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Local Hardy spaces on Chébli–Trimèche hypergroups

by

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Abstract. We investigate the local Hardy spaces $\mathbf{h}^p$ on Chébli–Trimèche hypergroups, and establish the equivalence of various characterizations of these in terms of maximal functions and atomic decomposition.

In this paper we continue the study of Hardy spaces on Chébli–Trimèche hypergroups of exponential growth begun in [BX2], devoting our attention to a study of the local Hardy spaces $\mathbf{h}^p$.

The theory of Hardy spaces became important in the study of harmonic analysis on euclidean spaces and other homogeneous spaces through its better understanding of related topics such as singular integrals, multiplier operators, maximal functions and, more generally, real-variable methods (see [FeS], [Coi], [Lat], [CW], [MS] and [FoS]). Hardy spaces H^p can be regarded as good "substitutes" for L^p $(0 < p < 1)$. Indeed, while the L^p spaces for $0 < p < 1$ are quite pathological, the corresponding H^p spaces enjoy many of the properties of L^p for $p > 1$. In addition, as would be expected, $H^p = L^p$ for $p > 1$. However, for $0 < p < 1$ this comparison breaks down in some aspects: H^p does not contain the Schwartz class of rapidly decreasing test functions, and pseudo-differential operators are not bounded on H^p. This deficiency can be overcome through the use of local Hardy spaces $\mathbf{h}^p$, which were introduced in [G]. The spaces $\mathbf{h}^p$ can also be identified with L^p when $p > 1$, they contain the Schwartz class, and any smooth quasi-homogeneous multiplier is bounded in $\mathbf{h}^p$. While H^p sits well within Fourier analysis, the $\mathbf{h}^p$ theory is more suited to problems associated with partial differential equations (see [G], [Cha] and [PS]).

The moment condition of an H^p function plays an essential role in the theory of Hardy spaces. In fact, it is cancellation that makes various maximal functions integrable. However, this property does not remain significant in the setting of exponential volume growth where the radial maximal

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function of an atom may not be integrable (a counterexample is given in [BX3]). There is some work on local Hardy spaces in [K1] and [K2] for a noncompact symmetric space of rank 1 which is of exponential growth, and it was proposed as an open problem in [K2] whether the characterizations of $\mathbf{h}^p$ in terms of maximal functions and atomic decomposition are equivalent. In this paper we investigate $\mathbf{h}^p$ on Chébli–Trimèche hypergroups of exponential growth, and establish the equivalence of the real Hardy spaces and the atomic Hardy spaces. The main problems arise from the difficulty in handling the generalized translation and also the exponential growth of the underlying hypergroup. Some natural properties of the translation are largely unavailable, and the Haar measure does not satisfy the doubling condition enjoyed by euclidean spaces or homogeneous spaces, both of which are of polynomial growth.

Our paper is organized as follows. In §1 we give a short introduction to some basic results of harmonic analysis on Chébli–Trimèche hypergroups, and §2 is devoted to some estimates for the characters. Various maximal operators are introduced and investigated in §3. Finally, in §4 we define the local Hardy spaces in terms of maximal functions, and establish the identification of the real Hardy spaces with the atomic Hardy spaces.

1. Harmonic analysis on Chébli–Trimèche hypergroups. Let $(\mathbb{R}_+, *(A))$ denote the Chébli–Trimèche hypergroup associated with a function A that is continuous on $\mathbb{R}_+$, twice continuously differentiable on $\mathbb{R}_+^* =]0,\infty[$, and satisfies the following conditions (see [Z] and, for general details on hypergroups, [BH]):

(1.1) $A(0) = 0$ and $A(x) > 0$ for $x > 0$;

(1.2) A is increasing and unbounded;

(1.3) $A'(x)/A(x) = (2\alpha+1)/x + B(x)$ on a neighbourhood of 0 where $\alpha > -1/2$ and B is an odd C^∞-function on $\mathbb{R}$;

(1.4) $A'(x)/A(x)$ is a decreasing C^∞-function on $\mathbb{R}_+^*$, and hence the following limit exists:

$$\varrho := \frac{1}{2}\lim_{x\to+\infty}\frac{A'(x)}{A(x)} \ge 0.$$

The hypergroup $(\mathbb{R}_+, *(A))$ is noncompact and commutative with neutral element 0 and the identity mapping as the involution. The Haar measure on $(\mathbb{R}_+, *(A))$ is given by $m := A\lambda_{\mathbb{R}_+}$ where $\lambda_{\mathbb{R}_+}$ is the Lebesgue measure on $\mathbb{R}_+$. The growth of the hypergroup is determined by the number ϱ in (1.4). If $\varrho > 0$ then (1.4) implies that $A(x) \ge A(1)e^{2\varrho(x-1)}$ for $x \ge 1$ and so the hypergroup is of *exponential growth*. Otherwise we say that the hypergroup is of *subexponential growth*. In this paper we restrict ourselves to Chébli–Trimèche hypergroups of exponential growth.

Let $L = L_A$ be the differential operator defined for $x > 0$ by

$$Lf(x) = -f''(x) - \frac{A'(x)}{A(x)}f'(x) \tag{1.5}$$

for each function f twice differentiable on $\mathbb{R}_+^*$. The multiplicative functions on $(\mathbb{R}_+, *(A))$ coincide with all the solutions φ_λ ($\lambda \in \mathbb{C}$) of the differential equation

$$L\varphi_\lambda(x) = (\lambda^2+\varrho^2)\varphi_\lambda(x), \quad \varphi_\lambda(0) = 1, \quad \varphi_\lambda'(0) = 0, \tag{1.6}$$

and the dual space $\mathbb{R}_+^\wedge$ can be identified with the parameter set $\mathbb{R}_+ \cup i[0,\varrho]$.

For $0 < p \le \infty$ the Lebesgue space $L^p(\mathbb{R}_+, A dx)$ is defined as usual, and we denote by $\|f\|_{p,A}$ the L^p-norm of $f \in L^p(\mathbb{R}_+, A dx)$. For $f \in L^1(\mathbb{R}_+, A dx)$ the Fourier transform of f is given by

$$\widehat{f}(\lambda) = \int_{\mathbb{R}_+} f(x)\varphi_\lambda(x)A(x)\,dx.$$

1.7. Theorem (Levitan–Plancherel; see [BH, Theorem 2.2.13]). *There exists a unique nonnegative measure σ on $\mathbb{R}_+^\wedge$ with support $[\varrho^2,\infty[$ such that the Fourier transform induces an isometric isomorphism from $L^2(\mathbb{R}_+, A dx)$ onto $L^2(\mathbb{R}_+^\wedge, \sigma)$, and for any $f \in L^1(\mathbb{R}_+, A dx) \cap L^2(\mathbb{R}_+, A dx)$,*

$$\int_{\mathbb{R}_+} |f(x)|^2 A(x)\,dx = \int_{\mathbb{R}_+^\wedge} |\widehat{f}(\lambda)|^2\,\sigma(d\lambda).$$

To determine the Plancherel measure σ we must place a further restriction on A. A function f is said to satisfy *condition* (H) if for some $a > 0$, f can be expressed as

$$f(x) = \frac{a^2 - 1/4}{x^2} + \zeta(x)$$

for all large x where

$$\int_{x_0}^{\infty} x^{\gamma(a)}|\zeta(x)|\,dx < \infty$$

for some $x_0 > 0$ and $\zeta(x)$ is bounded for $x > x_0$; here $\gamma(a) = a + 1/2$ if $a \ge 1/2$ and $\gamma(a) = 1$ otherwise. For $x > 0$ we put

$$G(x) := \frac{1}{4}\left(\frac{A'(x)}{A(x)}\right)^2 + \frac{1}{2}\left(\frac{A'(x)}{A(x)}\right)' - \varrho^2.$$

1.8. Theorem (see [BX1, Proposition 3.17]). *Suppose that G satisfies condition* (H) *together with one of the following conditions*:

(i) $a > 1/2$;

(ii) $a \ne |\alpha|$;

(iii) $a = \alpha \le 1/2$ *and*

$$\int_0^\infty t^{1/2-\alpha}\zeta(t)\varphi_0(t)A(t)^{1/2}\,dt \neq -2\alpha\sqrt{M_A}$$

or

$$\int_0^\infty t^{\alpha+1/2}\zeta(t)\varphi_0(t)A(t)^{1/2}\,dt = 0$$

where

$$M_A := \lim_{x\to 0^+} x^{-2\alpha-1}A(x) \quad \text{and} \quad \zeta(x) = G(x) + \frac{1/4-a^2}{x^2}.$$

Then the Plancherel measure σ is absolutely continuous with respect to the Lebesgue measure and has density $|c(\lambda)|^{-2}$ where the function $c(\lambda)$ satisfies the following: There exist positive constants C_1, C_2, K such that for any $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) \le 0$,

$$C_1|\lambda|^{a+1/2} \le |c(\lambda)|^{-1} \le C_2|\lambda|^{a+1/2}, \quad |\lambda| \le K,\ a > 0,$$
$$C_1|\lambda|^{\alpha+1/2} \le |c(\lambda)|^{-1} \le C_2|\lambda|^{\alpha+1/2}, \quad |\lambda| > K.$$

In the sequel we assume that A satisfies the conditions of Theorem 1.8. In addition we assume that for each $k \in \mathbb{N}$, $(A'(x)/A(x))^{(k)}$ is bounded for large $x \in \mathbb{R}_+$. The following result can be found in [BX1, Lemmas 2.5 and 3.28].

1.9. LEMMA. *We have*

$$A(x) \sim x^{2\alpha+1}\ (x \to 0^+), \qquad A(x) \sim e^{2\varrho x}\ (x \to \infty).$$

Let ε_x be the unit point mass at $x \in \mathbb{R}_+$. For any $x, y \in \mathbb{R}_+$ the probability measure $\varepsilon_x * \varepsilon_y$ is m-absolutely continuous with

$$\mathrm{supp}(\varepsilon_x * \varepsilon_y) \subset [|x-y|, x+y]. \tag{1.10}$$

We denote by $T_x f$ the generalized translation of a function f by $x \in \mathbb{R}_+$ defined by

$$T_x f(y) := f(x*y) := \int_{\mathbb{R}_+} f(z)\,(\varepsilon_x * \varepsilon_y)(dz). \tag{1.11}$$

The convolution of two functions f and g is defined by

$$f * g(x) = \int_{\mathbb{R}_+} T_x f(y) g(y) A(y)\,dy. \tag{1.12}$$

Finally, we use C to denote a positive constant whose value may vary from line to line. Dependence of such constants upon parameters of interest will be indicated through the use of subscripts.

2. Some estimates for characters. In this section we establish some estimates for the derivatives of characters.

2.1. LEMMA (see [BX1, Lemmas 2.8 and 3.27]). *There exists a positive constant C_A such that*

$$|\varphi_\lambda(x)| \le \begin{cases} 1, & \lambda, x \in \mathbb{R}_+, \\ C_A x A(x)^{-1/2}, & \lambda \in \mathbb{R}_+,\ x > 1, \\ C_A A(x)^{-1/2}(\lambda x)^{1/2-a}|c(\lambda)|, & \lambda x \le 1,\ x > 1, \\ C_A A(x)^{-1/2}|c(\lambda)|, & \lambda x > 1,\ x > 1. \end{cases}$$

2.2. LEMMA. *There exist $K_1, K_2 > 0$ such that for $\lambda \in \mathbb{C}$, $|\lambda| > K_1$,*

$$\varphi_\lambda(x) = \begin{cases} C_A A(x)^{-1/2} x^{\alpha+1/2}(j_\alpha(\lambda x) + O(\lambda x)), & |\lambda x| \le K_2, \\ C_A A(x)^{-1/2}\lambda^{-(\alpha+1/2)}(c_1 e^{-i\lambda x} + c_2 e^{i\lambda x}) & \\ \quad \times(1 + O(\lambda^{-1}) + O((\lambda x)^{-1})), & |\lambda x| > K_2, \end{cases}$$

and

$$\varphi'_\lambda(x) = \begin{cases} -\dfrac{1}{2}\dfrac{A'(x)}{A(x)}\varphi_\lambda(x) & \\ \quad + C_A(\alpha)A(x)^{-1/2}(x^{\alpha-1/2} + x^{\alpha+3/2})O(1), & |\lambda x| \le K_2, \\ -\dfrac{1}{2}\dfrac{A'(x)}{A(x)}\varphi_\lambda(x) + iC_A(\alpha)\lambda^{-\alpha+1/2}A(x)^{-1/2}(-c_1 e^{-i\lambda x} + c_2 e^{i\lambda x}) & \\ \quad \times(1 + O(\lambda^{-1}) + O((\lambda x)^{-1})), & |\lambda x| > K_2, \end{cases}$$

where $j_\alpha(z) := 2^\alpha \Gamma(\alpha+1) z^{-\alpha} J_\alpha(z)$ and J_α is the Bessel function of order α, and c_1, c_2 are constants which can be determined explicitly.

Proof. For each $\lambda \in \mathbb{C}$ consider the differential equation

$$Lu = (\lambda^2 + \varrho^2)u,$$

which becomes

$$v''(x) = \left(\chi(x) + \frac{\alpha^2 - 1/4}{x^2} - \lambda^2\right)v(x) \tag{2.3}$$

under the transform

$$v(x) = \sqrt{A(x)}\,u(x)$$

where

$$\chi(x) = \frac{1}{4}\left(\frac{A'(x)}{A(x)}\right)^2 + \frac{1}{2}\left(\frac{A'(x)}{A(x)}\right)' - \varrho^2 + \frac{1/4-\alpha}{x^2}.$$

Let $t = \frac{1}{2}x^2$ and $w = \sqrt{x}v$. Then

$$w''(t) + \left(\frac{\lambda^2}{2t} + \frac{1/4 - \alpha^2/4}{t^2} + \frac{\chi(0)}{2t} + \frac{\chi(\sqrt{2t}) - \chi(0)}{2t}\right)w = 0.$$

Note that $C_1(t) := (\chi(\sqrt{2t}) - \chi(0))/2t \in C^\infty(\mathbb{R}_+^*)$ and $C_1(t) \in C(\mathbb{R}_+)$. Therefore by Theorem 1 in [Lan], (2.3) has a solution $v = v(x, \lambda)$ with the property that there exist $K_1, K_2 > 0$ such that for $\lambda \in \mathbb{C}$, $|\lambda| > K_1$,

$$v(x,\lambda) = \begin{cases} (\lambda x)^{\alpha+1/2}(j_\alpha(\lambda x) + O(\lambda^2 x^2)), & |\lambda x| \le K_2, \\ (c_1 e^{-i\lambda x} + c_2 e^{i\lambda x})(1 + O((\lambda x)^{-1}) + O(\lambda^{-1})), & |\lambda x| > K_2, \end{cases}$$

and

$$\frac{d}{dx}v(x,\lambda) = \begin{cases} \frac{d}{dx}((\lambda x)^{\alpha+1/2} j_\alpha(\lambda x)) + \frac{(\lambda x)^{\alpha+3/2}}{\lambda} O(1), & |\lambda x| \le K_2, \\ i\lambda(-c_1 e^{-i\lambda x} + c_2 e^{i\lambda x}) \\ \quad \times (1 + O((\lambda x)^{-1}) + O(\lambda^{-1})), & |\lambda x| \ge K_2, \end{cases}$$

where c_1, c_2 are constants which can be determined explicitly. Now

$$u(x,\lambda) = C_A(\alpha)\lambda^{-\alpha-1/2}, \quad u(0,\lambda) = 1, \quad \frac{d}{dx}u(x,\lambda) = O(x) \quad (x \to 0^+),$$

which implies $\varphi_\lambda(x) = u(x,\lambda)$, and the lemma follows. ∎

2.4. Lemma. *For each $k \in \mathbb{N}$ we have*

$$|\varphi_\lambda^{(k)}(x)| \le \begin{cases} C_A(1+\lambda)^k, & \lambda x \le 1,\ x \le 1, \\ C_A x A(x)^{-1/2}, & \lambda x \le 1,\ x > 1, \\ C_A A(x)^{-1/2}|c(\lambda)|(1+\lambda)^k, & \lambda x > 1. \end{cases}$$

We also have the following alternative estimate:

$$|\varphi_\lambda^{(k)}(x)| \le C_A A(x)^{-1/2}(\lambda x)^{1/2-a}|c(\lambda)|(1+\lambda)^k, \quad \lambda x \le 1,\ x > 1.$$

Proof. Appealing to (1.5) and (1.6) we have

$$\varphi_\lambda'(x) = -\frac{\lambda^2+\varrho^2}{A(x)} \int_0^x \varphi_\lambda(t)A(t)\,dt \tag{2.5}$$

and

$$\varphi_\lambda^{(k)}(x) = -\sum_{j=0}^{k-2} \binom{k-2}{j} \left(\frac{A'(x)}{A(x)}\right)^{(j)} \varphi_\lambda^{(k-1-j)}(x) - (\lambda^2+\varrho^2)\varphi_\lambda^{(k-2)}(x).$$

Therefore by induction we obtain, using Lemmas 1.9, 2.1 and 2.2 and Theorem 1.8 together with our assumption on the derivatives of $A'(x)/A(x)$

$$|\varphi_\lambda^{(k)}(x)| \le \begin{cases} C_A x A(x)^{-1/2}, & \lambda x \le 1,\ x > 1, \\ C_A A(x)^{-1/2}\lambda^{-\alpha-1/2}(1+\lambda)^k, & \lambda x > 1,\ \lambda > 1. \end{cases}$$

Now we consider the case when $\lambda x \le 1$ and $x \le 1$. For any $\beta > 0$ and differentiable function f we define

$$H_\beta(f)(x) := \frac{1}{x^\beta}\int_0^x f(u)u^{\beta-1}\,du.$$

Then integration by parts gives

$$H_\beta(f)^{(k)}(x) = H_{\beta+k}(f^{(k)})(x). \tag{2.6}$$

Let $C(x) = x^{-2\alpha-1}A(x)$. By (1.3), $C(x)$ is an even and positive C^∞-function for $0 \le x \le 1$. In view of (2.5) we have

$$\varphi_\lambda'(x) = -\frac{(\lambda^2+\varrho^2)x}{C(x)} H_{2\alpha+2}(f_\lambda)(x)$$

where $f_\lambda(x) = \varphi_\lambda(x)C(x)$. Hence applying Lemma 2.1, (1.2) and (2.6) we obtain by induction

$$|\varphi_\lambda^{(k)}(x)| \le C_A(1+\lambda)^k, \quad \lambda x \le 1,\ x \le 1.$$

In view of Theorem 1.8 it remains to show that for $x > 1$,

$$|\varphi_\lambda^{(k)}(x)| \le \begin{cases} C_A A(x)^{-1/2}(\lambda x)^{1/2-a}|c(\lambda)|(1+\lambda)^k, & \lambda x \le 1, \\ C_A A(x)^{-1/2}|c(\lambda)|(1+\lambda)^k, & \lambda x > 1. \end{cases} \tag{2.7}$$

We refer to the proof of [BX1, Lemma 3.4]. The differential equation $Lu = (\lambda^2+\varrho^2)u$ has two linearly independent solutions Φ_λ and $\Phi_{-\lambda}$ such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x > 0.$$

Thus we have

$$|\varphi_\lambda^{(k)}(x)| \le 2|c(\lambda)||\Phi_{-\lambda}^{(k)}(x)|.$$

Let $H_\nu^{(2)}$ be the second class of Hankel functions of order ν, and let $w_1(x,\lambda) = (\lambda x)^{1/2}H_a^{(2)}(\lambda x)$, $w_2(x,\lambda) = (\lambda x)^{1/2}J_a(\lambda x)$. Now $\Phi_{-\lambda}$ can be written as

$$\Phi_{-\lambda}(x) = A(x)^{-1/2}e^{-i\lambda x}W(x,\lambda)$$

where $W(x,\lambda)$ satisfies the integral equation

$$W(x,\lambda) = C(a)w_1(x,\lambda)e^{i\lambda x} + \int_x^\infty k(x,t,\lambda)W(t,\lambda)\,dt$$

and

$$k(x,t,\lambda) = \frac{\pi}{2i\lambda}\zeta(t)e^{i\lambda(x-t)} \times (w_1(t,\lambda)w_2(x,\lambda) - w_2(t,\lambda)w_1(x,\lambda)), \quad 0 < x \le t.$$

By applying the properties of the derivatives of the Bessel and Hankel functions and the method of successive approximation, (2.7) now follows readily. ∎

3. Maximal functions. The purpose of this section is to investigate various maximal functions, which will lead us to our definition of local Hardy spaces. After reviewing some facts concerning maximal functions on L^p spaces, $p > 1$, we turn to the grand maximal function and establish the relationship between it and the heat and radial maximal functions.

We begin with introducing Schwartz functions and distributions. For $0 < q \le 2$ the generalized Schwartz space $\mathcal{S}_q(\mathbb{R}_+, *(A))$ consists of the restrictions to $\mathbb{R}_+$ of all functions in $\mathcal{S}_q(\mathbb{R})$ where

$$\mathcal{S}_q(\mathbb{R}) := \{g \in C^\infty(\mathbb{R}) : g \text{ is even and } \mu^q_{k,l}(g) < \infty,\ k,l \in \mathbb{N}_0\}$$

and

$$\mu^q_{k,l}(g) := \sup_{x\in\mathbb{R}_+} (1+x)^l \varphi_0(x)^{-2/q} |g^{(k)}(x)|.$$

A *q-distribution* on $\mathbb{R}_+$ is a continuous linear functional on $\mathcal{S}_q(\mathbb{R}_+,*(A))$; the totality of q-distributions on $\mathbb{R}_+$ is denoted by $\mathcal{S}'_q(\mathbb{R}_+,*(A))$. For $u \in \mathcal{S}'_q(\mathbb{R}_+,*(A))$ and $\phi \in \mathcal{S}_q(\mathbb{R}_+,*(A))$ the *convolution* of u and ϕ is a q-distribution defined by

$$u * \phi(\psi) := u(\phi * \psi), \qquad \psi \in \mathcal{S}_q(\mathbb{R}_+,*(A)). \tag{3.1}$$

Let $f \in \mathcal{S}'_q(\mathbb{R}_+,*(A))$. The *heat maximal function* H^+f is defined by

$$H^+f(x) := \sup_{t>0} |f * h_t(x)|$$

where h_t is the heat kernel (see [AT]). For a reasonably well-behaved function ϕ the *radial maximal function* $M_\phi f$ is defined by

$$M_\phi f(x) := \sup_{t>0} |f * \phi_t(x)|$$

where

$$\phi_t(x) = \frac{A(x/t)}{tA(x)}\phi(x/t). \tag{3.2}$$

Denote by H_0^+f the *local heat maximal function* defined by

$$H_0^+f(x) := \sup_{0<t\le 1} |f * h_t(x)|.$$

The *local radial maximal function* $M_{\phi,0}f$ is defined similarly with $\sup_{0<t\le1}$ replacing $\sup_{t>0}$ in the definition of $M_\phi f$.

The L^p-behaviour of these maximal functions is investigated in [BX2]. For $s > 1$ and $n \in \mathbb{N}_0$ let $B_{s,n}(\mathbb{R}_+,*(A))$ denote the set of functions ϕ in $\mathcal{S}_1(\mathbb{R}_+,*(A))$ satisfying, for $k = 0,1,\ldots,n$,

$$|\phi^{(k)}(x)| \le C_{A,k}(1+m([0,x]))^{-s} \quad\text{and}\quad \int_0^\infty \phi(x)A(x)\,dx = 1. \tag{3.3}$$

We write $B_{s,n}(\mathbb{R}_+,*(A))$ as $B_s(\mathbb{R}_+,*(A))$ if $n = 0$. The starting point of Hardy space theory is the following version of the classical maximal theorem:

3.4. THEOREM (see [BX2, Theorem 3.11]). *Let $\phi \in B_s(\mathbb{R}_+,*(A))$. Then the maximal operators H^+ and M_ϕ are of weak type (L^1,L^1) and bounded on $L^p(\mathbb{R}_+,A dx)$ for $1 < p \le \infty$.*

The converse of the maximal theorem (for $p > 1$) is also true.

3.5. THEOREM (see [BX3, Corollary 2.13]). *Let $f \in \mathcal{S}'_q(\mathbb{R}_+,*(A))$ $(0 < q \le 2)$ and $1 < p < \infty$. If M is any of the maximal operators H^+, H_0^+, M_ϕ and $M_{\phi,0}$ where $\phi \in \mathcal{S}_q(\mathbb{R}_+,*(A)) \cap B_s(\mathbb{R}_+, A dx)$, then $f \in L^p(\mathbb{R}_+,A dx)$ if and only if $Mf \in L^p(\mathbb{R}_+,A dx)$. Moreover,*

$$\|Mf\|_{p,A} \sim \|f\|_{p,A}.$$

The theorem is not valid for $p = 1$ (a counterexample was constructed in [BX3]). However, we can exhibit large classes of distributions whose local maximal functions are in L^p for any $p > 0$. For this purpose we first introduce the local grand maximal function f^*_m. Set $\mathcal{D}_*(\mathbb{R}_+) = \mathcal{D}_*(\mathbb{R})|_{\mathbb{R}_+}$ where

$$\mathcal{D}_*(\mathbb{R}) = \{g \in C^\infty(\mathbb{R}) : g \text{ is even with compact support}\}.$$

Let $m \in \mathbb{N}_0$ and $x \in \mathbb{R}_+$, and denote by $K_m(x)$ the set of functions $\psi \in \mathcal{D}_*(\mathbb{R}_+)$ such that for some $0 < r < r_0$ (where r_0 is a fixed constant independent of x and m),

$$\operatorname{supp}(\psi) \subset B(x,r), \qquad \int_0^\infty |\psi(u)|A(u)\,du \le 1$$

and

$$|\psi^{(k)}(u)| \le C_{A,k} r^{-k} |B(x,r)|^{-1}, \qquad k = 0,1,\ldots,m+[2\alpha+2]+2, \tag{3.6}$$

where $B(x,r) := \{y \in \mathbb{R} : |y-x| < r\}$ and $|B(x,r)|$ is the Haar measure of $B(x,r)$. In the following we implicitly associate such an r with each function $\psi \in K_m(x)$. The (*local*) *grand maximal function* of $f \in \mathcal{S}'(\mathbb{R}_+,*(A))$ is given by

$$f^*_m(x) = \sup\{|f(\psi)| : \psi \in K_m(x)\}.$$

3.7. THEOREM. (i) (see [BX2, Theorem 4.52]) *The grand maximal function f^*_m is of weak type (L^1,L^1) for $m > 2\alpha+3$, and strong type (L^p,L^p) for $p > 1$ and $m > (2\alpha+2)/p+1$.*

(ii) *Let $f \in \mathcal{S}'_1(\mathbb{R}_+,*(A))$ and $1 \le p < \infty$. If $f^*_m \in L^p(\mathbb{R}_+,A dx)$ then $f \in L^p(\mathbb{R}_+,A dx)$, and for m-almost all $x \in \mathbb{R}_+$,*

$$|f(x)| \le f^*_m(x).$$

Proof. We only give the proof of (ii). Choose $\phi \in \mathcal{D}_*(\mathbb{R}_+)$ such that $\operatorname{supp}(\phi) \subset [0,1]$, $\phi(u) = 1$ for $u \in [0,1/2]$ and $\|\phi\|_{1,A} = 1$. For $0 < t < 1$ it is straightforward to verify $T_x\phi_t \in K_m(x)$ using (1.3), (1.10), (1.11) and Lemma 1.9. Hence

$$|f(T_x\phi_t)| \le f^*_m(x). \tag{3.8}$$

By (3.1) observe that $f * \phi_t \to f$ in $\mathcal{S}'_1(\mathbb{R}_+,*(A))$ as $t \to 0^+$. Now proceeding as in the proof of [FoS, Theorem (2.7)] we obtain $f \in L^p(\mathbb{R}_+,A dx)$. Finally, we use (3.8) and the fact that $f * \phi_t(x) \to f(x)$ for m-almost every $x \in \mathbb{R}_+$

(see [BX2, Corollary 3.16]) to obtain

$$|f(x)| \le f_m^*(x)$$

as required. ∎

As in the case of euclidean spaces the local grand maximal function is dominated by the local heat and radial maximal functions.

3.9. THEOREM (see [BX2, Corollary 4.51]). *Suppose that* $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$ *and* $0 < p < \infty$. *If* $H_0^+ f \in L^p(\mathbb{R}_+, A dx)$ *then* $f_m^* \in L^p(\mathbb{R}_+, A dx)$ *for* $m > (2\alpha+2)/p + 1$ *and*

$$\|f_m^*\|_{p,A} \le C_{p,A} \|H_0^+ f\|_{p,A}.$$

Now we proceed to prove the domination of f_m^* by the radial maximal function. The main difficulty comes from the fact that (ϕ_t) does not form a semigroup, nor does $\frac{\partial}{\partial t}\phi_t = L\phi_t$ hold. Hence some of the techniques in proving Theorem 3.9 cannot be employed.

3.10. LEMMA. *If* $\psi \in K_m(x_0)$ *then for any* $k \in \mathbb{N}$, $k \le m + [2\alpha+2] + 2$ *with* k *even*,

$$|\widehat{\psi}(\lambda)| \le \begin{cases} C_{A,k} r^{-k+1} |B(x_0,r)|^{-1} (1+\lambda)^{-k} x_0^{2\alpha+1}, & x_0 \le 2, \\ C_{A,k} r^{-k+1} |B(x_0,r)|^{-1} (1+\lambda)^{-k} x_0 e^{\varrho x_0}, & x_0 > 2, \\ C_{A,k} r^{-k+1} |B(x_0,r)|^{-1} (1+\lambda)^{-k-\alpha-1/2} x_0^{\alpha+1/2}, & \lambda x_0 > 2,\ x_0 > 2r. \end{cases}$$

Proof. By (1.3) and (1.5) there exists $\delta > 0$ such that

$$Lf(x) = -f''(x) - \frac{2\alpha+1}{x} f'(x) - B(x) f'(x), \quad 0 \le x \le \delta. \tag{3.11}$$

Notice that ψ is an even C^∞-function on $\mathbb{R}$. Therefore by (1.5), (3.11) and (3.6) we obtain

$$|L^{k/2}\psi(x)| \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}. \tag{3.12}$$

In view of (1.5), (1.6) and (3.12) integration by parts gives

$$\begin{aligned} |(\lambda^2+\varrho^2)^{k/2} \widehat{\psi}(\lambda)| &= \Big| \int_0^\infty \psi(x) L^{k/2} \varphi_\lambda(x) A(x)\,dx \Big| \\ &= \Big| \int_0^\infty L^{k/2}\psi(x) \varphi_\lambda(x) A(x)\,dx \Big| \\ &\le C_{A,k} r^{-k} |B(x_0,r)|^{-1} \int_{B(x_0,r)} |\varphi_\lambda(x)| A(x)\,dx. \end{aligned}$$

Hence the lemma follows immediately from Lemmas 2.1 and 2.2. ∎

3.13. LEMMA. *Let* $\psi \in K_m(x_0)$. *Then for* $x \in B(x_0,r)$ *and* $y \in \mathbb{R}_+$,

$$|(T_x\psi)^{(k)}(y)| \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}, \quad k = 0, 1, \ldots, m.$$

Proof. Appealing to [BX1, (3.27) and (2.17)] and Theorem 1.7 we have

$$(T_x\psi)^{(k)}(y) = \int_0^\infty \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda. \tag{3.14}$$

We only give the proof for the case where $y \le x \le x_0 \le 2$. The proof for the other cases can be carried out similarly using (3.14), Lemma 3.10 and the estimates for characters given in §2. By (3.14) we write

$$\begin{aligned} (T_x\psi)^{(k)}(y) &= \int_0^1 \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda \\ &\quad + \int_1^{1/x_0} \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda \\ &\quad + \int_{1/x_0}^{1/x} \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda \\ &\quad + \int_{1/x}^{1/y} \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda \\ &\quad + \int_{1/y}^{\infty} \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda \\ &:= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

Using Lemmas 3.10, 2.1 and 2.4 together with Theorem 1.8 we immediately obtain

$$I_1 \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}$$

and if $x_0 > r$ then

$$I_2 \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}.$$

If $x_0 \le r$ then we write

$$\begin{aligned} I_2 &= \int_1^{1/r} \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda + \int_{1/r}^{1/x_0} \widehat{\psi}(\lambda) \varphi_\lambda(x) \varphi_\lambda^{(k)}(y) |c(\lambda)|^{-2}\,d\lambda \\ &:= I_2^{(1)} + I_2^{(2)}. \end{aligned}$$

Assume that both k and $k + [2\alpha+2]$ are even (otherwise consider $k-1$ or $k + [2\alpha+2] + 1$). We now apply Theorem 1.8, Lemmas 2.1 and 2.4, and Lemma 3.10 (with k replaced by $k + [2\alpha+2] + 2$ for $I_2^{(2)}$) to obtain

$$I_2^{(i)} \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}, \quad i = 1, 2.$$

For I_j ($j = 3,4,5$), if $x_0 \le 2r$ then we apply Theorem 1.8, Lemmas 2.1 and 2.4, and Lemma 3.10 (with k replaced by $k + [2\alpha + 2] + 2$) to obtain

$$I_j \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}, \quad j = 3,4,5.$$

If $x_0 > 2r$ then $|x - x_0| < 1$ implies that $x_0/x \le 2$ and $r/x \le 1$. Thus a similar argument gives

$$I_3 \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}$$

and if $y \ge r$ then

$$I_j \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}, \quad j = 4,5.$$

If $y < r$ then write

$$I_4 = \int_{1/x}^{1/r} \widehat{\psi}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k)}(y)|c(\lambda)|^{-2}\,d\lambda + \int_{1/r}^{1/y} \widehat{\psi}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k)}(y)|c(\lambda)|^{-2}\,d\lambda$$
$$:= I_4^{(1)} + I_4^{(2)}.$$

Now applying Theorem 1.8, Lemmas 2.1 and 2.4, and Lemma 3.10 (with k replaced by $k + 2$ for $I_4^{(2)}$ and I_5) we obtain

$$I_j \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}, \quad j = 4,5. \ \blacksquare$$

Let n_0 be a positive integer and let $l_0, l_1, \ldots, l_{n_0}$ be distinct numbers (with $1/4 < l_j < 1/2$ for $j = 0,1,\ldots,n_0$). Then (by a standard argument of linear interpolation theory) there exist $c_0, c_1, \ldots, c_{n_0} \in \mathbb{R}$ such that

$$\sum_{j=0}^{n_0} c_j = 1 \quad \text{and} \quad \sum_{j=0}^{n_0} c_j l_j^i = 0 \tag{3.15}$$

for $i = 1,2,\ldots,n_0$. For $\phi \in B_s(\mathbb{R}_+, *(A))$ and $t > 0$ define σ_t by

$$\sigma_t(x) := \sum_{j=0}^{n_0} c_j \phi_{l_j t}(x) \tag{3.16}$$

and set

$$\sigma_t^- := \sigma_{t/2} - \sigma_t \quad \text{and} \quad \sigma_t^+ := \sigma_{t/2} + \sigma_t. \tag{3.17}$$

3.18. Lemma. *Suppose that n_0 is a nonnegative integer and $\psi \in K_{n_0+1}(x_0)$. Then for $x \in B(x_0,r)$,*

$$|\sigma_t^- * \psi(x)| \le C_{A,n_0} |B(x_0,r)|^{-1} (t/r)^{n_0+1}.$$

Proof. By (3.16) we have

$$\sigma_t * \phi(x) = \sum_{j=1}^{n_0} c_j \psi * \phi_{l_j t}(x).$$

By (1.12) and (3.2) we observe that for any positive integer k,

$$\frac{\partial^k}{\partial t^k} \psi * \phi_t(x) = \int_0^\infty \phi(y) y^k (T_x\psi)^{(k)}(ty) A(y)\,dy. \tag{3.19}$$

Hence using Lemma 3.13 we have

$$\left| \frac{\partial^k}{\partial t^k} \psi * \phi_t(x) \right| \le C_{A,k} r^{-k} |B(x_0,r)|^{-1}. \tag{3.20}$$

Therefore expanding $\psi * \phi_{l_j t}(x)$ about $t = 0$ the lemma follows readily from (3.15), (3.17), (3.19) and (3.20). $\blacksquare$

3.21. Lemma. *Let $\phi \in B_{s,n}(\mathbb{R}_+, *(A))$. Then for any nonnegative integer $k \le n$ and $t \le 1$,*

$$|\phi_t^{(k)}(x)| \le \begin{cases} C_{A,k} t^{-k-2\alpha-2}, & x \le t, \\ C_{A,k} t^{-k-1} A(x)^{-1} A(x/t)^{1-s}, & x > t. \end{cases}$$

Proof. Let $C(x) = x^{-2\alpha-1}A(x)$ and $C_t(x) = C(x/t)/C(x)$. Then $\phi_t(x) = t^{-2\alpha-2} C_t(x) \phi(x/t)$. By our assumption on A we see that $C'(x)/C(x) \in C^\infty(\mathbb{R})$ and

$$\left| \left(\frac{C'(x)}{C(x)} \right)^{(k)} \right| \le C_{A,k}.$$

Thus in view of Lemma 1.9 we obtain by induction

$$|C_t^{(k)}(x)| \le \begin{cases} C_{A,k} t^{-k}, & x \le t, \\ C_{A,k} t^{-k+2\alpha+1} A(x)^{-1} A(x/t), & x > t. \ \blacksquare \end{cases}$$

For any $N > 0$ and $\phi \in B_s(\mathbb{R}_+, *(A))$ we introduce the following local tangential and nontangential maximal functions:

$$\phi_N^{**}(f)(x) := \sup_{\substack{y \ge 0 \\ 0 < t \le 1}} |f * \phi_t(y)| \left(\frac{|B(y,t)|}{|B(x, |x-y| + t)|} \right)^N \tag{3.22}$$

$$\phi_\nabla^*(f)(x) := \sup_{\substack{|y-x| \le t \\ 0 < t \le 1}} |f * \phi_t(y)| \quad \text{and} \quad \phi^*(f)(x) := \sup_{\substack{|y-x| \le t \\ 0 < t \le 1}} t \left| \frac{d}{dy} f * \phi_t(y) \right|$$

where $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$.

We now compare these with the local grand maximal function.

3.23. Theorem. *For any $N > 0$ choose $n_0, m \in \mathbb{N}$ such that $n_0 \ge (2\alpha + 2)N - 1$ and $m \ge n_0 + 1$. If $\phi \in B_s(\mathbb{R}_+, *(A))$ with $s > 2N + 1$ then there exists a positive constant C depending only on A, N, n_0 and m such that for $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$,*

$$f_m^*(x) \le C \phi_N^{**}(f)(x).$$

Proof. From (3.3) we see that $\lim_{t\to 0^+}\widehat{\phi}(\lambda)=1$. Using the Fourier inversion formula ([BH, Theorem 2.2.36]) and Theorems 1.7 and 1.8 we have, for any $\psi\in\mathcal{D}_*(\mathbb{R}_+)$,

$$\lim_{t\to 0^+}\phi_t*\phi_t*\psi(y)=\psi(y) \tag{3.24}$$

uniformly for $y\in\mathbb{R}_+$. Consequently we can write, by (3.15)–(3.17),

$$\psi(y)=\psi*\sigma_r*\sigma_r(y)+\sum_{k=0}^{\infty}\sigma^+_{2^{-k}r}*\sigma^-_{2^{-k}r}*\psi(y)$$

where $r<r_0$ is as in the definition of f^*_m. Now the theorem can be proved in a similar way to the proof of [BX2, Proposition 4.10] using Lemmas 3.18 and 3.21. ■

3.25. THEOREM. *Suppose that* $f\in S_1'(\mathbb{R}_+,*(A))$ *and* $0<p<\infty$. *If* $\phi^*_\nabla(f)\in L^p(\mathbb{R}_+,Adx)$ *then* $f^*_m\in L^p(\mathbb{R}_+,Adx)$ *for* $m>(2\alpha+2)/p+1$ *and*

$$\|f^*_m\|_{p,A}\le C_{p,A}\|\phi^*_\nabla(f)\|_{p,A}.$$

Proof. This is similar to the proof of [BX2, Theorem 4.36]. ■

To compare $\phi^*(f)$ with $\phi^{**}_N(f)$ the corresponding method for the non-tangential maximal functions defined by the heat kernel h_t fails to work. We consider the following estimates.

3.26. LEMMA. *Let* $\phi\in B_{s,n}(\mathbb{R}_+,*(A))$. *Then for any given constant* c_0 *and* $k=0,1,\ldots,n-[2\alpha+2]-3$,

$$\left|\left(T_z\frac{\partial}{\partial y}T_y\phi_t(z)\right)^{(k)}(u)\right| \le\begin{cases}C_{A,k}t^{-k-2\alpha-3}, & y,z\le c_0t \text{ and } t\le 1,\\ C_{A,k}t^{-k-2}(A(t)A(z))^{-1/2}, & y\le t\le z\le 1,\\ C_{A,k}t^{-k-2}(A(t)A(y))^{-1/2}, & z\le t\le y\le 1,\\ C_{A,k}t^{-k-2}(A(y)A(z))^{-1/2} & \text{otherwise},\end{cases}$$

where $C_{A,k}$ *depends only on* A *and* k.

Proof. Applying the Fourier inversion formula ([BH, Theorem 2.2.36]) and [BX1, (2.17) and (2.18)] we obtain

$$\left(T_z\frac{\partial}{\partial y}T_y\phi_t(z)\right)^{(k)}(u)=\int_0^\infty\widehat{\phi}_t(\lambda)\varphi'_\lambda(y)\varphi_\lambda(z)\varphi^{(k)}_\lambda(u)|c(\lambda)|^{-2}\,d\lambda.$$

Also, in view of Lemma 3.21 the same argument as in the proof of Lemma 3.10 gives, for any positive even integer $j\le n$,

$$|\widehat{\phi}_t(\lambda)|\le C_{A,j}t^{-j}(1+\lambda)^{-j}. \tag{3.27}$$

Therefore we can proceed similarly to the proof of Lemma 3.13 to obtain the result using (3.27), Lemmas 2.1, 2.2, 2.4 and Theorem 1.8. ■

LEMMA 3.28. *Suppose that* n_0 *is a nonnegative integer and* $\phi\in B_{n,s}(\mathbb{R}_+,*(A))$ *with* $n\ge n_0+[2\alpha+2]+3$. *Then for any given constant* $c_0>1$ *and* $y,z\in\mathbb{R}_+$,

$$\left|\frac{\partial}{\partial y}T_y\sigma^-_\tau*\phi_t(z)\right|\le\begin{cases}C_{A,n_0}\tau^{n_0+1}t^{-n_0-2\alpha-4}, & y,z\le c_0t,\\ C_{A,n_0}(A(y)A(z))^{-1/2}\tau^{n_0+1}t^{-n_0-3} & \text{otherwise},\end{cases}$$

where C_{A,n_0} *depends only on* A *and* n_0.

Proof. By (1.12) and (3.2) we have, for any positive integer k,

$$\frac{\partial^k}{\partial\tau^k}\left(\phi_\tau*\frac{\partial}{\partial y}T_y\phi_t(z)\right)=\int_0^\infty\phi(u)u^k\left(T_z\frac{\partial}{\partial y}T_y\phi_t(z)\right)^{(k)}(\tau u)A(u)\,du,\qquad y,z\in\mathbb{R}_+.$$

Applying Lemma 3.26 gives

$$\left|\frac{\partial^k}{\partial\tau^k}\left(\phi_\tau*\frac{\partial}{\partial y}T_y\phi_t(z)\right)\right|\le\begin{cases}C_{A,k}t^{-k-2\alpha-3}, & y,z\le c_0t,\\ C_{A,k}t^{-k-2}(A(y)A(z))^{-1/2} & \text{otherwise}.\end{cases}$$

Hence the result follows using a similar argument to the proof of Lemma 3.18. ■

3.29. LEMMA. *Let* $0<t\le 1$ *and* $\phi\in B_{s,n}(\mathbb{R}_+,*(A))$ *with* $s>1$ *and* $n\ge[2\alpha+2]+2$. *Then for any* $0<\delta<2(s-1)$ *and* $k=0,1,\ldots,n-[2\alpha+2]-2$,

$$\left|\frac{\partial^k}{\partial y^k}T_x\phi_t(y)\right|\le\begin{cases}C_{A,k}t^{-k-1}\Delta(x,y,t), & |x-y|\le ct,\\ C_{A,k}t^{-k-1}\Delta(x,y,t)e^{-\delta\varrho|x-y|/t}, & |x-y|>ct,\end{cases}$$

where $\Delta(x,y,t):=\min\{(A(x)A(y))^{-1/2},(A(t)A(x))^{-1/2},(A(t)A(y))^{-1/2}\}$, $c>1$ *is any given constant and* $C_{A,k}$ *depends only on* A, k *and* c.

Proof. We first consider the case where $|x-y|\le ct$. As in the proof of Lemma 3.26 we have

$$\frac{\partial^k}{\partial y^k}T_x\phi_t(y)=\int_0^\infty\widehat{\phi}_t(\lambda)\varphi_\lambda(x)\varphi^{(k)}_\lambda(y)|c(\lambda)|^{-2}\,d\lambda.$$

Thus a similar argument to the proof of Lemma 3.13 shows the lemma for $|x-y|\le ct$ with the use of (3.27), Lemmas 2.1, 2.2, 2.4 and Theorem 1.8.

For $|x-y|>ct$ we use the idea in [A] and choose $\omega\in C^\infty(\mathbb{R})$ such that $\omega(x)=0$ for $x\le 1/2$ and $\omega(x)=1$ for $x\ge 1$. For any fixed $x,y\in\mathbb{R}_+$ with $|x-y|>t$ define

$$\omega_{|x-y|,t}(u) = \omega(t^{-1}(|x-y|+u))\omega(t^{-1}(|x-y|-u)).$$

Then $\omega_{|x-y|,t} \in \mathcal{D}_*(\mathbb{R}_+)$, $\omega_{|x-y|,t}(u) = 1$ for $u < |x-y|-t$ and $\omega_{|x-y|,t}(u) = 0$ for $u > |x-y|-t/2$. Set $\Omega_{|x-y|,t} = 1 - \omega_{|x-y|,t}$. Then $\Omega_{|x-y|,t} \in \mathcal{D}_*(\mathbb{R}_+)$ satisfies

$$\Omega_{|x-y|,t}(u) = \begin{cases} 0, & 0 \le u < |x-y|-t, \\ 1, & u > |x-y|-t/2, \end{cases}$$

and

$$|\Omega^{(j)}_{|x-y|,t}(u)| \le C_j t^{-j}, \quad j \in \mathbb{N}_0.$$

Let $k = k_t = \phi_t$, $l = \mathcal{A}k$ and $m = \mathcal{F}k = \mathcal{F}_0 l$ where $\mathcal{A}$ is the Abel transform and $\mathcal{F}_0$ is the classical Fourier transform on $\mathbb{R}$ (see [T]). Put $l_{|x-y|,t} = l\Omega_{|x-y|,t}$, $k_{|x-y|,t} = \mathcal{A}^{-1} l_{|x-y|,t}$ and $m_{|x-y|,t} = \mathcal{F}_0 l_{|x-y|,t}$. Now $l - l_{|x-y|,t} \in \mathcal{D}_*(\mathbb{R}_+)$ is supported in $[0, |x-y|-t/2]$. Hence by [T] we have $\operatorname{supp}(k - k_{|x-y|,t}) \subset [0, |x-y|-t/2]$, which implies that

$$\phi_t(u) = k(u) = k_{|x-y|,t}(u), \quad u > |x-y|-t/2.$$

Thus $T_x\phi_t(y) = T_x k_{|x-y|,t}(y)$ and by [BH, Theorem 2.2.36] and [BX1, (2.17) and (2.18)],

$$T_x\phi_t(y) = \int_0^\infty m_{|x-y|,t}(\lambda)\varphi_\lambda(x)\varphi_\lambda(y)|c(\lambda)|^{-2}\,d\lambda. \tag{3.30}$$

For $j = 0, 1, \ldots, k$ put

$$\Omega_{|x-y|,t,j}(u) = \begin{cases} (-1)^{j+1}\sum_{i=0}^{j}\binom{j}{i}\omega^{(i)}\Big(\frac{|x-y|+u}{t}\Big)\omega^{(j-i)}\Big(\frac{|x-y|-u}{t}\Big), & x > y, \\ -\sum_{i=0}^{j}\binom{j}{i}\omega^{(i)}\Big(\frac{|x-y|+u}{t}\Big)\omega^{(j-i)}\Big(\frac{|x-y|-u}{t}\Big), & x < y, \end{cases}$$

and $l_{|x-y|,t,j}(u) = l(u)\Omega_{|x-y|,t,j}(u)$. Then

$$\frac{\partial^j}{\partial y^j}\Omega_{|x-y|,t}(u) = t^{-j}\Omega_{|x-y|,t,j}(u).$$

Recall that $\mathcal{F} = \mathcal{F}_0\mathcal{A}$ (see [T]). Therefore

$$\frac{\partial^j}{\partial y^j} m_{|x-y|,t}(\lambda) = t^{-j}\int_0^\infty l_{|x-y|,t,j}\cos\lambda u\,du = t^{-j}\mathcal{F}_0 l_{|x-y|,t,j}(\lambda)$$

and hence by (3.30),

$$\frac{\partial^k}{\partial y^k}T_x\phi_t(y) = \sum_{j=0}^{k}\binom{k}{j}t^{-j}\int_0^\infty m_{|x-y|,t,j}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k-j)}(y)|c(\lambda)|^{-2}\,d\lambda. \tag{3.31}$$

We claim now that for any integer $0 \le L \le k$,

$$\Big\{\int_0^\infty |m_{|x-y|,t,j}(\lambda)(1+\lambda)^L|^2\,d\lambda\Big\}^{1/2} \le C_{A,j,L}t^{-L-1/2}e^{-\delta\varrho|x-y|/t}. \tag{3.32}$$

In fact, because of the properties of the classical Fourier transform and

$$|l^{(i)}_{|x-y|,t,j}(u)| \le C_{A,i}\sum_{q=0}^{i}t^{-i+q}|l^{(q)}(u)|, \quad i = 0, 1, \ldots, L,$$

$$l_{|x-y|,t,j}(u) = 0 \quad \text{if } u < |x-y|-t/2$$

we have

$$\begin{aligned} \Big\{\int_0^\infty |m_{|x-y|,t,j}(\lambda)(1+\lambda)^L|^2\,d\lambda\Big\}^{1/2} &\le C_{A,j,L}\sum_{i=0}^{L}\Big\{\int_0^\infty |l^{(i)}_{|x-y|,t,j}(u)|^2\,du\Big\}^{1/2} \\ &\le C_{A,j,L}\sum_{i=0}^{L}\sum_{q=0}^{i}t^{-i+q}\Big\{\int_{|x-y|-t}^{\infty}|l^{(q)}(u)|^2\,du\Big\}^{1/2} \\ &\le C_{A,j,L}\sum_{i=0}^{L}\sum_{q=0}^{i}t^{-i+q}e^{-\delta\varrho|x-y|/t}\Big\{t\int_0^\infty |l^{(q)}(tu)e^{\delta\varrho u}|^2\,du\Big\}^{1/2} \\ &:= C_{A,j,L}\sum_{i=0}^{L}\sum_{q=0}^{i}t^{-i+q}e^{-\delta\varrho|x-y|/t}I_q. \end{aligned} \tag{3.33}$$

Appealing to the properties of the classical Fourier transform and the analyticity of $\widehat{\phi_t}$ we obtain

$$\begin{aligned} I_q = \Big\{t\int_0^\infty |l^{(q)}(tu)e^{\delta\varrho u}|^2\,du\Big\}^{1/2} &= \Big\{\int_0^\infty |l^{(q)}(v)e^{\delta\varrho t^{-1}v}|^2\,dv\Big\}^{1/2} \\ &= \Big\{\int_0^\infty \Big|\Big(\lambda + i\frac{\delta\varrho}{t}\Big)^q\widehat{\phi_t}\Big(\lambda + i\frac{\delta\varrho}{t}\Big)\Big|^2\,d\lambda\Big\}^{1/2}. \end{aligned}$$

By the Laplace representation of characters (see [Ché]),

$$\varphi_\lambda(x) = \int_{-x}^{x} e^{(i\lambda-\varrho)t}\,\nu_x(dt), \quad x \in \mathbb{R}_+,\ \lambda \in \mathbb{C},$$

we see that for $x \in \mathbb{R}_+$ and $\lambda = \xi + i\eta \in \mathbb{C}$,

$$|\varphi_\lambda(x)| \le e^{|\eta|x}\varphi_0(x). \tag{3.34}$$

Thus by (3.34) and Lemmas 1.9, 2.1 and 3.21 we argue similarly to the proof of Lemma 3.10 to obtain, for any positive even integer m and $\lambda = \xi + i\delta\varrho/t \in \mathbb{C}$,

$$\begin{aligned}|(\lambda^2+\varrho^2)^{m/2}\widehat{\phi}_t| &= \Big|\int_0^\infty L^{m/2}\phi_t(v)\varphi_\lambda(v)A(v)\,dv\Big|\\ &\le C_{A,m}\Big(t^{-m-2\alpha-2}\int_0^t e^{\delta\varrho v/t}\varphi_0(v)A(v)\,dv\\ &\quad + t^{-m-1}\int_t^\infty \frac{A(v/t)^{1-s}}{A(v)}e^{\delta\varrho v/t}\varphi_0(v)A(v)\,dv\Big)\\ &\le C_{A,m}t^{-m}\end{aligned}$$

provided $\delta < 2s-2$. Consequently we have

$$I_q \le C_{A,q}t^{-q-1/2} \tag{3.35}$$

and (3.32) follows from (3.33) and (3.35).

Now we use (3.31) and (3.32) to prove the lemma for $|x-y|>ct$. We only consider the case where $t \le y < x \le c$ and $|x-y|>ct$; the other cases (for $|x-y|>ct$) can be proved similarly. By (3.31) we write

$$\begin{aligned}\frac{\partial^k}{\partial y^k}T_x\phi_t(y) &= \sum_{j=0}^k \binom{k}{j}t^{-j}\Big(\int_0^{c^{-1}} m_{|x-y|,t,j}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k-j)}(y)|c(\lambda)|^{-2}\,d\lambda\\ &\quad+\int_{c^{-1}}^{x^{-1}} m_{|x-y|,t,j}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k-j)}(y)|c(\lambda)|^{-2}\,d\lambda\\ &\quad+\int_{x^{-1}}^{y^{-1}} m_{|x-y|,t,j}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k-j)}(y)|c(\lambda)|^{-2}\,d\lambda\\ &\quad+\int_{y^{-1}}^{t^{-1}} m_{|x-y|,t,j}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k-j)}(y)|c(\lambda)|^{-2}\,d\lambda\\ &\quad+\int_{t^{-1}}^{\infty} m_{|x-y|,t,j}(\lambda)\varphi_\lambda(x)\varphi_\lambda^{(k-j)}(y)|c(\lambda)|^{-2}\,d\lambda\Big)\\ &:= \sum_{j=0}^k\binom{k}{j}t^{-j}(J_1+J_2+J_3+J_4+J_5).\end{aligned}$$

Applying Lemmas 2.1, 2.2, 2.4, Theorem 1.8, (3.32) and the Cauchy–Schwarz inequality we see that

$$\begin{aligned}|J_1| &\le C_{A,k}\int_0^{c^{-1}} |m_{|x-y|,t,j}(\lambda)|\lambda^{k-j+2\alpha+1}\,d\lambda\\ &\le \Big\{\int_0^\infty |m_{|x-y|,t,j}(\lambda)|^2\,d\lambda\Big\}^{1/2} \le C_{A,k}t^{-k+j-1/2}e^{-\delta\varrho|x-y|/t},\\ |J_2| &\le C_{A,k}\int_{c^{-1}}^{x^{-1}} |m_{|x-y|,t,j}(\lambda)|\lambda^{k-j+2\alpha+1}\,d\lambda\\ &\le \Big\{\int_{c^{-1}}^\infty |m_{|x-y|,t,j}(\lambda)(\lambda+1)^{k-j}|^2\,d\lambda\Big\}^{1/2}\Big\{\int_{c^{-1}}^{x^{-1}}(1+\lambda)^{2(2\alpha+1)}\,d\lambda\Big\}^{1/2}\\ &\le C_{A,k}t^{-k-1}(A(x)A(y))^{-1/2}e^{-\delta\varrho|x-y|/t}\end{aligned}$$

and similarly

$$|J_i| \le C_{A,k}t^{-k-1}(A(x)A(y))^{-1/2}e^{-\delta\varrho|x-y|/t}, \quad i=3,4,5. \ \blacksquare$$

3.36. Theorem. *For any $N>0$ choose $n_0 \in \mathbb{N}$ such that $n_0 > N(2\alpha+2)-1$. Let $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$ and $\phi \in B_{n,s}(\mathbb{R}_+, *(A))$ with $s > 2N+1$ and $n \ge n_0 + [2\alpha+2]+3$. Then there exists a constant $C>0$ depending only on A, N, n_0 such that*

$$\phi^*(f)(x) \le C\phi_N^{**}(f)(x).$$

Proof. This is similar to the proof of [BX2, Propositions 4.10 and 4.15]. By (3.24) and (3.16)–(3.17) we can write

$$\phi_t(y) = \phi_t * \sigma_t * \sigma_t(y) + \sum_{k=0}^\infty \sigma_{2^{-k}t}^+ * \sigma_{2^{-k}t}^- * \phi_t(y)$$

and hence

$$\begin{aligned}\frac{d}{dy}f*\phi_t(y) &= \int_0^\infty f*\sigma_t(z)\frac{\partial}{\partial y}T_y\sigma_t^- * \phi_t(z)A(z)\,dz\\ &\quad+\sum_{k=0}^\infty\int_0^\infty f*\sigma_{2^{-k}t}(z)\frac{\partial}{\partial y}T_y\sigma_{2^{-k}t}^- * \phi_t(z)A(z)\,dz.\end{aligned}$$

Consequently, by (3.22) we have, for any $x, y \in \mathbb{R}_+$ and $|x-y| \le t$,

$$(3.37)\quad \left|t\frac{d}{dy} f * \phi_t(y)\right| \le Ct\phi_N^{**}(f)(x)\Big\{\int_0^\infty g(x,z,t,N)\left|\frac{\partial}{\partial y}T_y\sigma_t^- * \phi_t(z)\right| A(z)\,dz + \sum_{k=0}^\infty \int_0^\infty g(x,z,2^{-k}t,N)\left|\frac{\partial}{\partial y}T_y\sigma_{2^{-k}t}^- * \phi_t(z)\right|A(z)\,dz\Big\}$$

where $g(x,z,t,N) = (|B(x,|x-z|+t)|/|B(z,t)|)^N$. Applying Lemma 3.26 (with $k=0$) and Lemma 3.29 (with $k=1$) gives the following estimates:

$$(3.38)\quad \int_{|z-x|\le 2t} \left|\frac{\partial}{\partial y}T_y\sigma_t^- * \phi_t(z)\right| A(z)\,dz \le C_A t^{-1}, \quad \text{and} \quad \int_0^\infty \left|\frac{\partial}{\partial y}T_y\sigma_t^-(z)\right| A(z)\,dz \le C_A t^{-1}.$$

For any $s > 2N+1$ there exists $\delta > 0$ such that $4N < \delta < 2s-2$. The theorem now follows from a similar argument to the proof of [BX2, Proposition 4.10] with the use of (3.37), (3.38), Lemma 3.29 (with $k=1$ and $4N < \delta < 2s-2$) and Lemma 3.28. ∎

Now using Theorems 3.23, 3.25 and 3.36 the standard method in euclidean spaces (see [FeS] and [BX2]) gives the following results.

3.39. THEOREM. *Suppose that $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$ and $\phi \in B_{s,n}(\mathbb{R}_+, *(A))$ with $s > 2/p+1$ where $0 < p < \infty$ and $n \ge (2\alpha+2)/p + [2\alpha+2]+4$. If $M_{\phi,0}f \in L^p(\mathbb{R}_+, A dx)$ then $\phi_\nabla^*(f) \in L^p(\mathbb{R}_+, A dx)$ and*

$$\|\phi_\nabla^*(f)\|_{p,A} \le C_{p,A}\|M_{\phi,0}f\|_{p,A}.$$

3.40. COROLLARY. *Suppose that $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$ and $\phi \in B_{s,n}(\mathbb{R}_+, *(A))$ with $s > 2/p+1$ where $0 < p < \infty$ and $n \ge (2\alpha+2)/p + [2\alpha+2]+4$. If $M_{\phi,0}f \in L^p(\mathbb{R}_+, A dx)$ then $f_m^* \in L^p(\mathbb{R}_+, A dx)$ for $m > (2\alpha+2)/p+1$ and*

$$\|f_m^*\|_{p,A} \le C_{p,A}\|M_{\phi,0}f\|_{p,A}.$$

4. Local Hardy spaces. In this section we introduce the local Hardy spaces $\mathbf{h}^p$ by means of the maximal functions and the atomic Hardy spaces $\mathbf{h}_a^p$, and then prove that they are identical; the methods used parallel those in the euclidean case (see [FoS]).

4.1. DEFINITION. Let $0 < p < \infty$.

(i) The *Hardy space* $H^p = H^p(\mathbb{R}_+, *(A))$ is defined by

$$H^p := \{f \in \mathcal{S}_1'(\mathbb{R}_+, *(A)) : H^+ f \in L^p(\mathbb{R}_+, A dx)\}.$$

(ii) The *local Hardy space* $\mathbf{h}^p = \mathbf{h}^p(\mathbb{R}_+, *(A))$ is defined by

$$\mathbf{h}^p := \{f \in \mathcal{S}_1'(\mathbb{R}_+, *(A)) : H_0^+ f \in L^p(\mathbb{R}_+, A dx)\}.$$

Moreover, we introduce the quasi-norms $\|f\|_{H^p} := \|H^+ f\|_{p,A}$ and $\|f\|_{\mathbf{h}^p} := \|H_0^+ f\|_{p,A}$ to define topologies on H^p and $\mathbf{h}^p$ respectively.

For $p > 1$, H^p and $\mathbf{h}^p$ coincide with $L^p(\mathbb{R}_+, A dx)$.

4.2. THEOREM (see [BX3, Proposition 2.16]). *If $1 < p < \infty$ then*

$$H^p = \mathbf{h}^p = L^p(\mathbb{R}_+, A dx).$$

4.3. THEOREM. *For $0 < p < \infty$, $\mathbf{h}^p$ is complete.*

Proof. It suffices to show the inclusion $\mathbf{h}^p \subset \mathcal{S}_1'(\mathbb{R}_+, *(A))$ is continuous. The theorem can then be proved by mimicking the proof of [FoS, Proposition (2.16)]. Note that $\mathcal{D}_*(\mathbb{R}_+)$ is dense in $\mathcal{S}_1(\mathbb{R}_+, *(A))$ (this can be proved in the same way as for noncompact symmetric spaces; see [GV, p. 254]), so we need only show that if $f_n \to f$ in $\mathbf{h}^p$ then $(f_n, \psi) \to (f, \psi)$ for all $\psi \in \mathcal{D}_*(\mathbb{R}_+)$.

For any $\psi \in \mathcal{D}_*(\mathbb{R}_+)$ there exists a positive integer k_ψ such that $\mathrm{supp}(\psi) \subset \bigcup_{j=1}^{k_\psi} B(x_j, 1)$ where $x_j \in \mathrm{supp}(\psi)$, $j = 1, \ldots, k_\psi$. Let $\{\phi_j\}_{j=1}^{k_\psi}$ be a partition of unity satisfying $\phi_j \in \mathcal{D}_*(\mathbb{R}_+, *(A))$, $\mathrm{supp}(\phi_j) \subset B(x_j, 1)$, $0 < \phi_j \le 1$ and $\phi_j = 1$ on $B(x_j, 1/2)$. Thus

$$(f, \psi) = \sum_{j=1}^{k_\psi} (f, \psi_j)$$

where $\psi_j = \psi\phi_j$, and there exists a constant $C_\psi > 0$ such that $\psi_j/C_\psi \in K_m(y)$ for any $y \in B(x_j, 1)$. Therefore

$$|(f,\psi)|^p \le C_\psi \sum_{j=1}^{k_\psi} |B(x_j,1)| \int_{B(x_j,1)} f_m^*(y)^p A(y)\,dy \le C_{A,\psi} \int_0^\infty f_m^*(y)^p A(y)\,dy, \quad \psi \in \mathcal{D}_*(\mathbb{R}_+, *(A))$$

and by Theorem 3.9 the proof is complete. ∎

We assume throughout that the exponents p and q are admissible in the sense that $0 < p \le 1$, $1 \le q \le \infty$ and $p < q$, and put $d = [(2\alpha+2)(1/p-1)]$.

4.4. DEFINITION. (i) A *local (p,q)-atom* is a function $a \in L^q(\mathbb{R}_+, A dx)$ such that for some $x_0 \in \mathbb{R}_+$ and $r > 0$, $\mathrm{supp}(a) \subset B(x_0, r)$,

$$(4.5)\quad \|a\|_{q,A} \le m(B(x_0, r))^{1/q-1/p}$$

together with the following (local) moment condition: if r can be chosen not exceeding 1 then

$$\int_0^\infty a(x)x^k A(x)\,dx = 0 \tag{4.6}$$

for all integers k satisfying $0 \le k \le d$.

(ii) The *local atomic Hardy space* $\mathbf{h}_a^p = \mathbf{h}_a^p(\mathbb{R}_+, *(A))$ is the space of all distributions $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$ having a representation $f = \sum_{j=1}^\infty \lambda_j a_j$ where the a_j are local (p,q)-atoms and $\sum_{j=1}^\infty |\lambda_j|^p < \infty$. Write

$$\|f\|_{\mathbf{h}_a^p} := \inf\Big\{\sum_{j=1}^\infty |\lambda_j|^p : \sum_{j=1}^\infty \lambda_j a_j \text{ is an atomic representation of } f \text{ using local } (p,q)\text{-atoms}\Big\}.$$

We first proceed to prove that the maximal operators are bounded from $\mathbf{h}_a^p$ to $L^p(\mathbb{R}_+, A dx)$ for $0 < p \le 1$.

4.7. Lemma. *Let $0 < t \le 1$ and $\phi \in B_{s,n}(\mathbb{R}_+, *(A))$ with $s > 2/p + 1$ where $0 < p \le 1$ and $n \ge (2\alpha+2)/p + [2\alpha+2] + 4$. Then there exists $4/p < \delta < 2(s-1)$ such that for $k = 0, 1, \ldots, n - [2\alpha+2] - 2$,*

$$\left|\frac{\partial^k}{\partial y^k} T_x\phi_t(y)\right| \le \begin{cases} C_{A,p}|x-y|^{-k-1}\Delta_1(x,y), & |x-y| \le c,\ |x-y| \le \min\{x,y\}, \\ C_{A,p}|x-y|^{-k-\alpha-3/2}\Delta_2(x,y), & \min\{x,y\} < |x-y| \le c, \\ C_{A,p}|x-y|^{-k-1}(A(x)A(y))^{-1/2}e^{-\delta\varrho|x-y|}, & |x-y| > c, \end{cases}$$

where $\Delta_k(x,y) = \min\{A(x)^{-1/k}, A(y)^{-1/k}\}$, $k = 1, 2$.

Proof. This follows immediately from Lemma 3.29. ∎

4.8. Theorem. *Suppose that $\phi \in B_{s,n}(\mathbb{R}_+, *(A))$ with $s > 2/p+1$ where $0 < p \le 1$ and $n \ge (2\alpha+2)/p + [2\alpha+2] + 4$. Then there exists a positive constant $C_{A,p}$ such that for each local (p,q)-atom a, $\|M_{\phi,0}a\|_{p,A} \le C_{A,p}$.*

Proof. Let a be a local $(1,q)$-atom supported in $B(x_0, r)$ for some $x_0 \in \mathbb{R}_+$ and $r > 0$.

First we assume $r > 1$ and write

$$\|M_{\phi,0}a\|_{p,A}^p = \int_0^{x_0+r+1} M_{\phi,0}a(x)^p A(x)\,dx + \int_{x_0+r+1}^\infty M_{\phi,0}a(x)^p A(x)\,dx := I_1 + I_2.$$

By Lemma 1.9 we know that

$$|B(x_0,r)| \ge \begin{cases} C(x_0+r)^{2\alpha+2}, & x_0 + r \le C_0,\ x_0 \le r, \\ Cx_0^{2\alpha+1}r, & x_0 + r \le C_0,\ x_0 > r, \\ Ce^{2\varrho(x_0+r)}, & x_0 + r > C_0 \end{cases} \tag{4.9}$$

and

$$|B(0, x_0+r+1)| \le \begin{cases} C(x_0+r+1)^{2\alpha+2}, & x_0+r \le C_0, \\ Ce^{2\varrho(x_0+r)}, & x_0 + r > C_0 \end{cases} \tag{4.10}$$

where $C_0 > 1$ is a sufficiently large constant. Thus using Hölder's inequality, Theorem 3.4 and the size condition (4.5) we have

$$I_1 \le C_{A,p}|B(x_0,r)|^{p/q-1}|B(0,x_0+r+1)|^{1-p/q} \le C_{A,p}.$$

To estimate I_2 we observe that

$$a * \phi_t(x) = \int_{\max\{0, x_0-r\}}^{x_0+r} a(y)T_x\phi_t(y)A(y)\,dy, \quad x \ge x_0 + r + 1.$$

By Lemmas 2.1 and 4.7 (with $k = 0$) we have

$$|T_x\phi_t(y)| \le \begin{cases} C_A x^{-\alpha-3/2}e^{-(\delta+1)\varrho x}, \\ C_A(x - x_0 - r)^{-1}(A(x)A(y))^{-1/2}e^{-\delta\varrho(x-x_0-r)}. \end{cases}$$

Consequently, using (4.5) and (4.9) gives

$$\begin{aligned} |a * \phi_t(x)| &\le \int_{\max\{x_0-r,0\}}^1 |a(y)T_x\phi_t(y)|A(y)\,dy + \int_1^{x_0+r} |a(y)T_x\phi_t(y)|A(y)\,dy \\ &\le C_A(\|a\|_{1,A}x^{-\alpha-3/2}e^{-(\delta+1)\varrho x} + \|a\|_\infty A(x)^{-1/2}e^{-\delta\varrho(x-x_0-r)}e^{\varrho(x_0+r)}) \\ &\le C_A(x^{-\alpha-3/2}e^{-(\delta+1)\varrho x} + A(x)^{-1/2}e^{-\delta\varrho(x-x_0-r)}e^{(1-2/p)\varrho(x_0+r)}). \end{aligned}$$

Hence by Lemma 1.9, $|I_2| \le C_{A,p}$ and this completes the proof of the theorem in the case of $r > 1$.

Now we assume $r \le 1$. Let $P_{d,t,x}$ denote the Taylor polynomial of order d of $T_x\phi_t$ at x_0. Applying the moment condition (4.6) gives

$$(4.11)\quad |a*\phi_t(x)| = \Big|\int_0^\infty (T_x\phi_t(y) - P_{d,t,x}(y))A(y)\,dy\Big| \le \int_{B(x_0,r)} |a(y)||y-x_0|^{d+1} \sup_{w\in B(x_0,r)} \Big|\frac{\partial^{d+1}}{\partial w^{d+1}} T_x\phi_t(w)\Big| A(y)\,dy.$$

For $x_0 \le 3r$ we write

$$\|M_{\phi,0}a\|_{p,A} = \int_0^{8r} M_{\phi,0}a(x)^p A(x)\,dx + \int_{8r}^\infty M_{\phi,0}a(x)^p A(x)\,dx := J_1 + J_2.$$

Using Hölder's inequality, Theorem 3.4, Lemma 1.9, (4.5), (4.9) and (4.10) we see that

$$J_1 \le C_{A,q}.$$

To estimate J_2 we appeal to (4.11), (4.9), (4.10) and Lemmas 4.7 and 1.9 to obtain

$$|a*\phi_t(x)| \le \begin{cases} C_{A,p} r^{2\alpha+2-(2\alpha+2)/p+d+1} x^{-d-2\alpha-3}, & x \le C_0,\\ C_{A,p} r^{2\alpha+2-(2\alpha+2)/p+d+1} x^{-d-\alpha-5/2} e^{-(\delta+1)\varrho x}, & x > C_0,\end{cases}$$

where $C_0 > 1$ is a sufficiently large constant. Therefore by Lemma 1.9,

$$J_2 = \int_{8r}^{C_0} M_{\phi,0}a(x)^p A(x)\,dx + \int_{C_0}^\infty M_{\phi,0}a(x)^p A(x)\,dx \le C_{A,p}.$$

For $x_0 > 3r$ we write

$$\|M_{\phi,0}a\|_{p,A} = \int_{B(x_0,2r)} M_{\phi,0}a(x)^p A(x)\,dx + \int_{\mathbb{R}_+\setminus B(x_0,2r)} M_{\phi,0}a(x)^p A(x)\,dx := J_3 + J_4.$$

Proceeding as for the estimate of J_1 we get $J_3 \le C_{A,p}$.

It remains to obtain a bound for J_4. In view of (4.11), (4.5) and Lemmas 1.9 and 4.7 a straightforward calculation leads to the following estimates: if $3r < x_0 \le C_0$ then

$$|a*\phi_t(x)| \le \begin{cases} C_{A,p} r^{d+2-1/p}|x-x_0|^{-d-2}A(x)^{-1/p}, & 0 < x \le x_0 - 2r,\\ C_{A,p} r^{d+2-1/p}(|x-x_0|^{-d-2}A(x)^{-1/p} + A(x_0)^{1-1/p}x^{-d-2\alpha-3}), & x_0+2r < x \le 2C_0,\\ C_{A,p} r^{d+2-1/p}A(x_0)^{1-1/p}x^{-d-2\alpha-3}e^{-(\delta+1)\varrho x}, & x > 2C_0,\end{cases}$$

and if $x_0 > C_0$ then

$$|a*\phi_t(x)| \le \begin{cases} C_{A,p} r^{d+2-1/p}A(x)^{-1/p}, & 0 < x \le C_0/2,\\ C_{A,p} r^{d+2-1/p}(|x-x_0|^{-d-2}A(x)^{-1/p} + A(x_0)^{1/2-1/p}A(x)^{1/2}), & C_0/2 < x \le x_0 - 2r,\\ C_{A,p} r^{d+2-1/p}|x-x_0|^{-d-2}(A(x)^{-1/p} + A(x)^{-1/2}A(x_0)^{1/2-1/p}e^{-(\delta+1)\varrho x}), & x > x_0 + 2r.\end{cases}$$

Consequently, by Lemma 1.9 we have, for $3r < x_0 \le C_0$,

$$J_4 = \int_0^{x_0-2r} M_{\phi,0}a(x)^p A(x)\,dx + \int_{x_0+2r}^{2C_0} M_{\phi,0}a(x)^p A(x)\,dx + \int_{2C_0}^\infty M_{\phi,0}a(x)^p A(x)\,dx \le C_{A,p},$$

and for $x_0 > C_0$,

$$J_4 = \int_0^{C_0/2} M_{\phi,0}a(x)^p A(x)\,dx + \int_{C_0/2}^{x_0-2r} M_{\phi,0}a(x)^p A(x)\,dx + \int_{x_0+2r}^\infty M_{\phi,0}a(x)^p A(x)\,dx \le C_{A,p}. \ \blacksquare$$

4.12. Lemma. *Let k be a nonnegative integer and $c_1 > 1$ and $c_2 > 1$ any given constants. Then for $x, y \in \mathbb{R}_+$ with $x \ne y$ and for $0 < t \le 1$,*

$$\Big|\frac{\partial^k}{\partial y^k} T_x h_t(y)\Big| \le \begin{cases} C_A \sqrt{t}^{-k-1}\widetilde{\Delta}(x,y,t), & |x-y| \le c_1\sqrt{t},\\ C_A\Big(\frac{|x-y|}{t}\Big)^{k+[2\alpha+2]}\frac{e^{-|x-y|^2/(4t)}}{|x-y|^{1/2}}, & c_1\sqrt{t} < |x-y| \le c_2,\ x,y \le c_2,\\ C_A(A(x)A(y))^{-1/2}\Big(\frac{|x-y|}{t}\Big)^{k+1}\frac{e^{-|x-y|^2/(4t)}}{|x-y|^{1/2}}, & c_1\sqrt{t} < |x-y| \le c_2 \text{ and } x,y > c_2, \text{ or } |x-y| > c_2,\end{cases}$$

where

$$\widetilde{\Delta}(x,y,t) = \min\{\sqrt{t}^{-2\alpha-1}, (A(x)A(\sqrt{t}))^{-1/2}, (A(y)A(\sqrt{t}))^{-1/2}, (A(x)A(y))^{-1/2}\}.$$

Proof. The lemma can be proved using a similar argument to the proofs of Lemma 3.29 and [BX2, Theorem 2.17]. $\blacksquare$

Using Lemma 4.12 and arguing as in the proof of Theorem 4.8 we immediately obtain the following result.

4.13. THEOREM. *Let a be a local (p,q)-atom. Then there is a positive constant $C_{A,p}$ independent of a such that $\|H_0^+ a\|_{p,A} \le C_{A,p}$.*

4.14. THEOREM. *Suppose that $\phi \in B_{s,n}(\mathbb{R}_+, *(A))$ with $s > 2/p+1$ where $0<p\le 1$ and $n \ge (2\alpha+2)/p+[2\alpha+2]+4$. Then the grand maximal operator $f \mapsto f_m^*$ (where $m > 2\alpha+3$) and the maximal operators H_0^+ and $M_{\phi,0}$ are bounded from $\mathbf{h}_{\mathrm{a}}^p$ to $L^p(\mathbb{R}_+, A\,dx)$.*

Proof. This follows readily from Corollary 3.40 and Theorems 4.8 and 4.13. ∎

We now show that the local Hardy spaces $\mathbf{h}^p$ coincide with the atomic Hardy spaces $\mathbf{h}_{\mathrm{a}}^p$. By Theorem 4.14 it suffices to give the atomic decomposition for each function f in $\mathbf{h}^p$. We start by presenting a useful covering lemma which is a variant of a classical result of Whitney on $\mathbb{R}^n$.

4.15. LEMMA (Whitney-type covering lemma). *Suppose that E is open in $\mathbb{R}_+$ such that $m(E)<\infty$. Then there exist $x_1, x_2, \ldots$ in E, positive numbers $r_1, r_2, \ldots$ and $N = N(m(E)) \in \mathbb{N}$ with $r_i = 1$ for $1 \le i \le N$ and $r_i < 1$ otherwise, satisfying the following conditions:*

(a) $E = \bigcup_{j=1}^{\infty} B(x_j, r_j)$,

(b) *the intervals $B(x_j, \frac14 r_j)$ are disjoint,*

(c) *if $i > N$ then $B(x_j, 18r_j) \cap E^c = \emptyset$, but $B(x_j, 54r_j)\cap E^c \ne \emptyset$,*

(d) *no point of E belongs to more than M of the intervals $B(x_j, 18r_j)$, where $M = M(A)$ is a positive constant depending only on A.*

Proof. The lemma can be proved following the proof of [FoS, (1.67)]. ∎

4.16. LEMMA. *Let $f \in \mathcal{S}_q'(\mathbb{R}_+, *(A))$ $(0<q\le 2)$. Then f_m^* is lower semicontinuous.*

Proof. For any $\gamma>0$, if $f_m^*(x) > \gamma$ then there exists $\psi \in K_m(x)$ with $\mathrm{supp}(\psi) \subset B(x,r)$ such that $|f(\psi)| > \gamma$. Let $0<\delta<\min\{r, (r_0-r)/2\}$. Then by (1.10) and (1.11) it is straightforward to verify that $T_\delta \psi \in K_m(x\pm\delta)$. Since $f \in \mathcal{S}_q'(\mathbb{R}_+, *(A))$ and $T_\delta\psi \to \psi$ in $\mathcal{S}_q(\mathbb{R}_+, *(A))$ we obtain $f(T_\delta\psi) \to f(\psi)$ as $\delta \to 0^+$. Hence there exists $\delta_0 > 0$ such that $f_m^*(y) > \gamma$ for all $y \in B(x, \delta_0)$. ∎

Fix $f \in \mathbf{h}^p$ $(0<p\le 1)$ and for any $k \in \mathbb{Z}$ put

$$\Omega_k := \{x \in \mathbb{R}_+ : f_m^*(x) > 2^k\}.$$

In the sequel we write from time to time $\int_0^\infty f(x)\psi(x)A(x)\,dx$ instead of (f,ψ) for $f \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$ and $\psi \in \mathcal{S}_1'(\mathbb{R}_+, *(A))$. By Lemma 4.16 and Theorem 3.9, Ω_k is open and, since $f_m^* \in L^p(\mathbb{R}_+, A\,dx)$,

$$m(\Omega_k) \le 2^{-pk}\|f_m^*\|_{p,A}^p. \tag{4.17}$$

We apply Lemma 4.15 to these sets Ω_k $(k\in\mathbb{Z})$ obtaining (x_i^k) in Ω and a sequence of positive numbers $r_i^k \le 1$ satisfying (a)–(d) in Lemma 4.15. Choose once for all an even C^∞-function θ on $\mathbb{R}$ such that $\mathrm{supp}(\theta) \subset [-2,2]$, $0 \le \theta \le 1$ and $\theta(x) = 1$ for $x \in [-1,1]$, and set $\theta_i^k(x) = \theta((x-x_i^k)/r_i^k)$. Then $\theta_i^k \in \mathcal{D}_*(\mathbb{R}_+)$, $\mathrm{supp}(\theta_i^k) \subset B(x_i^k, 2r_i^k)$, $\theta_i^k(x) = 1$ for $x \in B(x_i^k, r_i^k)$ and

$$\left|\frac{d^j}{dx^j}\theta_i(x)\right| \le C_j r_i^{-j}, \quad j = 0,1,\ldots,$$

where C_j depends only on j. Set

$$\zeta_i^k(x) := \begin{cases} \dfrac{\theta_i^k(x)}{\sum_j \theta_j^k(x)}, & x\in\Omega,\\ 0 & \text{otherwise.}\end{cases}$$

By Lemma 4.15 we have

$$\begin{gathered}\zeta_i^k \in \mathcal{D}_*(\mathbb{R}_+) \ \text{ if } r_i^k < 1, \quad \mathrm{supp}(\zeta_i^k) \subset B(x_i^k, 2r_i^k),\\ 0 \le \zeta_i^k \le 1, \quad \zeta_i^k(x) = 1 \ \text{ on } B(x_i^k, r_i^k), \quad \sum_i \zeta_i^k(x) = \chi_\Omega(x).\end{gathered} \tag{4.18}$$

For each pair i,k if $r_i^k<1$ we let $\mathcal{P}_{d,i,k}$ denote the space of polynomials on $\mathbb{R}_+$ of degree $\le d$ with the Hilbert space norm

$$\|P\| := \Big(\int_0^\infty \zeta_i^k(y)A(y)\,dy\Big)^{-1}\int_0^\infty |P(x)|^2\zeta_i^k(x)A(x)\,dx$$

and denote by P_i^k the projection of f into $\mathcal{P}_{d,i,k}$, that is,

$$\int_0^\infty f(y)Q(y)\zeta_i^k(y)A(y)\,dy = \int_0^\infty P_i^k(y)Q(y)\zeta_i^k(y)A(y)\,dy, \quad Q \in \mathcal{P}_{d,i,k}. \tag{4.19}$$

If $r_i^k = 1$ we define $P_i^k = 0$. Furthermore, for each i,k we define

$$b_i^k = (f-P_i^k)\zeta_i^k \quad\text{and}\quad g_k = f - \sum_i b_i^k. \tag{4.20}$$

4.21. LEMMA. *There is a constant C_A independent of f, i, k such that*

$$|P_i^k(y)| \le C_A 2^k, \quad y \in B(x_i^k, 2r_i^k).$$

Proof. Let $\pi_1, \ldots, \pi_L$ $(L = \dim \mathcal{P}_{d,i,k})$ be an orthonormal basis for $\mathcal{P}_{d,i,k}$. Then by (1.2), (4.18) and Lemma 1.9,

$$\begin{aligned} 1 &= \Big(\int_0^\infty \zeta_i^k(y)A(y)\,dy\Big)^{-1}\int_0^\infty |\pi_l(y)|^2\zeta_i^k(x)A(x)\,dx \\ &\ge C_A\frac{1}{|B(x_i^k, r_i^k)|}\int_{B(x_i^k, r_i^k)} |\pi_l(y)|^2 A(y)\,dy\end{aligned} \tag{4.22}$$

$$\geq \begin{cases} C_A \int\limits_0^{1/2} |\pi_l(r_i^k y + x_i^k)|^2 y^{2\alpha+1}\,dy, & x_i^k \leq r_i^k, \\ C_A \int\limits_0^{1/2} |\pi_l(r_i^k y + x_i^k)|^2\,dy, & x_i^k > r_i^k. \end{cases}$$

Since $\mathcal{P}_{d,i,k}$ is finite-dimensional it is easy to see from (4.22), using the equivalence of the L^2 and L^∞ norms, that

$$\sup_{y\in B(x_i^k,2r_i^k)} |\pi_l(y)| \leq \sup_{0\leq u\leq 2} |\pi_l(r_i^k u + x_i^k)| \leq C_A, \tag{4.23}$$

and applying Bernstein's inequality

$$\begin{aligned} |\pi_l^{(m)}(y)| &\leq C_m(r_i^k)^{-m} \sup_{-2\leq u\leq 2} |\pi_l(x_i^k + r_i^k u)| \\ &\leq C_m(r_i^k)^{-m}, \quad y \in B(x_i^k, 2r_i^k). \end{aligned} \tag{4.24}$$

In view of (4.19) we have, for $r_i^k < 1$,

$$P_i^k(y) = \sum_{l=1}^{L} \Big(\int\limits_0^\infty f(x)\phi_i^k(x)\pi_l(x)A(x)\,dx \Big)\pi_l(y) \tag{4.25}$$

where

$$\phi_i^k(x) = \Big(\int\limits_0^\infty \zeta_i^k(u)A(u)\,du \Big)\zeta_i^k(x).$$

By Lemma 4.15 we can take $z \in B(x_i^k, 54r_i^k) \cap \Omega_k^{\mathrm{c}}$. Thus using (4.18), (4.23) and (4.24) a straightforward calculation shows that $\phi_i^k \in K_m(z)$, and hence by (4.23) and (4.25),

$$|P_i^k(y)| \leq C_A f_m^*(z) \leq C_A 2^k. \ ■$$

Now by analogy with P_i^k we define P_{ij}^{k+1} to be the orthogonal projection of $(f - P_j^{k+1})\zeta_i^k$ on $\mathcal{P}_{d,j,k+1}$ if $r_j^{k+1} < 1$ and $P_{ij}^{k+1} = 0$ otherwise. The following result follows from a similar argument to the proof of Lemma 4.21.

4.26. LEMMA. $|P_{ij}^{k+1}| \leq C_A 2^k$.

4.27. LEMMA. *There exists $k_0 > 0$ such that for any $k > k_0$ the series $\sum_i b_i^k$ converges in* $\mathbf{h}^p$ *and*

$$\int\limits_0^\infty \Big(\sum_i b_i^k \Big)_m^*(x)^p A(x)\,dx \leq C_A \int\limits_{\Omega_k} f_m^*(x)^p A(x)\,dx.$$

Proof. We first note that $r^{2\alpha+2} \leq C_A|B(x,r)|$ if $r \leq 1$. Hence by (4.17) there exists $k_0 > 0$ such that for any $k > k_0$, $r_i^k < 1$ for all i. In the sequel we assume $k > k_0$.

Let $B(x; a, b) = \{y \in \mathbb{R}_+ : a < |y - x| < b\}$. We claim that

$$(b_i^k)_m^*(x) \leq \begin{cases} C_A f_m^*(x), & x \in B(x_i^k, 2r_i^k), \\ C_A(f_m^*(x) + 2^k)\chi_{B(x_i^k; 2r_i^k, 18r_i^k)}(x) & \\ \quad + C_A 2^k \Big(\dfrac{r_i^k}{|x - x_i^k|} \Big)^{d+1} \chi_{B(x_i^k; 18r_i^k, 2)}(x), & x \notin B(x_i^k, 2r_i^k). \end{cases} \tag{4.28}$$

In fact, if $x \in B(x_i^k, 2r_i^k)$ then by Lemma 1.9 and (4.18) it is straightforward to verify $\psi\zeta_i^k \in K_m(x)$ for any $\psi \in K_m(x)$, and hence (4.28) follows from (4.20) and Lemma 4.21.

Now we prove (4.28) for $x \notin B(x_i^k, 2r_i^k)$. If $x \in B(x_i^k, 18r_i^k)$ then for any $\psi \in K_m(x)$ we can use Lemma 1.9 and (4.18) to get $\psi\zeta_i^k \in K_m(x)$. Thus (4.28) follows from (4.20) and Lemma 4.21. Assume now $x \notin B(x_i^k, 18r_i^k)$ and $\psi \in K_m(x)$. We need only consider the case where $r > 16r_i^k$ and $x \in B(x_i^k; 18r_i^k, r + 2r_i^k)$; otherwise by (4.18), $\psi\zeta_i^k = 0$. For $r > 16r_i^k$ and $x \in B(x_i^k; 18r_i^k, r + 2r_i^k)$ let $P_{d,x}$ be the Taylor polynomial of ψ at x_i^k of degree d and $R_{d,x} = \psi - P_{d,x}$. Then the integral form of the remainder of the Taylor expansion gives

$$R_{d,x}^{(j)}(u) = \begin{cases} \Big(\int\limits_0^1 (1-y)^{d-j}\psi^{(d+1)}((u - x_i^k)y + x_i^k)\,dy \Big)(u - x_i^k)^{d-j+1}, & j \leq d, \\ \psi^{(j)}(u), & j > d. \end{cases}$$

Thus by (3.6),

$$|R_{d,x}^{(j)}(u)| \leq \begin{cases} C_A \dfrac{|u - x_i^k|^{d+1-j}}{|x - x_i^k|^{d+1}|B(x,r)|}, & j \leq d, \\ C_A \dfrac{(r_i^k)^{d+1-j}}{|x - x_i^k|^{d+1}|B(x,r)|}, & d+1 \leq j \leq m. \end{cases} \tag{4.29}$$

Let

$$\Phi(u) = \frac{|x - x_i^k|^{d+1}|B(x,r)|}{(r_i^k)^{d+1-j}} R_{d,x}(u)\phi_i^k(u)$$

where $\phi_i^k = (\int_0^\infty \zeta_i^k(v)A(v)\,dv)^{-1}\zeta_i^k$. By Lemma 4.15 we can choose $z \in B(x_i^k, 54r_i^k) \cap \Omega_k^{\mathrm{c}}$. Then applying Lemma 1.9, (4.18) and (4.29) we can verify $\Phi \in K_m(z)$. Therefore by (4.19), (4.20), (4.18) (4.29) and Lemmas 1.9 and 4.21 we obtain, for $x \in B(x_i^k; 18r_i^k, r + 2r_i^k)$,

$$\begin{aligned} |(b_i^k, \psi)| &= |(b_i^k, R_{d,x})| \leq |(f\zeta_i^k, R_{d,x})| + |(P_i^k\zeta_i^k, \psi)| \\ &= \frac{(r_i^k)^{d+1-j}}{|x - x_i^k|^{d+1}|B(x,r)|} \Big(\int\limits_0^\infty \zeta_i^k(v)A(v)\,dv \Big) |(f, \Phi)| \end{aligned}$$

$$+\Big|\int_0^\infty P_i^k(u)\zeta_i^k(u)R_{d,x}(u)A(u)\,du\Big| \le C_A 2^k (r_i^k)^{d+1}|x-x_i^k|^{-d-1}.$$

This gives for $x \notin B(x_i^k, 18r_i^k)$,

$$(4.30)\qquad (b_i^k)_m^*(x) \le 2^k (r_i^k)^{d+1}|x-x_i^k|^{-d-1}\chi_{B(x_i^k;18r_i^k,2)}(x),$$

which completes the proof of (4.28).

We now apply (4.28) to prove the lemma. For any i we write

$$\begin{aligned}\int_0^\infty (b_i^k)_m^*(x)^p A(x)\,dx &= \int_{B(x_i^k,2r_i^k)} (b_i^k)_m^*(x)^p A(x)\,dx \\ &\quad + \int_{(B(x_i^k,2r_i^k))^c} (b_i^k)_m^*(x)^p A(x)\,dx \\ &:= I_{i,1} + I_{i,2}.\end{aligned}$$

By (4.29) we immediately obtain

$$I_{i,1} \le C_{A,p} \int_{B(x_i^k,18r_i^k)} f_m^*(x)^p A(x)\,dx$$

and

$$\begin{aligned}I_{i,2} &\le C_{A,p} \int_{B(x_i^k;2r_i^k,18r_i^k)} (f_m^*(x)^p + 2^{pk})A(x)\,dx \\ &\quad + C_{A,p} \int_{B(x_i^k;18r_i^k,2)} f_m^*(x)^p A(x)\,dx \\ &:= I_{i,2}^{(1)} + I_{i,2}^{(2)}.\end{aligned}$$

By Lemma 4.15 we have $B(x_i^k, 18r_i^k) \subset \Omega_k$. Hence

$$\begin{aligned}I_{i,2}^{(1)} &\le C_{A,p} \int_{B(x_i^k,18r_i^k)} f_m^*(x)^p A(x)\,dx + C_{A,p}2^{pk}|B(x_i^k,18r_i^k)| \\ &\le C_{A,p} \int_{B(x_i^k,18r_i^k)} f_m^*(x)^p A(x)\,dx.\end{aligned}$$

A straightforward calculation using Lemma 1.9 gives

$$I_{i,2}^{(2)} \le C_{A,p}2^{pk}|B(x_i^k,2r_i^k)| \le C_{A,p} \int_{B(x_i^k,18r_i^k)} f_m^*(x)^p A(x)\,dx.$$

Consequently, by Lemma 4.15 we obtain

$$\begin{aligned}(4.31)\qquad \sum_i \int_0^\infty (b_i^k)_m^*(x)^p A(x)\,dx &\le C_{A,p} \sum_i \int_{B(x_i^k,18r_i^k)} f_m^*(x)^p A(x)\,dx \\ &\le C_{A,p} \int_{\Omega_k} f_m^*(x)^p A(x)\,dx.\end{aligned}$$

The result now follows from (4.31) and Theorem 4.3. ■

4.32. LEMMA. *There exists $k_0 > 0$ such that for any $k > k_0$, $(g_k)_m^* \in L^1(\mathbb{R}_+, Adx)$ and*

$$\|(g_k)_m^*\|_{1,A} \le C_{A,p} 2^{(1-p)k} \|f_m^*\|_{p,A}^p.$$

Proof. Let $F_{ik}(x) = 2^k (r_i^k)^{d+1}|x-x_i^k|^{-d-1}\chi_{B(x_i^k;18r_i^k,2)}(x)$. If $x \in \Omega_k^c$ then by Lemma 4.15, $x \notin B(x_i^k, r_i^k)$ for all i, and hence applying (4.20) and (4.30) leads to the following estimate:

$$(4.33)\qquad (g_k)_m^*(x) \le f_m^*(x) + C_A \sum_i F_{ik}(x), \qquad x \in \Omega_k^c.$$

If $x \in \Omega_k$ then by Lemma 4.15, $x \in B(x_i^k, r_i^k)$ for some i. Let $J := J_i := \{j :\ B(x_i^k, 2r_i^k) \cap B(x_i^k, 18r_i^k) \ne \emptyset\}$. By Lemma 4.15, $\mathrm{card}(\psi) \le M$. In view of (4.30) we have

$$\sum_{j\notin J} (b_j^k)_m^*(x) \le C_A \sum_{j\notin J} F_{jk}(x).$$

For $f - \sum_{j\in J} b_j^k$ let $z \in B(x_i^k, 54r_i^k) \cap \Omega_k^c$ and $\psi \in K_m(x)$. Observe that $1 - \sum_{j\in J}\zeta_j^k = 0$ on $B(x_j^k, 18r_j^k)$ by Lemma 4.15 and hence $(1-\sum_{j\in J}\zeta_j^k)\psi = 0$ if $r \le 16r_j^k$. Thus if $r \le 16r_j^k$ we have by (4.20), (4.18) and Lemma 4.21,

$$\Big|\Big(f - \sum_{j\in J} b_j^k, \psi\Big)\Big| = \Big|\Big(\sum_{j\in J} P_j^k \zeta_j^k, \psi\Big)\Big| \le C_A 2^k.$$

If $r > 16r_j^k$ then by Lemma 1.9 we verify $\psi \in K_m(z)$. Thus

$$|(f,\psi)| \le f_m^*(z) \le 2^k$$

and by (4.30) and Lemma 4.15,

$$\Big|\Big(\sum_{j\in J} b_j^k, \psi\Big)\Big| \le \sum_{j\in J} (b_j^k)_m^*(z) \le C_A \sum_{j\in J} F_{jk}(z) \le C_A 2^k.$$

Consequently,

$$(4.34)\qquad (g_k)_m^*(x) \le C_A\Big(\sum_{j\notin J} F_{jk}(x) + 2^k\Big), \qquad x \in B(x_i^k, r_i^k).$$

Now we apply (4.33), (4.34), (4.17) and Lemma 4.15 to obtain

$$\int_0^\infty (g_k)_m^*(x)A(x)\,dx \le C_{A,p}\int_{\Omega_k^c} f_m^*(x)A(x)\,dx + C_{A,p}\sum_i \int_{\Omega_k^c} F_{ik}(x)A(x)\,dx$$
$$+ C_{A,p}\sum_i \int_{B(x_i^k,r_i^k)} \Big(2^k + \sum_{j\notin J} F_{jk}(x)\Big)A(x)\,dx$$
$$\le C_{A,p}2^{(1-p)k}\int_0^\infty f_m^*(x)^p A(x)\,dx + C_{A,p}2^k|B(x_i^k,r_i^k)|$$
$$+ C_{A,p}\sum_i \int_0^\infty F_{ik}(x)A(x)\,dx$$
$$\le C_{A,p}2^{(1-p)k}\int_0^\infty f_m^*(x)^p A(x)\,dx + C_{A,p}2^k|B(x_i^k,r_i^k)|$$
$$\le C_{A,p}2^{(1-p)k}\|f_m^*\|_{p,A}. \ ■$$

The following result is an immediate corollary of Theorem 3.7 and Lemmas 4.27 and 4.32.

4.35. THEOREM. *For $0 < p \le 1$, $\mathbf{h}^p \cap L^1(\mathbb{R}_+, Adx)$ is dense in $L^1(\mathbb{R}_+, Adx)$.*

Now using Lemmas 4.21 and 4.26 and Theorem 4.35 we can argue similarly to the case of euclidean spaces (see [FoS, Chapter 3] and [JSW]) to obtain

4.36. THEOREM. *For $0<p\le 1$, $\mathbf{h}^p \subset \mathbf{h}^p_{\mathrm{a}}$. Moreover, if $f \in \mathbf{h}^p$ then*

$$\|f\|_{\mathbf{h}^p_{\mathrm{a}}} \le C_{A,p}\|f\|_{\mathbf{h}^p}.$$

4.37. COROLLARY. *For $0 < p \le 1$ we have $\mathbf{h}^p = \mathbf{h}^p_{\mathrm{a}}$, and for $f \in \mathbf{h}^p$, $\|f\|_{\mathbf{h}^p} \sim \|f\|_{\mathbf{h}^p_{\mathrm{a}}}$.*

Proof. This is a consequence of Theorems 4.14 and 4.36. ■

The following result is immediate from Theorems 3.9, 3.25, 3.39 and 4.14 and Corollaries 3.40 and 4.37.

4.38. COROLLARY. *Suppose that $\phi \in B_{s,n}(\mathbb{R}_+, *(A))$ with $s > 2/p+1$ and $n \ge (2\alpha+2)/p + [2\alpha+2]+4$ where $0 < p \le 1$. Then the grand maximal operator $f \mapsto f_m^*$ $(m > 2\alpha+3)$ and the maximal operators H_0^+ and $M_{\phi,0}$ are bounded from $\mathbf{h}^p$ to $L^p(\mathbb{R}_+, Adx)$ and*

$$\|f_m^*\|_{p,A} \sim \|H_0^+ f\|_{p,A} \sim \|M_{\phi,0}f\|_{p,A} \sim \|\phi_\nabla^*(f)\|_{p,A} \sim \|f\|_{\mathbf{h}^p}.$$

References

[A] J.-Ph. Anker, *Sharp estimates for some functions of the Laplacian on noncompact symmetric spaces*, Duke Math. J. 65 (1992), 257–297.

[AT] A. Achour and K. Trimèche, *La g-fonction de Littlewood–Paley associée à un opérateur différentiel singulier sur* $(0,\infty)$, Ann. Inst. Fourier (Grenoble) 33 (1983), 203–226.

[BH] W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter Stud. Math. 20, Walter de Gruyter, Berlin, 1995.

[BX1] W. R. Bloom and Z. Xu, *The Hardy–Littlewood maximal function for Chébli–Trimèche hypergroups*, in: Applications of Hypergroups and Related Measure Algebras (Joint Summer Research Conference (AMS-IMS-SIAM), Seattle, 1993), Contemp. Math. 183, Amer. Math. Soc., Providence, R.I., 1995, 45–70.

[BX2] —, —, *Maximal functions on Chébli–Trimèche hypergroups*, preprint.

[BX3] —, —, *Hardy spaces on Chébli–Trimèche hypergroups*, Methods Funct. Anal. Topology 3 (1997).

[Cha] D. C. Chang, *Estimates for singular integral operators with mixed type homogeneities in Hardy classes*, Math. Ann. 287 (1990), 303–322.

[Ché] H. Chébli, *Sur un théorème de Paley–Wiener associé à la décomposition spectrale d'un opérateur de Sturm–Liouville sur* $]0,\infty[$, J. Funct. Anal. 17 (1974), 447–461.

[Coi] R. R. Coifman, *A real variable characterization of* H^p, Studia Math. 51 (1974), 269–274.

[CW] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 549–645.

[FeS] C. Fefferman and E. Stein, H^p *spaces of several variables*, Acta Math. 129 (1972), 137–193.

[FoS] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes 28, Princeton Univ. Press, Princeton, N.J., 1982.

[G] D. Goldberg, *A local version of real Hardy spaces*, Duke Math. J. 46 (1979), 27–42.

[GV] R. Gangolli and V. S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Ergeb. Math. Grenzgeb. 101, Springer, Berlin, 1988.

[JSW] A. Jonsson, P. Sjögren and H. Wallin, *Hardy and Lipschitz spaces on subsets of* $\mathbb{R}^n$, Studia Math. 80 (1984), 141–166.

[K1] T. Kawazoe, *Maximal functions on non-compact, rank one symmetric spaces*, in: Adv. Stud. Pure Math. 4, Kinokuniya, Tokyo, 1984, 121–138.

[K2] —, *Atomic Hardy spaces on semisimple Lie groups*, Japan. J. Math. 11 (1985), 293–343.

[Lan] R. E. Langer, *On the asymptotic solutions of ordinary differential equations, with reference to the Stokes' phenomenon about a singular point*, Trans. Amer. Math. Soc. 37 (1935), 397–416.

[Lat] R. H. Latter, *A characterization of* $H^p(\mathbb{R}^n)$ *in terms of atoms*, Studia Math. 62 (1978), 93–101.

[MS] R. A. Macias and C. Segovia, *A decomposition into atoms of distributions on spaces of homogeneous type*, Adv. Math. 33 (1979), 271–309.

[PS] L. Päivärinta and E. Somersalo, *A generalization of the Calderón–Vaillancourt theorem to* L^p *and* h^p, Math. Nachr. 138 (1988), 145–156.

[T] K. Trimèche, *Transformation intégrale de Weyl et théorème de Paley–Wiener associés à un opérateur différentiel singulier sur* $(0,\infty)$, J. Math. Pures Appl. (9) 60 (1981), 51–98.
[Z] H. Zeuner, *One-dimensional hypergroups*, Adv. Math. 76 (1989), 1–18.

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On strongly asymptotically developable functions and the Borel–Ritt theorem

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Abstract. We show that the holomorphic functions on polysectors whose derivatives remain bounded on proper subpolysectors are precisely those strongly asymptotically developable in the sense of Majima. This fact allows us to solve two Borel–Ritt type interpolation problems from a functional-analytic viewpoint.

Introduction. It is well known that, for a function f holomorphic on a sector S in the complex plane with vertex at 0, the existence of asymptotic expansion as the variable tends to 0 amounts to the boundedness of the derivatives of f on bounded proper subsectors of S. The Borel–Ritt theorem assures the existence of holomorphic functions on a given sector S admitting a prescribed asymptotic expansion at 0 in S. There are several classical proofs of this result in the literature (see, e.g., [Ol, Chapter 1, §9, p. 22], [Wa, Chapter III, §9.2, p. 43]). One of them (based on the ideas of [Ol, Chapter 4, §1.1, p. 106]; see Theorem 5.1 in this paper) has the particular feature that the derivatives of the solution are in fact bounded on unbounded proper subsectors of S. So, the Borel–Ritt interpolation problem is solvable in a different setting.

The aim of this paper is to transfer this characterization and results to the case of strongly asymptotically developable holomorphic functions of several complex variables, as defined by Majima [Ma]. To this end, Section 3 is devoted to the study of the space $\mathcal{A}(S)$ of holomorphic functions on a polysector S of $\mathbb{C}^n$ whose derivatives remain bounded in bounded proper subpolysectors of S; we give $\mathcal{A}(S)$ a natural Fréchet space topology, and prove that it is precisely the space of holomorphic functions on S strongly asymptotically developable at the origin. This equivalence allows us to obtain many properties of these functions in an elementary way. The main ideas in this section first appeared, for the Gevrey case, in the paper of Haraoka [Ha]; the results, in the present terms, come from the work of Hernández [He].

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