Define $H : \mathcal{D} \to \mathbb{C}$ by $H(\mathcal{G}') = L(G)$, $\mathcal{G}' \in \mathcal{D}$. H is well defined, as can be easily deduced from the hypothesis imposed on L. For the same reason, it is clear that $L = H \circ \psi$.

To prove the continuity of H, consider a sequence $\{\mathcal{G}'_k\}_{k\in\mathbb{N}} \subset \mathcal{D}$ converging to 0, where $\mathcal{G}'_k = \{g_{jm,k}\}$, $k \in \mathbb{N}$. This implies the convergence of $\{g_{jm,k}\}_{k\in\mathbb{N}}$ to 0 in the corresponding spaces $\mathcal{B}_{jm}$. As shown in Proposition 5.3, there exists a sequence $\{G_k\}_{k\in\mathbb{N}} \subset \mathcal{B}(S)$ converging to 0 such that if $\mathrm{TA}'(G_k) = \{G_{jm,k}\}$, $k \in \mathbb{N}$, then $G_{jm,k} = g_{jm,k}$ for $j \in N$ and $m \in \mathbb{N}$ with $m \le r$. So, $H(\mathcal{G}'_k) = L(G_k)$, $k \in \mathbb{N}$. Now, the continuity of L implies $\lim_{k\to\infty} L(G_k) = 0$, and so H is continuous. Its linearity results immediately. ■

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Simple systems are disjoint from Gaussian systems

by

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Abstract. We prove the theorem promised in the title. Gaussians can be distinguished from simple maps by their property of divisibility. Roughly speaking, a system is divisible if it has a rich supply of direct product splittings. Gaussians are divisible and weakly mixing simple maps have no splittings at all so they cannot be isomorphic. The proof that they are disjoint consists of an elaboration of this idea, which involves, among other things, the notion of virtual divisibility, which is, more or less, divisibility up to distal extensions. The theory of Kronecker Gaussians also plays a crucial role.

1. Main result and overview of the proof. We deal throughout with (dynamical) systems $\mathbf{X} = (X, \mathcal{B}, \mu, T)$ and $\mathbf{Y} = (Y, \mathcal{C}, \nu, S)$, by which we understand that $(X, \mathcal{B}, \mu)$ is a Lebesgue probability space and $T : X \to X$ is a measurable invertible μ-preserving map. The purpose of this paper is to prove:

THEOREM 1. *If* $\mathbf{X}$ *is simple and* $\mathbf{Y}$ *is Gaussian then* $\mathbf{X}$ *and* $\mathbf{Y}$ *are disjoint.*

In the special case when $\mathbf{Y}$ has minimal self-joinings Theorem 1 was established by Thouvenot in [Th1]. After learning of our result Thouvenot has recently proved that the assumption that Y is Gaussian can be weakened to the assumption that Y is the time one map in a flow which is infinitely divisible (see §3 for the notion of divisibility).

Thouvenot [Th1] has shown that every horocycle flow is a factor of a simple flow, so Theorem 1 has the following corollary.

COROLLARY 2. *The time one map of any horocycle flow is disjoint from any Gaussian system.*

By a result of [J,R], Theorem 1 is equivalent to showing:

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PROPOSITION 3. *No factor of* $\mathbf{Y}$ *can be isomorphic to a symmetric Cartesian power* $(\mathbf{X}/K)^{n\odot}$ *of a factor* $\mathbf{X}/K$ *of* $\mathbf{X}$.

Here K denotes a compact subgroup of the centralizer of $\mathbf{X}$ and $\mathbf{X}/K$ is the factor system determined by the σ-algebra of K-invariant sets.

To make use of Proposition 3 we need some handle on an arbitrary factor of the Gaussian $\mathbf{Y}$. The fact that $\mathbf{Y}$ is Gaussian is expressed by the existence of an S-invariant Gaussian subspace $\mathcal{H}$ of $L^2_{\mathbb{R}}(\nu)$ generating $\mathcal{C}$, together with a unitary operator U on $\mathcal{H}$ such that $S = S(U)$ is the unique measure-preserving automorphism of $(Y,\mathcal{C},\nu)$ such that $f \circ S = Uf$ for all $f \in \mathcal{H}$. We will refer to U as the *unitary* underlying S. We will always assume that U has continuous spectrum so that S is weakly mixing. If E is a U-invariant subspace of $\mathcal{H}$ then the σ-algebra $\Sigma(E)$ generated by E defines a factor of Y which is again a Gaussian system. We denote this factor system by $\mathbf{Y}_E$ and we will refer to such factors as *natural factors*.

If V is any unitary commuting with U then the Gaussian automorphism $S(V)$ commutes with $S = S(U)$. Thus if L is any compact group of unitaries commuting with U we may view it as a compact subgroup of the centralizer of the system $\mathbf{Y}$ and so we can form the factor system $\mathbf{Y}/L$. We will refer to such factors as *compact factors*. Since any natural factor $\mathbf{Y}_E$ is again Gaussian we can combine these two constructions to form $\mathbf{Y}_E/L$ whenever L is a compact group of unitaries commuting with $U|_E$. In this way we get a large supply of factors of $\mathbf{Y}$ which we will refer to as *classical factors*. In the special case when $\mathbf{Y}$ is a Kronecker Gaussian system (see §4 for the definition) Thouvenot has shown that every factor of $\mathbf{Y}$ is classical (see [Th3], [Th2]). In the infinite entropy case there are many other factors. [I,L,R,S] contains an example of a zero-entropy Gaussian with a nonclassical factor but it seems to be a difficult problem to describe nonclassical factors in the zero-entropy case. This problem and the problem of disjointness in the class of Gaussian systems is studied in [L,P,Th].

We now use the observation that the centralizer of $\mathbf{Y}$ contains a Kronecker Gaussian (indeed, all Gaussians), which leads to the following result.

PROPOSITION 4. *Any factor of* $\mathbf{Y}$ *has a countable self-joining which is isomorphic to a classical factor of* $\mathbf{Y}$, *and hence to a compact factor of a Gaussian.*

If we use Proposition 4 and the fact that, by simplicity, a countable self-joining of $(\mathbf{X}/K)^{n\odot}$ is isomorphic to a factor of $\mathbf{X}^{\mathbb{N}}$, Proposition 3 may now be recast in the following form.

PROPOSITION 5. *No factor of* $\mathbf{X}^{\mathbb{N}}$ *can be isomorphic to a compact factor of a Gaussian.*

We first consider the following special case of Proposition 5.

PROPOSITION 6. $\mathbf{X}^{\mathbb{N}}$ *cannot be isomorphic to a Gaussian.*

Proposition 6 may be proved using the notion of (infinite) divisibility which has been studied by Katok and Thouvenot. A system $\mathbf{Y}$ is *divisible* if it splits as a direct product of two of its factors, each of which in turn splits, and so on ad infinitum, in such a way that all the decreasing sequences of σ-algebras which arise in this way have trivial intersection. Using natural factors it is easy to see that Gaussians are divisible. It is also easy to see that $\mathbf{X}^{\mathbb{N}}$ is not divisible.

However, a compact factor of a Gaussian will not be divisible so we introduce the notion of virtual divisibility, which is, roughly speaking, divisibility up to distal extension. We then show that any compact factor of a Gaussian enjoys the property of virtual divisibility but that no factor of $\mathbf{X}^{\mathbb{N}}$ does. This concludes our overview of the proof of Theorem 1.

2. Some notation and background material. If $\mathbf{X} = (X,\mathcal{B},\mu,T)$ is a system we will sometimes specify it by the σ-algebra $\mathcal{B}$ alone, or by the map T alone, when the other data are clear from the context. Thus, for example, if $\mathcal{A}_1 \supset \mathcal{A}_2$ are factor algebras of $\mathbf{X}$, that is, T-invariant sub-σ-algebras of $\mathcal{B}$, we may speak of the extension $\mathcal{A}_1 \to \mathcal{A}_2$ by which we mean the extension $(X,\mathcal{A}_1,\mu,T) \to (X,\mathcal{A}_2,\mu,T)$. The *centralizer* of a system $\mathbf{X}$, denoted by $C(\mathbf{X})$, consists of all automorphisms of $(X,\mathcal{B},\mu)$ which commute with T a.e. Whenever K is a compact subgroup of $C(\mathbf{X})$, then $\mathcal{I}(K)$ will denote the factor algebra of K-invariant sets.

We assume that the reader is familiar with the basic facts about joinings, disjointness and simple systems (see for example [J,R]). We will need the following result about factors $\mathcal{A}$ of a countable Cartesian product $\mathbf{X}^{\mathbb{N}}$ with $\mathbf{X}$ simple. It is a special case of the theory of semisimplicity developed in [J,L,M]. One can also prove it directly using Veech's characterization of group extensions ([J,R], Theorem 1.8.2). For any $I \subset \mathbb{N}$ we let $\mathcal{B}^I$ denote the factor algebra generated by the ith co-ordinate projections $\pi_i : (X^{\mathbb{N}},\mathcal{B}^{\mathbb{N}}) \to (X,\mathcal{B})$, $i \in I$. If $\mathcal{A}$ is any factor of $\mathbf{X}^{\mathbb{N}}$ we let $\widehat{\mathcal{A}}$ denote the smallest $\mathcal{B}^I$ containing $\mathcal{A}$.

PROPOSITION 2.1. *If* $\mathbf{X}$ *is simple and* $\mathcal{A}$ *is a factor algebra of* $\mathbf{X}^{\mathbb{N}}$ *then* $\widehat{\mathcal{A}} \to \mathcal{A}$ *is a group extension. Moreover, for each* $S \in C(\mathbf{X}^{\mathbb{N}})$ *we have* $S(\widehat{\mathcal{A}}) = \widehat{S\mathcal{A}}$.

We will need some facts about extensions of various kinds. We assume that the reader is familiar with the notions of isometric, distal and weakly mixing extensions. In the following proposition, $\mathcal{A},\mathcal{B},\mathcal{C}$ etc. denote factor algebras of a given arbitrary system $\mathbf{X}$.

PROPOSITION 2.2. (i) *If $\mathcal{A} \to \mathcal{C}$ is weakly mixing and $\mathcal{B} \subset \mathcal{A}$ then $\mathcal{B} \to \mathcal{B} \cap \mathcal{C}$ is also weakly mixing.*

(ii) *If $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ and $\mathcal{A} \to \mathcal{C}$ is distal then $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{C}$ are both distal.*

(iii) *If $\mathcal{C}_0$ and $\mathcal{C}_1$ are independent and $\mathcal{C}_i \to \mathcal{A}_i$ are group extensions for $i = 0, 1$ then $\mathcal{C}_0 \vee \mathcal{C}_1 \to \mathcal{A}_0 \vee \mathcal{A}_1$ is also a group extension.*

Proof. (i) can be found in [J,L,M], (ii) is standard and (iii) is easy. ∎

We will be dealing with a unitary operator U on a real Hilbert space $\mathcal{H}$. When $\mathcal{H}$ needs to be identified we write $(U, \mathcal{H})$. We let $C(U)$ denote the unitary centralizer of U, that is, it consists of all unitaries $V : \mathcal{H} \to \mathcal{H}$ such that $UV = VU$. We say that U has *continuous spectrum* if its complexification has no eigenvalues. We say that U has *simple spectrum* if it has a cyclic vector.

We will need a formulation of the spectral theorem for real unitaries. A finite measure σ on the circle $\mathbb{T}$ is called *symmetric* if it is invariant under conjugation, and a complex-valued function f is called *(hermitian) symmetric* if $\overline{f(z)} = f(\overline{z})$. We let $L^2_{\mathrm{h}}(\sigma)$ denote the real subspace of $L^2_{\mathbb{C}}(\sigma)$ consisting of symmetric functions. If f is a bounded symmetric function on $\mathbb{T}$ then M_f denotes the operator of multiplication by f on $L^2_{\mathrm{h}}(\sigma)$.

PROPOSITION 2.3. *Suppose U is a real unitary operator with simple spectrum. Then there exists a symmetric σ such that U is unitarily equivalent to $(M_z, L^2_{\mathrm{h}}(\sigma))$. Moreover, any two bounded operators commuting with U must commute with each other.*

The following result actually holds without the assumption of simple spectrum but this is all we shall need.

PROPOSITION 2.4. *Suppose U and V are unitary operators with continuous spectrum on separable real Hilbert spaces and that V has simple spectrum. Then there is a $V' \in C(U)$ which is unitarily equivalent to V.*

Proof. We may assume that V is $(M_z, L^2_{\mathrm{h}}(\tau))$ for some symmetric τ. We first deal with the case when U has simple spectrum as well. Then U is $(M_z, L^2_{\mathrm{h}}(\sigma))$ for some symmetric σ. Both σ and τ must be nonatomic measures by the assumption of continuous spectrum. By changing σ to an equivalent symmetric measure we may assume that $\sigma(\mathbb{T}) = \tau(\mathbb{T})$. Then we can find a measure-preserving isomorphism f between the nonatomic Lebesgue spaces $(\mathbb{T}, \sigma)$ and $(\mathbb{T}, \tau)$ such that $f(\overline{z}) = \overline{f(z)}$. Clearly, $(M_f, L^2_{\mathrm{h}}(\sigma))$ and $(M_z, L^2_{\mathrm{h}}(\tau))$ are unitarily equivalent so M_f is the desired element in $C(U)$.

The general case follows easily by decomposing the space $\mathcal{H}$ on which U acts as an orthogonal direct sum of cyclic subspaces. ∎

3. Divisibility. A *splitting* of an arbitrary system Y is a pair of factor algebras $(\mathcal{C}_0, \mathcal{C}_1)$ which are independent and together generate $\mathcal{C}$. We say that $\mathbf{Y}$ is *(infinitely) divisible* if it has a system of factor algebras

$$\left\{\mathcal{C}_\delta : \delta \in \bigcup_{n=0}^{\infty} \{0,1\}^n\right\}$$

(we adopt the convention that $\{0,1\}^0 = \{\emptyset\}$ consists of the "empty sequence") such that

$$\mathcal{C}_\emptyset = \mathcal{C}, \quad (\mathcal{C}_{\delta 0}, \mathcal{C}_{\delta 1}) \text{ is a splitting of } \mathcal{C}_\delta \text{ for all } \delta,$$

and

$$\bigcap_n \mathcal{C}_{x_1,\ldots,x_n} \text{ is trivial for all } x \in \{0,1\}^{\mathbb{N}}.$$

This notion has been studied by Katok and Thouvenot (private communication), who have constructed an example of a divisible system with discrete spectrum and divisible rank-1 smooth systems which are weakly mixing. The following result is well known. We include a proof for the reader's convenience and to facilitate the proof of Theorem 3.2.

THEOREM 3.1. *Any Gaussian system is divisible.*

Proof. Choose a system $\{A_\delta : \delta \in \bigcup_n \{0,1\}^n\}$ of Borel subsets of $\mathbb{T}$ such that $A_\emptyset = \mathbb{T}$, $\{A_{\delta 0}, A_{\delta 1}\}$ is a partition of A_δ for all δ, and for each $x \in \{0,1\}^{\mathbb{N}}$, $\bigcap_n A_{x_0,x_1,\ldots x_n}$ is either empty or a singleton. Let E_δ denote the spectral subspace of U corresponding to A_δ and define $\mathcal{C}_\delta = \Sigma(E_\delta)$.

Since U has continuous spectrum each decreasing intersection of E_δ's is trivial. It follows easily (by the zero-one law) that each decreasing intersection of $\mathcal{C}_\delta$'s is also trivial. Thus the $\mathcal{C}_\delta$ form a system of factors satisfying the definition of divisibility. ∎

To handle compact factors of a Gaussian we introduce the following weaker notion, in which the two factors required for a splitting must still be independent but need not generate, rather they must generate a factor which is within a distal extension of the full σ-algebra. Here is the formal definition.

A *virtual splitting* of $\mathbf{Y}$ is a pair of factor algebras $(\mathcal{A}_0, \mathcal{A}_1)$ such that

(i) $\mathcal{A}_0 \perp \mathcal{A}_1$,
(ii) the extension $\mathcal{A} \to \mathcal{A}_0 \vee \mathcal{A}_1$ is distal,
(iii) the extensions $\mathcal{A} \to \mathcal{A}_0$ and $\mathcal{A} \to \mathcal{A}_1$ are weakly mixing.

(Note that the last condition is automatic in the case of a genuine splitting.) We say that $\mathbf{Y}$ is *virtually divisible* if it has a system of factor algebras $\{\mathcal{A}_\delta : \delta \in \bigcup_n \{0,1\}^n\}$ just as in the definition of divisibility except that now $(\mathcal{A}_{\delta 0}, \mathcal{A}_{\delta 1})$ is only required to be a virtual splitting of $\mathcal{A}_\delta$ for each δ.

THEOREM 3.2. *Any compact factor of a Gaussian system* $\mathbf{Y}$ *is virtually divisible.*

Proof. Let $L \subset C(U)$ be the compact subgroup determining the factor, so the factor algebra is $\mathcal{A} = \mathcal{I}(L)$. Let $C_\delta = \Sigma(E_\delta)$ be factor algebras of $\mathbf{Y}$ as defined in the proof of Theorem 3.1. Recall that the E_δ are spectral subspaces of U and hence are invariant not only under U, but also under each $V \in C(U)$.

Now let

$$\mathcal{A}_\delta = \mathcal{C}_\delta \cap \mathcal{A} = \{C \in \mathcal{C}_\delta : lC = C \text{ for all } l \in L\}.$$

If $l \in L$ and $l = S(V)$ with $V \in C(U)$, then $V(E_\delta) = E_\delta$, which implies that $l(\mathcal{C}_\delta) = \mathcal{C}_\delta$. Thus the restrictions of the l's in L to $\mathcal{C}_\delta$ form a compact subgroup L_δ of the centralizer of the factor system $\mathcal{C}_\delta$ and $\mathcal{A}_\delta$ is the fixed algebra of L_δ. This means $\mathcal{C}_\delta \to \mathcal{A}_\delta$ is a group extension.

Fixing δ we have $\mathcal{C}_{\delta 0} \perp \mathcal{C}_{\delta 1}$ and $\mathcal{C}_{\delta 0} \to \mathcal{A}_{\delta 0}$ and $\mathcal{C}_{\delta 1} \to \mathcal{A}_{\delta 1}$ are both group extensions. It follows that $\mathcal{C}_\delta \to \mathcal{A}_{\delta 0} \vee \mathcal{A}_{\delta 1}$ is a group extension, so $\mathcal{A}_\delta \to \mathcal{A}_{\delta 0} \vee \mathcal{A}_{\delta 1}$ is distal (in fact isometric). Finally, $\mathcal{C}_\delta \to \mathcal{C}_{\delta i}$ is a weakly mixing extension so $\mathcal{A}_\delta = \mathcal{A} \cap \mathcal{C}_\delta \to \mathcal{A} \cap \mathcal{C}_{\delta i} = \mathcal{A}_{\delta i}$ is weakly mixing for $i = 0, 1$. ∎

THEOREM 3.3. *If* $\mathbf{X}$ *is simple then no factor* $\mathcal{A}$ *of the countable Cartesian product* $\mathbf{X}^{\mathbb{N}}$ *is virtually divisible.*

For the proof of Theorem 3.3 we will use the following lemma. Recall the notation of Proposition 2.1.

LEMMA 3.4. *Suppose that* $\widehat{\mathcal{A}} = \mathcal{B}^{\mathbb{N}}$, *so* $\mathcal{A} = \mathcal{I}(K)$ *with* K *a compact subgroup of* $C(\mathbf{X}^{\mathbb{N}})$. *Suppose further that* $(\mathcal{A}_0, \mathcal{A}_1)$ *is a virtual splitting of* $\mathcal{A}$ *such that* $\widehat{\mathcal{A}}_i = \mathcal{B}^{I_i}$. *Then* $\{I_0, I_1\}$ *is a partition of* $\mathbb{N}$ *and* $k(\mathcal{B}^{I_i}) = \mathcal{B}^{I_i}$ *for all* $k \in K$. *If* K_i *denotes the restriction of* K *to* $\mathcal{B}^{I_i}$ *then* $\mathcal{A}_i = \mathcal{I}(K_i) = \mathcal{A} \cap \mathcal{B}^{I_i}$.

Proof. Since $\mathcal{A}_0$ and $\mathcal{A}_1$ are independent and $\widehat{\mathcal{A}}_i \to \mathcal{A}_i$ are group extensions, we can conclude that $\widehat{\mathcal{A}}_0$ and $\widehat{\mathcal{A}}_1$ are independent as well, by [J,R], Theorem 5.1. This means that I_0 and I_1 are disjoint. Since $\mathcal{B}^{\mathbb{N}} \to \mathcal{A}$ is a group extension and $\mathcal{A} \to \mathcal{A}_0 \vee \mathcal{A}_1$ is distal, $\mathcal{B}^{\mathbb{N}} \to \mathcal{A}_0 \vee \mathcal{A}_1$ is again distal. It follows that $\mathcal{B}^{\mathbb{N}} \to \mathcal{B}^{I_0} \vee \mathcal{B}^{I_1}$ is also a distal extension, which certainly implies that $I_0 \cup I_1 = \mathbb{N}$. Since each $k \in K$ fixes the algebra $\mathcal{A}_i$ (in fact, fixes it setwise), k also fixes $\widehat{\mathcal{A}}_i$ (but no longer setwise).

Finally, to see that $\mathcal{B}^{I_i} \cap \mathcal{A} = \mathcal{A}_i$ note first that $\mathcal{A}_i \subset \mathcal{B}^{I_i} \cap \mathcal{A}$. Since $\mathcal{B}^{I_i} \to \mathcal{A}_i$ is a group extension, $\mathcal{B}^{I_i} \cap \mathcal{A} \to \mathcal{A}_i$ is an isometric extension. On the other hand, $\mathcal{A} \to \mathcal{A}_i$ is a weakly mixing extension so $\mathcal{B}^{I_i} \cap \mathcal{A} \to \mathcal{A}_i$ is also weakly mixing. Thus $\mathcal{B}^{I_i} \cap \mathcal{A} \to \mathcal{A}_i$ is both isometric and weakly mixing, which forces it to be trivial. ∎

Proof of Theorem 3.3. There is no harm in assuming, as in Lemma 3.4, that $\widehat{\mathcal{A}} = \mathcal{B}^{\mathbb{N}}$. Suppose we had a system of factors $\mathcal{A}_\delta$ as in the definition of virtual divisibility. Applying Lemma 3.4 to the initial splitting $(\mathcal{A}_0, \mathcal{A}_1)$ we obtain compact subgroups K_i of $C(\mathcal{B}^{I_i})$ such that $\mathcal{A}_i = \mathcal{I}(K_i)$. This means that Lemma 3.4 can be applied again to the splittings of $\mathcal{A}_0$ and $\mathcal{A}_1$. Proceeding inductively we obtain subsets I_δ, $\delta \in \bigcup_n \{0,1\}^n$, such that $\{I_{\delta 0}, I_{\delta 1}\}$ is a partition of I_δ for each δ, K fixes each $\mathcal{B}^{I_\delta}$ and $\mathcal{A}_\delta = \mathcal{A} \cap \mathcal{B}^{I_\delta}$.

Now choose an $x \in \bigcup_n \{0,1\}^n$ such that, if we set $\delta_n = (x_1, \ldots, x_n)$, then

$$J = \bigcap_n I_{\delta_n} \neq \emptyset.$$

(In other words, some decreasing intersection of the I_δ's must contain a point of $\mathbb{N}$.) Since each $k \in K$ fixes each $\mathcal{B}^{I_{\delta_n}}$, it fixes $\mathcal{B}^J$ as well and the restriction K_∞ of K to $\mathcal{B}^J$ is thus a compact subgroup of $C(\mathcal{B}^J)$. It follows that the nontrivial factor $\mathcal{I}(K_\infty)$ is contained in $\mathcal{A}_{\delta_n}$ for all n, contradicting triviality of the intersection of these algebras. ∎

4. Kronecker Gaussians and proof of Theorem 1. The $\mathbf{Y}$ is called a *Kronecker Gaussian* if its underlying unitary U has simple spectrum and its spectral type is a measure supported on a subset $E \cup \overline{E}$ where E is a Kronecker subset of the upper half-circle. We recall Thouvenot's result that every factor of a Kronecker Gaussian is classical. Although Kronecker Gaussians are very special we can use Thouvenot's result to obtain some weak but nonetheless useful information about factors of an arbitrary Gaussian.

THEOREM 4.1. *If* $\mathbf{Y}$ *is Gaussian and* $\mathbf{Z}$ *is a factor of* $\mathbf{Y}$ *then* $\mathbf{Z}$ *has a countable ergodic self-joining which is isomorphic to a classical factor of* $\mathbf{Y}$.

Proof. Let $(U, \mathcal{H})$ denote the underlying unitary of $\mathbf{Y}$. Suppose the factor is given by a factor algebra $\mathcal{A} \subset \mathcal{C}$. By Proposition 2.4 we can find a Kronecker Gaussian $R = S(V)$ with $V \in C(U)$. Now let

$$\widehat{\mathcal{A}} = \bigvee_{i \in \mathbb{Z}} R^i(\mathcal{A}),$$

a σ-algebra which is both S- and R-invariant. Since the action of S on each $R^i(\mathcal{A})$ is isomorphic to its action on $\mathcal{A}$, the system

$$\widehat{\mathbf{Z}} = (Y, \widehat{\mathcal{A}}, \nu, S)$$

is a countable self-joining of $\mathbf{Z}$. Since $\widehat{\mathcal{A}}$ is R-invariant, by Thouvenot's result $\widehat{\mathbf{Z}}$ must be of the form

$$\widehat{\mathbf{Z}} = \mathcal{C}_E / L$$

for some V-invariant subspace E and some compact subgroup L of $C(V|_E)$.

Since V has simple spectrum and U commutes with V, E must be U-invariant as well as V-invariant (Proposition 2.3). Since $V|_E$ again has simple spectrum, a similar argument shows that $L \subset C(U|_E)$, so we are done. ∎

We now have all the tools needed for the proof of Theorem 1. As observed in §1, if $\mathbf{X}$ and $\mathbf{Y}$ are not disjoint then $\mathbf{Y}$ has some $(\mathbf{X}/K)^{n\odot}$ as a factor. By Theorem 4.1 it follows that $(\mathbf{X}/K)^{n\odot}$ has a countable ergodic self-joining $\mathbf{Z}$ which is a classical factor of $\mathbf{Y}$. A classical factor of $\mathbf{Y}$ is a compact factor of some Gaussian factor of $\mathbf{Y}$, so it is virtually divisible by Theorem 3.2.

On the other hand, since $(\mathbf{X}/K)^{n\odot}$ is a factor of $\mathbf{X}^n$, a countable ergodic self-joining of $(\mathbf{X}/K)^{n\odot}$ lifts to a countable ergodic self-joining of $\mathbf{X}^n$, which must be isomorphic to $\mathbf{X}^{\mathbb{N}}$, by simplicity of $\mathbf{X}$. This means that $\mathbf{Z}$ is isomorphic to a factor of $\mathbf{X}^{\mathbb{N}}$ so Theorem 3.3 tells us that $\mathbf{Z}$ cannot be virtually divisible. This contradiction completes the proof of Theorem 1. ∎

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A theorem on isotropic spaces

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Abstract. Let X be a normed space and $G_F(X)$ the group of all linear surjective isometries of X that are finite-dimensional perturbations of the identity. We prove that if $G_F(X)$ acts transitively on the unit sphere then X must be an inner product space.

1. Introduction and statement of the result. During the thirties some people studied *isotropic* spaces. These are normed spaces in which the group of linear surjective isometries acts transitively on the unit sphere. Clearly, inner product spaces are isotropic. That the converse is also true for finite-dimensional spaces was proved by S. Mazur [7] in 1938 (see also [2]):

THEOREM 1. *Isotropic finite-dimensional normed spaces are euclidean (in the sense that the norm comes induced by an inner product).*

There are, however, isotropic normed spaces that are not isomorphic to inner product spaces (this was discovered in the sixties by A. Pełczyński and S. Rolewicz [8]): for instance, if μ is a homogeneous non-σ-finite measure, the space $L_p(\mu)$ is isotropic for every finite p (see also [6]). These examples are necessarily non-separable. Also, isotropic separable normed (not complete) non-euclidean spaces are known: for example, the subspace of all functions in $L_p(-\infty,\infty)$ having bounded support. In spite of these examples the Mazur problem on the existence of a separable isotropic Banach space which is not a Hilbert space remains open [3]. (A recent survey on this problem and related topics is [4], which contains an extensive bibliography.) In this note, we generalize Mazur's result replacing the hypothesis on the dimension of the space by a weaker one concerning the structure of the isometry group.

So, let X be a (real or complex) normed space with unit sphere $S(X)$. We denote by $G(X)$ the group of all isometric automorphisms of X. An operator $T : X \to X$ is said to be a *finite-dimensional perturbation* of the identity if the difference $T - \mathrm{Id}$ is a finite rank operator. If we write

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