})|) = \sum_{n=1}^{\infty}\frac{|\varphi^*(e^{i\theta})|^n}{n} \quad \text{for almost all } e^{i\theta}.$$

Finally, by Beppo Levi's theorem and the equality $\|f^*\|_p = \|f\|_p$ for all $f \in H^p$ (see [2]), we conclude that (1) and (2) are equivalent. ∎

We easily see from the definition and the inclusion $L^q \subset \log^+ L$ that the (H^p, L^q)-ob composition operators are necessarily $(H^p, \log^+ L)$-ob. But the converse is not true, as confirmed by the following proposition.

PROPOSITION 3.2. *There is a one-parameter family of composition operators which are* $(H^p, \log^+ L)$*-ob for all* $0 < p < \infty$ *and* (H^p, L^q)*-ob for no* $0 < p, q < \infty$.

Proof. In order to show the existence of such a family, it is sufficient to apply Corollary 2.3.4 to a one-parameter set of appropriate moment sequences $(F_\beta(n))_{n\in\mathbb{N}}$ satisfying

$$(\diamond) \qquad \sum_{n=1}^{\infty}\frac{F_\beta(n)}{n} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{F_\beta(n)}{n^\alpha} = \infty \quad \text{for all } 0 < \alpha < 1.$$

Then an appeal to Theorem 3.1 will complete the proof.

For example, any sequence similar to $(\log n)^{-\beta}$ with $\beta > 1$ satisfies $(\diamond)$. For each $\beta > 1$ the sequence $(F_\beta(n))_{n\in\mathbb{N}}$ defined by

$$F_\beta(n) = (1+\log(n+1))^{-\beta}$$

satisfies $(\diamond)$ and is a moment sequence (apply Proposition 2.3.2 and Faà di Bruno's formula recalled in [6] and used in the proof of Proposition 4.6).

By Corollary 2.3.4, there exists $\varphi_\beta \in H(D,D)$ such that $\|\varphi_\beta^n\|_1 \sim F_\beta(n)$. Clearly then, $(\|\varphi_\beta^n\|_1)_{n\in\mathbb{N}}$ satisfies $(\diamond)$. Finally, the conclusion follows from Theorem 3.1. ∎

Here is an explicit construction of many (H^p, L^q)-ob composition operators induced by functions $\varphi \in H(D,D)$ such that $\|\varphi\|_\infty := \sup_{|z|<1}|\varphi(z)| = 1$, for all $0 < p, q < \infty$.

Fix $\alpha > 0$. Take a measurable partition $(A_j)_{j\in\mathbb{N}^*}$ of the unit circle such that (m denoting the normalized Haar measure)

$$m(A_j) = e^\alpha(e^{-\alpha\sqrt{j}} - e^{-\alpha\sqrt{j+1}}).$$

Consider the function g_α defined on ∂D by

$$g_\alpha(e^{it}) := \sum_{j=1}^{\infty} e^{-\alpha/\sqrt{j}}\chi_j(e^{it}),$$

where χ_j denotes the indicator function of A_j. We have the following proposition.

PROPOSITION 3.3. *The outer function* φ_α *defined on* D *by*

$$\varphi_\alpha(z) := \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log g_\alpha(e^{it})\,dt\right)$$

induces an (H^p, L^q)*-ob composition operator for all* $0 < p, q < \infty$.

Proof. Since

$$-\log g_\alpha(e^{it}) = \sum_{j=1}^{\infty}\frac{\alpha}{\sqrt{j}}\chi_j(e^{it}),$$

by integration we get

$$\frac{1}{2\pi}\int_0^{2\pi} -\log g_\alpha(e^{it})\,dt = \sum_{j=1}^{\infty}\frac{\alpha}{\sqrt{j}}m(A_j) \le \alpha\sum_{j=1}^{\infty}m(A_j) = \alpha.$$

So $\log g_\alpha$ is integrable on ∂D and we can take the related outer function φ_α (defined as in the statement). We have

$$|\varphi_\alpha(z)| = \exp(u(z)),$$

where $u(z)$ is the Poisson integral of the non-positive function $\log g_\alpha$ $(0 < g_\alpha < 1)$. Therefore $\varphi_\alpha \in H(D,D)$. Recall (cf. [2], p. 5) that

$$u^*(e^{i\theta}) = \log g_\alpha(e^{i\theta}) \quad \text{for almost all } e^{i\theta}.$$

Consequently, we find that

$$|\varphi_\alpha^*(e^{i\theta})| = g_\alpha(e^{i\theta}) < 1 \quad \text{for almost all } e^{i\theta}.$$

In particular, we have $\|\varphi_\alpha\|_\infty = 1$. (Obviously, the case $\|\varphi\|_\infty < 1$ provides an (H^p, L^q)-ob composition operator for all $0 < p, q < \infty$. Indeed in this case, the maximal function $M_{\widetilde{C}_\varphi}$ is bounded and hence q-integrable on ∂D.)

For the rest of the proof, observe that

$$\begin{aligned}\|\varphi_\alpha^n\|_1 &= \frac{1}{2\pi}\int_0^{2\pi} g_\alpha^n(e^{it})\,dt\\ &= \sum_{j=1}^{n} e^{-\alpha n/\sqrt{j}} m(A_j) + \sum_{j=n+1}^{\infty} e^{-\alpha n/\sqrt{j}} m(A_j)\\ &\le e^{-\alpha\sqrt{n}} \sum_{j=1}^{n} m(A_j) + \sum_{j=n+1}^{\infty} m(A_j)\\ &\le e^{-\alpha\sqrt{n}} + e^{\alpha} e^{-\alpha\sqrt{n}} = (1+e^\alpha)e^{-\alpha\sqrt{n}}.\end{aligned}$$

This implies that

$$\sum_{n=0}^{\infty} n^{q/p-1}\|\varphi^n\|_1 < \infty \quad \text{for all } 0 < p, q < \infty.$$

An appeal to Theorem 3.1 completes the proof. ∎

REMARK. In the proof of the last proposition, we only need a majorization of the analytic moments. To find a minorant of the same form, we proceed as follows:

$$\begin{aligned}\|\varphi_\alpha^n\|_1 &= \frac{1}{2\pi}\int_0^{2\pi} g_\alpha^n(e^{it})\,dt \ge \sum_{j=n}^{2n-1} e^{-\alpha n/\sqrt{j}} m(A_j)\\ &\ge e^{-\alpha\sqrt{n}} \sum_{j=n}^{2n-1} m(A_j)\\ &= e^{-\alpha\sqrt{n}} e^{\alpha}(e^{-\alpha\sqrt{n}} - e^{-\alpha\sqrt{2n}}) \sim e^{\alpha} e^{-2\alpha\sqrt{n}}.\end{aligned}$$

4. $(\mathcal{N}, L_h)$-ob composition operators

LEMMA 4.1. (1) *For every $s > 0$, there are $b_s, c_s > 0$ such that*

$$b_s \exp\left(\frac{c_s}{1-|z|}\right) \le \sup_{f\in \overline{B}_{\mathcal{N}}(0,s)} |f(z)| \le \exp\left(\frac{2s}{1-|z|}\right) \quad \textit{for all } z \in D.$$

(2) *For every $p > 0$, there is $s_p > 0$ such that*

$$\sup_{f\in \overline{B}_{\mathcal{N}}(0,s)} |f(z)| \ge \exp\left(\frac{p}{1-|z|}\right) \quad \textit{for all } s \ge s_p \textit{ and } z \in D.$$

Proof. (1) Let $f \in \overline{B}_{\mathcal{N}}(0,s)$. By Lemma 2.2.1(2), we have

$$|f(z)| \le \exp\left(\frac{2s}{1-|z|}\right) - 1 \le \exp\left(\frac{2s}{1-|z|}\right),$$

which yields the right-hand inequality.

To show the left-hand one, set $\eta = 1/(1+2/\pi)$ and, for each $s > 0$, take a small number $\varepsilon = \varepsilon_s \in]0, s\eta[$. There exists $\delta = \delta_s > 0$ such that

$$(*) \qquad |e^\omega - 1| \le \varepsilon \quad \text{for all } \omega \text{ with } |\omega| \le \delta.$$

Set now $c = c_s := \min\{s - \varepsilon/\eta,\ \frac{1}{2}(1-\cos\varepsilon)\delta\}$. The function f_s defined by

$$f_s(\omega) := \exp\left(\frac{c(1+\omega)}{1-\omega}\right) - 1$$

belongs to $\overline{B}_{\mathcal{N}}(0,s)$. Indeed, for any $0 \le r < 1$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1+|f_s(re^{i\theta})|)\,d\theta &= \frac{1}{2\pi}\int_{|\theta|>\varepsilon} \log(1+|f_s(re^{i\theta})|)\,d\theta\\ &\quad + \frac{1}{2\pi}\int_{|\theta|\le\varepsilon} \log(1+|f_s(re^{i\theta})|)\,d\theta.\end{aligned}$$

On the arc $\{e^{i\theta} : |\theta| > \varepsilon\}$, we have

$$|1 - re^{i\theta}| \ge 1 - r\cos\theta \ge 1 - \cos\theta \ge 1 - \cos\varepsilon.$$

Therefore, we get

$$\left|\frac{c(1+re^{i\theta})}{1-re^{i\theta}}\right| \le \frac{2c}{1-\cos\varepsilon} \le \delta.$$

Hence, we deduce by $(*)$ that

$$|f_s(re^{i\theta})| \le \varepsilon \quad \text{for all } \theta \text{ with } |\theta| > \varepsilon.$$

On the complementary arc $\{e^{i\theta} : |\theta| \le \varepsilon\}$, we use the relation

$$\log(1+x) \le 1 + \log^+ x \quad (x \ge 0)$$

to obtain

$$\log(1+|f_s(re^{i\theta})|) \le 2 + cP_r(e^{i\theta}).$$

Consequently,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(1+|f_s(re^{i\theta})|)\,d\theta \le \log(1+\varepsilon) + \frac{2\varepsilon}{\pi} + c \le \frac{\varepsilon}{\eta} + c \le s.$$

Thus, $f_s \in \overline{B}_{\mathcal{N}}(0,s)$.

Now, for all $z = |z|e^{i\alpha} \in D$, the function g_s defined by $g_s(\omega) := f_s(e^{-i\alpha}\omega)$ is also in $\overline{B}_{\mathcal{N}}(0,s)$, since $\|g_s\|_{\mathcal{N}} = \|f_s\|_{\mathcal{N}}$. On the other hand, we have

$$|g_s(z)| = \exp\left(\frac{c(1+|z|)}{1-|z|}\right) - 1 \ge \exp\left(\frac{c}{1-|z|}\right) - 1 \ge (1-e^{-c})\exp\left(\frac{c}{1-|z|}\right).$$

Hence, for all $z \in D$,

$$\sup_{f\in\overline{B}_{\mathcal{N}}(0,s)} |f(z)| \ge |g_s(z)| \ge b\exp\left(\frac{c}{1-|z|}\right),$$

where $b = b_s := 1-e^{-c}$.

(2) For all $p > 0$, let k_p be the function defined on D by

$$k_p(\omega) := \exp\left(\frac{p(1+\omega)}{1-\omega}\right).$$

Since

$$\log(1+|k_p(re^{i\theta})|) \le 1 + pP_r(e^{i\theta}),$$

integrating and letting r tend to 1 ensures that $\|k_p\|_{\mathcal{N}} \le 1 + p =: s_p$. Now as in the proof of (1), for all $z = |z|e^{i\alpha} \in D$, the function $l_p : \omega \mapsto k_p(e^{-i\alpha}\omega)$ belongs to $\overline{B}_{\mathcal{N}}(0,s_p)$ and satisfies

$$|l_p(z)| = \exp\left(\frac{p(1+|z|)}{1-|z|}\right) \ge \exp\left(\frac{p}{1-|z|}\right).$$

Hence, for all $s \ge s_p$ and $z \in D$, we have

$$\sup_{f\in\overline{B}_{\mathcal{N}}(0,s)} |f(z)| \ge |l_p(z)| \ge \exp\left(\frac{p}{1-|z|}\right).$$

This completes the proof. ∎

REMARK. We can show that the constants b_s and c_s given in Lemma 4.1(1) satisfy

$$b_s \sim c_s = O(s^{5/2}).$$

THEOREM 4.2. *The following are equivalent.*

(1) *C_φ is $(\mathcal{N}, \log^+ L)$-ob.*
(2) *C_φ is (H^p, L^p)-ob for some and hence all $0 < p < \infty$.*
(3) *$C_\varphi : H^2 \to H^2$ is Hilbert–Schmidt.*
(4) *$\sum_{n=0}^{\infty} \|\varphi^n\|_1 < \infty$.*

Proof. We recall that (2) and (3) are equivalent to (4) (see [5] and [14]).

(1) is nothing else but the integrability of all functions $\log^+(M(\widetilde{C}_\varphi, s))$ where $s > 0$. For almost all $e^{i\theta}$, we have

$$M(\widetilde{C}_\varphi, s)(e^{i\theta}) := \sup_{f\in\overline{B}_{\mathcal{N}}(0,s)} |f(\varphi^*(e^{i\theta}))|.$$

By Lemma 4.1 there exist $b_s, c_s > 0$ such that

$$b_s\exp\left(\frac{c_s}{1-|\varphi^*(e^{i\theta})|}\right) \le M(\widetilde{C}_\varphi, s)(e^{i\theta}) \le \exp\left(\frac{2s}{1-|\varphi^*(e^{i\theta})|}\right)$$

almost everywhere on ∂D. On the other hand, since

$$\log^+\left(b_s\exp\left(\frac{c_s}{1-|\varphi^*(e^{i\theta})|}\right)\right) \ge \log\left(1+b_s\exp\left(\frac{c_s}{1-|\varphi^*(e^{i\theta})|}\right)\right) - 1 \ge \frac{c_s}{1-|\varphi^*(e^{i\theta})|} + \log b_s - 1,$$

we conclude that

$$\frac{c_s}{1-|\varphi^*(e^{i\theta})|} + \log b_s - 1 \le \log^+ M(\widetilde{C}_\varphi, s)(e^{i\theta}) \le \frac{2s}{1-|\varphi^*(e^{i\theta})|}$$

almost everywhere on ∂D. Now expand $\frac{1}{1-|\varphi^*(e^{i\theta})|}$ and apply Beppo Levi's theorem to conclude that

$$M(\widetilde{C}_\varphi, s) \in \log^+ L \quad \text{if and only if} \quad \sum_{n=0}^{\infty} \|\varphi^{*n}\|_{L^1} < \infty.$$

Finally, we deduce Theorem 4.2 by using the well known equality $\|\varphi^n\|_1 = \|\varphi^{*n}\|_{L^1}$. ∎

The next lemma will play a crucial role in getting a necessary and sufficient condition, in terms of the analytic moments, for the operator C_φ to be $(\mathcal{N}, L^q)$-ob.

LEMMA 4.3. *For all $p > 0$, there are constants $c_1(p), c_2(p) > 0$ such that*

$$\exp\left(\frac{p}{1-z}\right) = \sum_{n=0}^{\infty} a_n(p)z^n \quad \textit{for all } z \in D,$$

with

$$c_1(p)n^{-3/4}e^{2\sqrt{np}} \le a_n(p) \le c_2(p)n^{-3/4}e^{2\sqrt{np}} \quad \textit{for all } n \in \mathbb{N}^*.$$

Proof. For all $z \in D$, we have

$$\exp\left(\frac{p}{1-z}\right) = 1 + \sum_{k=1}^{\infty} p^k \frac{(1-z)^{-k}}{k!}$$
$$= 1 + \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{\infty} p^k \frac{(n+k-1)!}{(k-1)!k!} z^n$$
$$= 1 + \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} p^{k+1} \frac{(n+k)!}{k!(k+1)!} z^n$$
$$= \sum_{n=0}^{\infty} a_n(p) z^n,$$

with, for all $n \in \mathbb{N}^*$,

$$a_n(p) = \frac{1}{n!} \sum_{k=0}^{\infty} p^{k+1} \frac{(n+k)!}{k!(k+1)!}$$
$$= \sum_{k=0}^{\infty} p^{k+1} \frac{(n+1)(n+2)\ldots(n+k)}{k!(k+1)!}$$
$$\geq \sum_{k=0}^{\infty} \frac{p^{k+1}n^k}{k!(k+1)!} = \left(\frac{p}{n}\right)^{1/2} I_1(2\sqrt{np}).$$

Here I_1 is the modified Bessel function J_1, defined by

$$I_1(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{1+2k}}{k!(k+1)!}.$$

Now, it is known (cf. [7], p. 123) that

$$I_1(z) \sim e^z (2\pi z)^{-1/2} \quad \text{as } |z| \to \infty.$$

Consequently, we find a minorization of the form

$$a_n(p) \geq c_1(p) n^{-3/4} e^{2\sqrt{np}}.$$

Moreover, if we set

$$u_k = p^{k+1} \frac{(n+1)(n+2)\ldots(n+k)}{k!(k+1)!} \quad \text{for all } k \in \mathbb{N},$$

we can find $A_p > 0$ such that

$$\frac{u_{k+1}}{u_k} = \frac{p(n+k+1)}{(k+1)(k+2)} \leq \frac{p(n+k+1)}{k^2} \leq \frac{1}{2} \quad \text{for all } k \geq A_p\sqrt{n}.$$

So the rest of order $A_p\sqrt{n}$ of the series $\sum_{k\in\mathbb{N}} u_k$ satisfies

$$\sum_{k \geq A_p\sqrt{n}} u_k = o\Big(\sum_{k < A_p\sqrt{n}} u_k\Big) \quad \text{as } n \to \infty.$$

On the other hand, for all $k \in \mathbb{N}$, we have

$$u_k = \frac{p^{k+1}n^k}{k!(k+1)!}\left(1+\frac{1}{n}\right)\ldots\left(1+\frac{k}{n}\right) \leq \frac{p^{k+1}n^k}{k!(k+1)!} e^{k^2/n}.$$

Hence, we get

$$a_n(p) = \sum_{k < A_p\sqrt{n}} u_k + \sum_{k \geq A_p\sqrt{n}} u_k$$
$$\leq 2 \sum_{k < A_p\sqrt{n}} u_k \quad (\text{for } n \text{ large enough})$$
$$\leq 2 \sum_{k < A_p\sqrt{n}} \frac{p^{k+1}n^k}{k!(k+1)!} e^{k^2/n}$$
$$\leq c \sum_{k < A_p\sqrt{n}} \frac{p^{k+1}n^k}{k!(k+1)!} \quad (\text{for a constant } c > 0)$$
$$\leq c \sum_{k=0}^{\infty} \frac{p^{k+1}n^k}{k!(k+1)!} = c\left(\frac{p}{n}\right)^{1/2} I_1(2\sqrt{np}).$$

This and the property of the function I_1 lead to a majorization of the form

$$a_n(p) \leq c_2(p) n^{-3/4} e^{2\sqrt{np}}. \ \blacksquare$$

REMARK. We can show that the constants $c_1(p)$ and $c_2(p)$ in Lemma 4.3 satisfy

$$\begin{cases} c_1(p) \geq \delta_1 & \text{if } p \geq p_1 > 0, \\ c_2(p) \leq \delta_2 & \text{if } p \leq p_2 < \infty. \end{cases}$$

THEOREM 4.4. *The following are equivalent.*

(1) $\|\varphi^n\|_1 = O(e^{-t\sqrt{n}})$ *for all* $t > 0$.
(2) C_φ *is* $(\mathcal{N}, L^q)$*-ob for all* $0 < q < \infty$.
(3) C_φ *is* $(\mathcal{N}, L^q)$*-ob for some* $0 < q < \infty$.

Proof. (1)⇒(2). First of all, we remark that (1) is equivalent to

$$(1') \qquad \sum_{n=0}^{\infty} e^{t\sqrt{n}} \|\varphi^n\|_1 < \infty \quad \text{for all } t > 0.$$

Let $q, s \in]0, \infty[$. By Lemma 4.1(1) one has

$$(\mathrm{i}) \qquad M(\widetilde{C}_\varphi, s)(e^{i\theta}) = \sup_{f \in \overline{B}_{\mathcal{N}}(0,s)} |f(\varphi^*(e^{i\theta}))| \leq \exp\left(\frac{2s}{1-|\varphi^*(e^{i\theta})|}\right).$$

Moreover, Lemma 4.3 provides a positive constant $c_2(q,s)$ such that

$$\text{(ii)}\qquad \exp\left(\frac{2qs}{1-|\varphi^*(e^{i\theta})|}\right) \le c_2(q,s)\sum_{n=1}^{\infty} n^{-3/4}e^{2\sqrt{2nqs}}|\varphi^*(e^{i\theta})|^n.$$

Now, by $(1')$ we get the convergence of the series

$$\sum_{n\ge1} n^{-3/4}e^{2\sqrt{2nqs}}\|\varphi^n\|_1.$$

So by Beppo Levi's theorem and the equality $\|\varphi^n\|_1=\|\varphi^{*^n}\|_1$, the function

$$\sum_{n=1}^{\infty} n^{-3/4}e^{2\sqrt{2nqs}}|\varphi^*|^n$$

is integrable on ∂D. By (i) and (ii), this implies that the maximal function $M(\widetilde{C}_\varphi,s)$ is q-integrable on ∂D. Finally, (2) follows because s and q are arbitrary.

(2)⇒(3) is immediate.

(3)⇒(1). Let $t>0$. It follows from (3) that

$$M(\widetilde{C}_\varphi,s)\in L^q(\partial D,m)\quad \text{for all } s>0.$$

By Lemma 4.1(2), there is $s(t,q)>0$ such that, for all $s\ge s(t,q)$, we have

$$M(\widetilde{C}_\varphi,s)(e^{i\theta})=\sup_{f\in\overline{B}_{\mathcal N}(0,s)}|f(\varphi^*(e^{i\theta}))|\ge \exp\left(\frac{t^2/(4q)}{1-|\varphi^*(e^{i\theta})|}\right).$$

Now the q-integrability of $M(\widetilde{C}_\varphi,s)$ on ∂D (for $s\ge s(t,q)$) implies the integrability of the function $\exp\left(\frac{t^2/4}{1-|\varphi^*|}\right)$. Hence, by Lemma 4.3, we get the integrability on ∂D of $\sum_{n=1}^\infty n^{-3/4}e^{t\sqrt n}|\varphi^*|^n$. So, by Beppo Levi's theorem and the equality $\|\varphi^n\|_1=\|\varphi^{*n}\|_1$, we obtain the convergence of the series $\sum_{n=1}^\infty n^{-3/4}e^{t\sqrt n}\|\varphi^n\|_1$.

Finally, since t is arbitrary, $(1')$ holds and so does (1). ∎

The next corollary is an immediate consequence of Theorems 4.4 and 3.1.

COROLLARY 4.5. *If C_φ is $(\mathcal N, L^q)$-ob for some $0<q<\infty$, then it is (H^p,L^q)-ob for all $0<p,q<\infty$.*

The converse of Corollary 4.5 is not always true. More precisely, we have the following.

PROPOSITION 4.6. (1) *There is a one-parameter family of operators C_φ with $\|\varphi\|_\infty=1$ which are $(\mathcal N,L^q)$-ob for all $0<q<\infty$.*

(2) *There is a one-parameter family of composition operators which are (H^p,L^q)-ob for all $0<p,q<\infty$ and $(\mathcal N,L^q)$-ob for no $0<q<\infty$.*

Proof. We are going to show (1) and (2) simultaneously. In order to show the existence, it is sufficient to apply Corollary 2.3.4 to a one-parameter family of appropriate moment sequences. We get the first set once we exhibit moment sequences $(F_\gamma(n))_{n\in\mathbb N}$ satisfying

$$(\bullet)\qquad F_\gamma(n)=O(e^{-\alpha\sqrt n})\quad\text{for all }\alpha>0.$$

The existence of the second set will be ensured by those sequences $(F_\gamma(n))_{n\in\mathbb N}$ such that

$$c_\alpha e^{-\alpha\sqrt n}\le F_\gamma(n)\le c'_\alpha n^{-\alpha}\quad\text{for all }\alpha>0\text{ and }n\in\mathbb N^*.$$

Given $0<\gamma<1$, we consider the function $G=G_\gamma$ defined on $[0,\infty[$ by

$$G(x)=1-(x+1)^\gamma.$$

Then $F=F_\gamma=\exp\circ G$ is of class C^∞ on $[0,\infty[$. Apply the formula of Faà di Bruno:

$$F^{(n)}=\sum\frac{n!}{k_1!\cdots k_n!}(\exp^{(\sum k_i)}\circ G)\left(\frac{G'}{1!}\right)^{k_1}\left(\frac{G''}{2!}\right)^{k_2}\cdots\left(\frac{G^{(n)}}{n!}\right)^{k_n},$$

where summation is over all integers $k_1,\dots,k_n$ such that

$$\sum_{i=1}^n ik_i=n.$$

Noting that $\operatorname{sign}G^{(k)}=(-1)^k$ for each $k\in\mathbb N^*$, we deduce that

$$\operatorname{sign}F^{(n)}=(-1)^{k_1}(-1)^{2k_2}\cdots(-1)^{nk_n}=(-1)^n\quad\text{for all }n\in\mathbb N^*.$$

Now since $F(0)=1$ and $F>0$, Theorem 2.3.2 asserts that $(F(n))_{n\in\mathbb N}$ is a moment sequence. Hence, by Corollary 2.3.4, there is $\varphi=\varphi_\gamma\in H(D,D)$ such that

$$(\bullet\bullet)\qquad \|\varphi^n\|_1\sim F(n).$$

The sequences $(F_\gamma(n))_{n\in\mathbb N}$ with $1/2<\gamma<1$ satisfy $(\bullet)$ and so do (because of $(\bullet\bullet)$) the corresponding analytic moment sequences. Therefore, by Theorem 4.4, the operators C_{φ_γ} $(1/2<\gamma<1)$ are $(\mathcal N,L^q)$-ob for all $0<q<\infty$. This completes the proof of (1).

On the other hand, the sequences $(F_\gamma(n))_{n\in\mathbb N}$ with $0<\gamma<1/2$ satisfy

$$e^{-\alpha\sqrt n}=o(F_\gamma(n)),\qquad F_\gamma(n)=o(n^{-\alpha})\quad\text{for all }\alpha>0.$$

So, by $(\bullet\bullet)$ we deduce that, for each $0<\gamma<1/2$,

$$e^{-\alpha\sqrt n}\le\|\varphi_\gamma^n\|_1\le n^{-\alpha}\quad\text{for all }\alpha>0\text{ and }n\in\mathbb N^*.$$

Now by Theorem 3.1, one easily sees that the second inequality implies that every C_{φ_γ} $(0<\gamma<1/2)$ is (H^p,L^q)-ob for all $0<p,q<\infty$, while the first estimate, according to Theorem 4.4, shows that those composition operators fail to be $(\mathcal N,L^q)$-ob for any $0<q<\infty$. This completes the proof of (2). ∎

REMARK. There is another way to show (1) of Proposition 4.6. Indeed, as in Proposition 3.3 we can give an explicit construction. Let $\alpha > 0$ and $1/2 < \gamma < 1$. Consider a partition $(A_j)_{j\in\mathbb{N}^*}$ of the unit circle such that

$$m(A_j) = e^{\alpha}(e^{-\alpha j^{\gamma}} - e^{-\alpha(j+1)^{\gamma}}).$$

We define the function $g_{\alpha,\gamma}$ on ∂D by

$$g_{\alpha,\gamma}(e^{it}) := \sum_{j=1}^{\infty} e^{-\alpha j^{\gamma-1}} \chi_j(e^{it}),$$

where χ_j denotes the characteristic function of the set A_j. Now taking the outer function $\varphi_{\alpha,\gamma}$ as in Proposition 3.3 and using the same arguments given in the proof of that proposition, one can deduce by Theorem 4.4(2) that the operators $C_{\varphi_{\alpha,\gamma}}$ are $(\mathcal{N}, L^q)$-ob for all $0 < q < \infty$.

The following theorem says that the $(\mathcal{N}, L^q)$-order boundedness of C_φ is not stronger than its compactness from $\mathcal{N}$ into H^q.

THEOREM 4.7. *The following are equivalent.*

(1) $\|\varphi^n\|_1 = O(e^{-\lambda\sqrt{n}})$ *for all* $\lambda > 0$.
(2) $C_\varphi : \mathcal{N} \to H^q$ *is compact for all* $0 < q < \infty$.
(3) $C_\varphi : \mathcal{N} \to H^q$ *is compact for some* $0 < q < \infty$.
(4) $C_\varphi : \mathcal{N} \to H^q$ *is bounded on every bounded set for some* $0 < q < \infty$.

Proof. (1)⇒(2). As in [1], we say that C_φ is compact from $\mathcal{N}$ into H^q if, for every $s > 0$, the image under C_φ of $\overline{B}_{\mathcal{N}}(0, s)$ is relatively compact in H^q, and by a normal family argument this is equivalent to the following:

$$(*) \qquad f_n \xrightarrow{u.c} 0 \text{ and } \|f_n\|_{\mathcal{N}} \le s \Rightarrow \|C_\varphi f_n\|_q \to 0.$$

The hypothesis implies that $|\varphi^*(e^{i\theta})| < 1$ almost everywhere. So, if $g_n = f_n \circ \varphi$, then $g_n^* = f_n \circ \varphi^*$ almost everywhere and

$$\frac{1}{2\pi}\int_0^{2\pi} |g_n^*(e^{i\theta})|^q\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f_n(\varphi^*(e^{i\theta}))|^q\, d\theta$$
$$\le \frac{1}{2\pi}\int_0^{2\pi} \exp\left(\frac{2sq}{1-|\varphi^*(e^{i\theta})|}\right) d\theta =: \frac{1}{2\pi}\int_0^{2\pi} M(e^{i\theta})\, d\theta.$$

Now, $M \in L^1$, since by Lemma 4.3 and Beppo Levi's theorem,

$$\|M\|_1 = \sum_{n=0}^{\infty} a_n(2sq)\|\varphi^n\|_1 \le \sum_{n=0}^{\infty} c_2(2sq)e^{2\sqrt{2nsq}}\|\varphi^n\|_1 < \infty.$$

By the hypothesis (with $\lambda > 2\sqrt{2sq}$), this proves that g_n^* is in L^q, therefore

$g_n \in H^q$ with $\|g_n\|_q = \|g_n^*\|_q$. Moreover, if $0 < \lambda < 1$, then

$$\|g_n\|_q^q \le \frac{1}{2\pi}\int_{|\varphi^*|\le\lambda} |f_n(\varphi^*(e^{i\theta}))|^q\, d\theta + \frac{1}{2\pi}\int_{|\varphi^*|>\lambda} M(e^{i\theta})\, d\theta$$
$$\le \sup_{|\omega|\le\lambda} |f_n(\omega)|^q + \frac{1}{2\pi}\int_{|\varphi^*|>\lambda} M(e^{i\theta})\, d\theta.$$

It then follows from the hypothesis of $(*)$ that

$$(**) \qquad \overline{\lim}\, \|g_n\|_q^q \le \frac{1}{2\pi}\int_{|\varphi^*|>\lambda} M(e^{i\theta})\, d\theta =: \varrho(\lambda).$$

But $\varrho(\lambda) \to 0$ as $\lambda \to 1$, since

$$\varrho(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} M(e^{i\theta}) 1_{\{M>A(\lambda)\}}(e^{i\theta})\, d\theta,$$

where $A(\lambda) = \exp(2sq/(1-\lambda)) \to \infty$ as $\lambda \to 1$. Therefore, letting λ tend to 1 in $(**)$ gives $\overline{\lim}\, \|g_n\|_q^q \le 0$, which proves $(*)$ and thus the assertion (2).

(2)⇒(3)⇒(4) is obvious.

(4)⇒(1). Fix $\lambda > 0$, let $s > 0$ (to be chosen later) and set

$$g_\alpha(z) := \exp\left(\frac{s(1+e^{i\alpha}z)}{1-e^{i\alpha}z}\right).$$

From the inequality $\log(1+x) \le 1 + \log^+ x$, it follows that $\|g_\alpha\|_{\mathcal{N}} \le 1 + s$. Therefore, $\|g_\alpha \circ \varphi\|_q \le M_s$, where M_s depends only on s. That is to say,

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\exp\left(\frac{s(1+e^{i\alpha}\varphi^*(e^{i\theta}))}{1-e^{i\alpha}\varphi^*(e^{i\theta})}\right)\right|^q d\theta \le M_s^q$$

or equivalently (using the identity $\frac{1+z}{1-z} = -1 + \frac{2}{1-z}$)

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\exp\left(\frac{sq}{2}\frac{(1+e^{i\alpha}\varphi^*(e^{i\theta}))}{1-e^{i\alpha}\varphi^*(e^{i\theta})}\right)\right|^2 d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=0}^{\infty} a_n(sq)e^{in\alpha}\varphi^{*n}(e^{i\theta})\right|^2 d\theta \le M_s^q.$$

Now, integrate with respect to $d\alpha/(2\pi)$, and apply Fubini's and Parseval's theorems to get

$$\sum_{n=0}^{\infty} |a_n(sq)|^2 \|\varphi^{2n}\|_1 \le M_s^q.$$

Fixing q, we obtain in particular

$$\|\varphi^{2n}\|_1 = O(|a_n(sq/2)|^{-2}) = O(n^{3/2}e^{-4\sqrt{nsq}}).$$

Since $\|\varphi^n\|_1 \le \|\varphi^n\|_2 = \|\varphi^{2n}\|_1^{1/2}$, we have

$$\|\varphi^n\|_1 = O(n^{3/4}e^{-2\sqrt{nsq}}).$$

Now adjusting s so that $2\sqrt{sq} > \lambda$, we get

$$\|\varphi^n\|_1 = O(e^{-\lambda\sqrt{n}}),$$

as desired. ■

We conclude with the following question. If we assume that $C_\varphi : \mathcal{N} \to H^q$ is continuous, we can prove that $\|\varphi^n\|_1 = O(e^{-\lambda\sqrt{n}})$ for some $\lambda > 0$, which characterizes the continuity of $C_\varphi : F^+ \to H^q$ (see [9]). However, we have not been able to decide if this is true for all $\lambda > 0$.

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