References

[1] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. 6 (1956), 405–439.
[2] C.-M. Cho, *The metric approximation property and intersection properties of balls*, J. Korean Math. Soc. 31 (1994), 467-475.
[3] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1984.
[4] M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. 21 (1975), 38–49.
[5] T. Figiel, *Factorization of compact operators and applications to the approximation problem*, Studia Math. 45 (1973), 191–210.
[6] G. Godefroy, N. J. Kalton and P. D. Saphar, *Unconditional ideals in Banach spaces*, ibid. 104 (1993), 13–59.
[7] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[8] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.
[9] H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.
[10] W. B. Johnson, *Factoring compact operators*, Israel J. Math. 9 (1971), 337–345.
[11] Å. Lima, *Uniqueness of Hahn–Banach extensions and liftings of linear dependences*, Math. Scand. 53 (1983), 97–113.
[12] —, *The metric approximation property, norm-one projections and intersection properties of balls*, Israel J. Math. 84 (1993), 451–475.
[13] —, *Property* (wM^*) *and the unconditional metric compact approximation property*, Studia Math. 113 (1995), 249–263.
[14] J. Lindenstrauss, *On a problem of Nachbin concerning extension of operators*, Israel J. Math. 1 (1963), 75–84.
[15] —, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964).
[16] —, *On projections with norm 1—an example*, Proc. Amer. Math. Soc. 15 (1964), 403–406.
[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Ergeb. Math. Grenzgeb. 92, Springer, Berlin, 1977.
[18] E. Oja and M. Põldvere, *On subspaces of Banach spaces where every functional has a unique norm-preserving extension*, Studia Math. 117 (1996), 289–306.
[19] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer, Berlin, 1993.
[20] W. M. Ruess and C. P. Stegall, *Extreme points in duals of operator spaces*, Math. Ann. 261 (1982), 535–546.
[21] I. Singer, *Bases in Banach Spaces II*, Springer, Berlin, 1981.

Department of Mathematics
Agder College
Tordenskjoldsgate 65
N-4604 Kristiansand, Norway
E-mail: Asvald.Lima@hia.no

Faculty of Mathematics
Tartu University
Vanemuise 46
EE-2400 Tartu, Estonia
E-mail: eveoja@math.ut.ee

Received May 11, 1998
Revised version August 24, 1998 (4101)

STUDIA MATHEMATICA 133 (2) (1999)

Toeplitz operators in the commutant of a composition operator

by

BRUCE CLOAD (St. Catharines, Ont.)

Abstract. If ϕ is an analytic self-mapping of the unit disc D and if $H^2(D)$ is the Hardy–Hilbert space on D, the composition operator C_ϕ on $H^2(D)$ is defined by $C_\phi(f) = f \circ \phi$. In this article, we consider which Toeplitz operators T_f satisfy $T_f C_\phi = C_\phi T_f$.

1. Introduction. Denote the unit disc in the complex plane by D and the boundary of the unit disc by ∂D. We will be working with the following spaces: $H^2(D)$, the Hardy–Hilbert space which is defined as the set of analytic functions on D with square summable power series coefficients, $H^\infty(D)$, the set of bounded analytic functions on D, $L^2(\partial D)$, the set of L^2 functions on the unit circle with respect to normalized Lebesgue measure, and $L^\infty(\partial D)$, the corresponding L^∞ space on the unit circle. For notational convenience, throughout the paper, we may abbreviate the symbols for these spaces as H^2, H^∞, L^2, and L^∞. For more information on these spaces see [9] and [11].

If ϕ is an analytic function on D, we may define a bounded operator on $H^2(D)$, for all f in H^2, by $C_\phi(f) = f \circ \phi$. These composition operators provide a large class of examples of operators on Hilbert spaces and, moreover, there are excellent single variable complex analysis techniques with which to analyze them. For more information on composition operators see [7] and [16].

The commutant of a composition operator, C_ϕ, is the weakly closed algebra generated by all operators that commute with C_ϕ. For a brief discussion on the commutant of a composition operator see [8]. In [3], we consider the question of how to generate the commutant of C_ϕ for some specific functions ϕ. In [4], we examine other aspects of the commutant of a composition operator. In [5], Carl Cowen showed that if f is a covering map of D onto a bounded domain in the complex plane, then the commutant of the Toeplitz

1991 *Mathematics Subject Classification*: Primary 47B38; Secondary 47B35, 47A05.

The author would like to thank Peter Rosenthal, Joe Repka, and Ruben Martinez for their helpful comments and suggestions on this article.

operator, T_f, is generated by composition operators induced by linear fractional transformations ϕ that satisfy $f \circ \phi = f$ and by Toeplitz operators.

In this paper, we are interested in classifying the Toeplitz operators that commute with C_ϕ. In the second section, Theorem 1 shows that the only Toeplitz operators that commute with a composition operator induced by an analytic self-map of the disc which is not an elliptic disc automorphism of finite periodicity must be analytic Toeplitz operators. The third section concerns which multiplication operators commute with a given composition operator. The most important results in that section are Theorems 5 and 6. Theorem 5 provides a full solution to this problem in the parabolic disc automorphism case. Theorem 6 provides a partial solution in the hyperbolic disc automorphism case.

2. Toeplitz operators. A *Toeplitz operator* is a special type of generalized multiplication operator. If $f \in L^\infty(\partial D)$ and P is the orthogonal projection of $L^2(\partial D)$ onto H^2 (under the standard identification of an H^2 function with its boundary function in $L^2(\partial D)$), then we may define an operator on H^2 by $T_f(g) = P(fg)$ for $g \in H^2$. If the symbol f of the Toeplitz operator is in H^∞, then such a Toeplitz operator is called *analytic* and it corresponds to the multiplication operator with symbol f on H^2.

Let $\phi : D \to D$ be an analytic mapping. Denote by $\phi^{[n]}$ the nth term in the sequence of iterates of ϕ under composition, that is

$$\phi^{[n]} = \underbrace{\phi \circ \ldots \circ \phi}_{n \text{ times}}.$$

Disc automorphisms may be classified into three types: elliptic, parabolic and hyperbolic. Elliptic disc automorphisms have one fixed point in D, parabolic one fixed point on ∂D, and hyperbolic two distinct fixed points on ∂D. By conjugation by a Möbius transformation, an elliptic disc automorphism is equivalent to a rotation about zero (Schwarz's Lemma). They may be sub-classified into elliptic disc automorphisms of finite periodicity ($\phi^{[n]} \equiv z$ for some natural number n) and of infinite periodicity ([16], pp. 5–6).

Let $\phi : D \to D$ be an analytic mapping which is neither an elliptic disc automorphism nor the identity mapping. The *Denjoy–Wolff point* of ϕ is the unique point in the closed unit disc to which the iterates of $\phi^{[n]}$ converge uniformly on compacta ([16], p. 78, and [7], p. 58). Denote the Denjoy–Wolff point of ϕ by a. We note that, for each fixed positive integer l, $(\phi^{[n]})^l$ converges weakly to a^l ([7], p. 3, Corollary 1.3). Also, if A commutes with C_ϕ then A commutes with C_ϕ^n and hence $C_{\phi^{[n]}}$. Thus one possible technique to glean information about the commutant is to iterate ϕ under composition.

THEOREM 1. *Let $\phi : D \to D$ be an analytic mapping which is neither an elliptic disc automorphism of finite periodicity nor the identity mapping. If $f \in L^\infty(\partial D)$ and T_f commutes with C_ϕ, then f must be analytic.*

Proof. The proof breaks into three parts depending on the nature of ϕ.

(I) First suppose that ϕ is neither an elliptic disc automorphism nor a constant. Let the Denjoy–Wolff point of ϕ be a. Let $f = \sum_{n=-\infty}^{\infty} c_n z^n$ and decompose f as $f_1 + f_2$ where $f_1 \in (H^2)^\perp$ and $f_2 \in H^2$. Then the fact that T_f commutes with C_ϕ implies

$$T_f C_{\phi^{[n]}}(z) = C_{\phi^{[n]}} T_f(z).$$

Expand the right hand side to conclude that

$$T_f(\phi^{[n]}) = (c_{-1} + f_2 z) \circ \phi^{[n]}.$$

Note that $C_\phi T_f(1) = T_f C_\phi(1)$ implies $f_2 \circ \phi = f_2$ and thus

$$T_f(\phi^{[n]}) = c_{-1} + (f_2)\phi^{[n]}.$$

Taking the inner product of this equation with 1 yields

$$\langle T_f(\phi^{[n]}), 1\rangle = c_{-1} + f_2(0)\phi^{[n]}(0).$$

This is equivalent to

$$\langle \phi^{[n]}, T_f^*(1)\rangle = c_{-1} + f_2(0)\phi^{[n]}(0).$$

By the remark previous to this theorem, $\langle \phi^{[n]}, T_f^*(1)\rangle$ converges to $\langle a, T_f^*(1)\rangle = \langle af_2, 1\rangle = af_2(0)$. By the Denjoy–Wolff Theorem, $c_{-1} + f_2(0)\phi^{[n]}(0)$ converges to $c_{-1} + af_2(0)$. Thus $c_{-1} = 0$. Assume by strong induction that $c_{-1} = \ldots = c_{-l+1} = 0$ and consider $T_f(z^l)$ in the above argument. This results in the equation $T_f((\phi^{[n]})^l) = c_{-l} + (f_2)(\phi^{[n]})^l$. By taking the inner product with 1 and passing to the weak limit as above, we get $a^l f_2(0) = c_{-l} + a^l f_2(0)$. Thus $c_{-l} = 0$. Hence by induction, $c_{-n} = 0$ for all $n \ge 1$; that is, f is analytic.

(II) Let $\phi(z) = b$ for all $z \in D$, where $|b| < 1$. Let $f \in L^\infty(\partial D)$ with $f = f_1 + f_2$ where $f_1 \in (H^2)^\perp$ and $f_2 \in H^2$. First, consider $T_f C_\phi(1) = C_\phi T_f(1)$, which implies $f_2 = f_2(b)$. Thus f_2 is a constant; let $f_2 = c$. For any $g \in H^2$, $T_f C_\phi(g) = C_\phi T_f(g)$. After expanding both sides of this equation, we get $cg(b) = (P(f_1 g))(b) + cg(b)$, so $(P(f_1 g))(b) = 0$. In particular, take g to be successively z^k, where k is a natural number, to conclude that the negative Fourier coefficients of f are zero. Thus $f = f_2$, so f is a constant function, and T_f is an analytic Toeplitz operator.

(III) Let ϕ be an elliptic automorphism of infinite periodicity. First, assume that the fixed point is 0; then by Schwarz's Lemma, $\phi(z) = e^{i\theta} z$ where $e^{in\theta} \ne 1$ for all integers $n \ne 0$. A direct computation yields that if $AC_\phi = C_\phi A$, then A is an operator represented by a diagonal matrix with respect to the standard basis, $\{z^k : k = 0, 1, \ldots\}$, and every such operator must commute with C_ϕ. If T_f is a Toeplitz operator which commutes with C_ϕ, then since T_f must have constant diagonals (in the standard basis), f must be a constant function. Let the fixed point of ϕ be $b \neq 0$. Let $\alpha(z) = (b - z)/(1 - \bar{b}z)$, and note that α is a self inverse. Suppose that T_f commutes with C_ϕ where $f = f_1 + f_2$ with $f_1 \in (H^2)^\perp$ and $f_2 \in H^2$. $T_f C_\phi(1) = C_\phi T_f(1)$ implies that $f_2 = f_2 \circ \phi$ and since ϕ has infinite periodicity, f_2 is a constant. Thus f_1 induces a Toeplitz operator which commutes with C_ϕ. We claim $f_1 = 0$. Now T_{f_1} commutes with C_ϕ implies $C_\alpha T_{f_1} C_\alpha$ commutes with $C_\alpha C_\phi C_\alpha$ but $C_\alpha C_\phi C_\alpha = C_{\alpha\circ\phi\circ\alpha}$ and $\alpha\circ\phi\circ\alpha$ is an elliptic disc automorphism of infinite periodicity with fixed point 0. Thus $T_{f_1} = C_\alpha D C_\alpha$ where D is a diagonal matrix in the standard basis. Let the kth diagonal entry of D be λ_k. Represent $g = D(\alpha)$ by

$$g(z) = \lambda_0 b + \sum_{k=1}^{\infty} \lambda_k (\bar{b})^{k-1}(|b|^2 - 1)z^k.$$

Since $T_{f_1}(z)$ is a constant, we see that $g \circ \alpha$ is a constant and hence g is a constant; this implies $\lambda_k = 0$ for $k \geq 1$. Also, $\lambda_0 = 0$, because $T_{f_1}(1) = 0$. Thus $D = 0$ and hence $f_1 = 0$. ∎

As the next result shows, the previous theorem cannot be extended to all elliptic disc automorphisms.

THEOREM 2. *Let ϕ be an elliptic disc automorphism of period q $(q \geq 2)$ with $\phi(0) = 0$. Then T_f commutes with C_ϕ if and only if the Fourier expansion of f is of the form $\sum_{n=-\infty}^{\infty} a_{nq} z^{nq}$.*

Proof. Since ϕ is an elliptic disc automorphism of period q, we have $\phi(z) = e^{i\theta} z$ with $\theta = 2\pi\frac{p}{q}$ where p is an integer, q is a natural number, and $\gcd(p, q) = 1$. Note that $e^{2\pi i pn/q} = 1$ if and only if $q \mid n$. Let $f = \sum_{n=-\infty}^{\infty} a_n z^n = f_1 + f_2$ with $f_1 \in (H^2)^\perp$ and $f_2 \in H^2$. Note that $T_f C_{e^{2\pi ip/q}z}(1) = C_{e^{2\pi ip/q}z} T_f(1)$ implies

$$f_2(z) = f_2(e^{2\pi ip/q} z).$$

Thus f_2 is q-fold symmetric and hence $f_2 = \sum_{n=0}^{\infty} a_{nq} z^{nq}$. Similarly,

$$T_f C_{e^{2\pi ip/q}z}(z) = C_{e^{2\pi ip/q}z} T_f(z)$$

implies that

$$c_{-1} e^{2\pi ip/q} + z e^{2\pi ip/q} f_2(z) = c_{-1} + z e^{2\pi ip/q} f_2(z)$$

which implies that $c_{-1} = 0$. For n such that $q - n$ assume by strong induction that if $m < n$ and $q - m$ then $c_{-m} = 0$. Then

$$T_f C_{e^{2\pi ip/q}z}(z^n) = C_{e^{2\pi ip/q}z} T_f(z^n)$$

implies

$$a_{-n} e^{2\pi pn/q} + \sum_{0<m<n,\, q|m} a_{-m} z^{n-m} e^{2\pi ipn/q} + e^{2\pi ipn/q} z^n f_2(z)$$

is equal to

$$a_{-n} + \sum_{0<m<n,\, q|m} a_{-m} z^{n-m} e^{2\pi ipn/q} + e^{2\pi ipn/q} z^n f_2(z).$$

Comparing terms, we conclude that $a_{-n} = 0$. Thus $f = \sum_{n=-\infty}^{\infty} a_{nq} z^{nq}$. Conversely, if f is q-fold symmetric in its Fourier coefficients and if $f \in L^\infty(\partial D)$, then by straightforward calculation T_f commutes with $C_{e^{2\pi ip/q}z}$. ∎

3. Multiplication operators. A reasonable conjecture is that the commutant of a composition operator C_ϕ is the weakly closed algebra generated by the Toeplitz operators which commute with C_ϕ and the composition operators which commute with C_ϕ. In [3], we show that this conjecture is not true for certain examples, but we believe the question is still of interest in many cases. Carl Cowen in [6] characterizes for a given function f (not a conformal automorphism of D) which functions g commute with it under composition. In Section 1, we showed, except for elliptic disc automorphisms of finite periodicity, that the Toeplitz operators which commute with C_ϕ must be analytic. In order to emphasize this, we denote T_f by M_f if $f \in H^\infty$. We will now determine which multiplication operators commute in some special cases of C_ϕ. For $f \in H^\infty$, M_f commutes with C_ϕ is equivalent to $f \circ \phi = f$. Note that if α is a disc automorphism and f a function in H^∞, then $C_\alpha M_f C_{\alpha^{-1}} = M_{f\circ\alpha}$.

THEOREM 3. *Let ϕ be an elliptic disc automorphism with fixed point b. If ϕ is of infinite periodicity, then a multiplication operator M_f commutes with C_ϕ if and only if f is a constant. If ϕ is of period q, then a multiplication operator, M_f, commutes with C_ϕ if and only if $f \in H^\infty$ and f is of the form $\sum_{n=0}^{\infty} a_{nq}((b - z)/(1 - \bar{b}z))^{nq}$.*

Proof. Apply Theorem 1 (part III) and the remark preceding this theorem for the infinite periodic case; apply Theorem 2 and the same remark for the finite periodic case. ∎

THEOREM 4. *If $\phi : D \to D$ is an analytic mapping with an interior fixed point, and also if ϕ is neither an elliptic disc automorphism nor the identity mapping, then M_f commutes with C_ϕ implies that f is constant.*

Proof. Suppose $\phi(a) = a$ and $f \circ \phi = f$; then for any $z_0 \in D$, $f(\phi^{[n]}(z_0)) = f(z_0)$ for all natural numbers n. Now since, by the Denjoy–Wolff Theorem, $\phi^{[n]}(z_0)$ converges to a as n tends to ∞, it follows that $f(z_0) = f(a)$ for every $z_0 \in D$; thus f is constant. ∎

If (a_l) is a sequence of non-zero complex numbers in the unit disc such that $\sum_{l=1}^{\infty}(1-|a_l|)<\infty$, then the infinite product

$$\prod_{n=1}^{\infty}\frac{\overline{a}_n}{|a_n|}\frac{a_n-z}{1-\overline{a}_n z}$$

converges uniformly on compact subsets of D to a holomorphic function called a *Blaschke product* ([7], p. 32, or [16], p. 181).

Now suppose that ϕ is a parabolic disc automorphism with Denjoy–Wolff point 1, or a hyperbolic disc automorphism with fixed points 1 and -1 where 1 is the Denjoy–Wolff point. Define

$$\phi^{[0]}=z \quad\text{and}\quad \phi^{[-n]}=\underbrace{\phi^{-1}\circ\ldots\circ\phi^{-1}}_{n\text{ times}}.$$

Let $z_n=\phi^{[n]}(0)$. In [14], pp. 332–333, the authors prove that

$$B(z)=z\prod_{n=-\infty}^{\infty}\frac{\overline{z}_n}{|z_n|}\frac{z_n-z}{1-\overline{z}_n z}\quad(n\neq 0)$$

is a convergent Blaschke product (note that the index in the product is two-sided infinite) and that $B\circ\phi=B$ for such a parabolic disc automorphism, and $B\circ\phi=-B$ for such a hyperbolic disc automorphism.

THEOREM 5. *Let ϕ be a parabolic disc automorphism with Denjoy–Wolff point 1 and let B be the Blaschke product defined above. Let $f\in H^2$. Then $f\circ\phi=f$ if and only if $f=\sum_{k=0}^{\infty}a_kB^k$ where $\sum_{k=0}^{\infty}|a_k|^2<\infty$.*

Proof. Let M be the subspace of H^2 spanned by $\{B^k : k=0,1,\ldots\}$. By definition,

$$\langle B^k,B^j\rangle=\frac{1}{2\pi}\int_0^{2\pi}B^k(e^{i\theta})\overline{B}^j(e^{i\theta})\,d\theta,$$

which, when $k\geq j$, equals

$$\frac{1}{2\pi}\int_0^{2\pi}B^{k-j}(e^{i\theta})d\theta=\langle B^{k-j},1\rangle,$$

and when $j\geq k$, equals

$$\frac{1}{2\pi}\int_0^{2\pi}\overline{B}^{j-k}(e^{i\theta})d\theta=\langle 1,B^{j-k}\rangle.$$

Since $B(0)=0$, B^k and B^j are orthogonal when $k\neq j$. Thus

$$M=\Big\{\sum_{n=0}^{\infty}a_nB^n:\sum_{n=0}^{\infty}|a_n|^2<\infty\Big\}.$$

We want to show that if $f\circ\phi=f$, then $f\in M$. Let $P:H^2\to M$ be the orthogonal projection onto M and let $Q:H^2\to M^{\perp}$ be the orthogonal projection onto $M^{\perp}$. Decompose f with respect to these projections to get $f=P(f)+Q(f)$. Now if $g\in M$ then $g\circ\phi=g$ (since $B\circ\phi=B$); thus $P(f)\circ\phi=P(f)$. Also, since $f\circ\phi=f$, we have $Q(f)\circ\phi=Q(f)$, which also implies $Q(f)\circ\phi^{-1}=Q(f)$. The constant functions are a subspace of M, so $(Q(f))(0)=0$ and thus $(Q(f))(\phi^{[n]}(0))=(Q(f))(0)=0$ for all integers n. Hence B divides $Q(f)$ in the H^2 sense that B must be a factor of the inner part of $Q(f)$. Thus $Q(f)=Bh$ for some $h\in H^2$. For $i\geq 1$, $\langle B^i,Q(f)\rangle=0$ implies $\langle B^i,Bh\rangle=0$, which implies $\langle B^{i-1},h\rangle=0$, so $h\perp M$. Since, moreover, $Q(f)\circ\phi=Q(f)$ and $B\circ\phi=B$, it follows that $h\circ\phi=h$. Repeating the process on h, we conclude that $B\mid h$. By induction, $B^k\mid Q(f)$ for any arbitrary natural number k. Thus $z^k\mid Q(f)$ for all k, which implies $Q(f)=0$; thus $f\in M$. ∎

COROLLARY 1. *If ϕ is a parabolic disc automorphism with Denjoy–Wolff point 1, and $f\in H^{\infty}$, then M_f commutes with C_ϕ if and only if $f=\sum_{n=0}^{\infty}a_nB^n$ where $\sum_{n=0}^{\infty}|a_n|^2<\infty$.*

COROLLARY 2. *If ϕ is a parabolic disc automorphism with Denjoy–Wolff point 1, and if h is an eigenfunction associated with eigenvalue λ for C_ϕ, then*

$$h=\frac{\sum_{n=0}^{\infty}a_nB^n}{f_{\overline{\lambda}}}$$

for some a_n such that $\sum_{n=0}^{\infty}|a_n|^2<\infty$ and $f_{\overline{\lambda}}=e^{(idb^{-1}\gamma(z))}$ where $\gamma(z)=i(1+z)/(1-z)$, b is a real number determined by ϕ, and d is a real number of the same sign as b such that $e^{id}=\overline{\lambda}$. Conversely, if $h\circ\phi=h$, then $f_\lambda h$ is an eigenfunction for eigenvalue λ where f_λ is defined in a similar manner to $f_{\overline{\lambda}}$.

Proof. The following construction can be found in [13], p. 447, Theorem 5. Define $\gamma(z)=i(1+z)/(1-z)$. This is a conformal mapping of the unit disc onto the upper half plane. Let $\delta=\gamma\circ\phi\circ\gamma^{-1}$. Since δ has one and only one fixed point in the upper half plane, and is a conformal map, it must be of the form $\delta(w)=w+b$ for b a real number and $b\neq 0$. Now the set of eigenvalues for C_ϕ is ∂D ([13], p. 447, Theorem 5) so let $\mu\in\partial D$ and let $d\in[-2\pi,2\pi]$ with $e^{id}=\mu$ and $\operatorname{sign}(d)=\operatorname{sign}(b)$. Define

$$f_\mu=e^{(idb^{-1}\gamma(z))}.$$

Now

$$|f_\mu(z)|=e^{(-db^{-1}\operatorname{Im}(\gamma(z)))}\leq 1$$

and thus $f_\mu\in H^{\infty}$. Now $\gamma\circ\phi=\delta\circ\gamma=\gamma+b$ so

$$C_\phi(f_\mu)=e^{(idb^{-1}\gamma\circ\phi)}=e^{(id+idb^{-1}\gamma)}=\mu f_\mu.$$

This procedure holds for any μ in ∂D, in particular, for $\overline{\lambda}$ and λ. If $h\circ\phi=\lambda h$ where $h\in H^2$, $\lambda\in\partial D$, $\lambda\neq 1$, then $f_{\overline{\lambda}}h\in H^2$ (because $f_{\overline{\lambda}}\in H^\infty$) and $(f_{\overline{\lambda}}h)\circ\phi=|\lambda|^2 f_{\overline{\lambda}}h=f_{\overline{\lambda}}h$. Thus $f_{\overline{\lambda}}h=\sum_{n=0}^{\infty}a_nB^n$ for some square summable complex numbers a_n by Theorem 5. Note that $f_{\overline{\lambda}}$ is never zero on D even though $|f_{\overline{\lambda}}(z)|$ tends to 0 as z tends to 1. Thus $1/f_{\overline{\lambda}}$ is not bounded on D but it is an analytic function on D. Hence

$$h=\frac{\sum_{n=0}^{\infty}a_nB^n}{f_{\overline{\lambda}}}$$

as analytic functions on D but $h\in H^2$ so $(\sum_{n=0}^{\infty}a_nB^n)/f_{\overline{\lambda}}$ is in H^2. Conversely, if $h\circ\phi=h$, then $(f_\lambda h)\circ\phi=\lambda f_\lambda h$ and hence $f_\lambda h$ is an eigenvector with eigenvalue λ. ■

If ϕ is a parabolic disc automorphism that has Denjoy–Wolff point $e^{i\theta}\neq 1$, then we may conjugate C_ϕ by $\alpha(z)=e^{-i\theta}z$, so that $\alpha^{-1}\circ\phi\circ\alpha$ has Denjoy–Wolff point 1. Thus if $f\in H^2$ is such that $f\circ\phi=f$, then f must be of the form $g\circ\alpha^{-1}$ where g belongs to the subspace M as defined in the proof of Theorem 5 with C_ϕ replaced by $C_{\alpha^{-1}\circ\phi\circ\alpha}$.

It is not so easy to deal with the hyperbolic case. There is a pesky negative sign which occurs when B is composed with ϕ. For hyperbolic composition operators, the fact that $B\circ\phi=-B$ rather than $B\circ\phi=B$ has also caused problems in other results relying on the Blaschke product ([12]). Here we provide a partial classification of the solution to $f\circ\phi=f$ for ϕ a hyperbolic disc automorphism.

THEOREM 6. *Let ϕ be a hyperbolic disc automorphism with Denjoy–Wolff point* 1. *For each integer $i\geq 0$, let B_i be the Blaschke product formed from the iterates of* 0 *under the hyperbolic disc automorphism $\phi^{[2^i]}$. If $f\circ\phi=f$, then $f=\sum_{k=0}^{\infty}a_kB_0\ldots B_k$, where $\sum_{k=0}^{\infty}|a_k|^2<\infty$.*

Proof. First of all, since $B_0\ldots B_k$ divides $B_0\ldots B_l$ as inner functions when $k\leq l$ and each B_k for every k has a factor of z, by following the integral norm procedure in Theorem 5, we conclude $B_0\ldots B_k$ is perpendicular to $B_0\ldots B_l$ when $k\neq l$.

Since B_k for any k is not fixed by ϕ under composition, a slightly different technique is required than in the parabolic case. Let $a_0=f(0)$ and consider $g_1=f-a_0$. Now $g_1\circ\phi=g_1$ since f and a_0 are fixed by ϕ under composition. Thus g_1 is zero at all points $\phi^{[k]}(0)$ and hence $B_0|g$. Hence f may be represented as $a_0+B_0f_1$. Now since $B_0\circ\phi=-B_0$ and $f\circ\phi=f$, we have $f_1\circ\phi=-f_1$. We cannot work with this directly; however, $f_1\circ\phi\circ\phi=-f_1\circ\phi=f_1$; thus f_1 is fixed by $\phi^{[2]}$. Replacing ϕ by $\phi^{[2]}$ we may, using the above procedure, work with f_1 and write f_1 as $a_1+B_1f_2$. Thus $f=a_0+a_1B_0+B_0B_1f_2$; here f_2 is fixed by $\phi^{[4]}$. By continuing this

procedure f may be written at the Nth step as

$$f=\sum_{k=0}^{N}a_kB_0\ldots B_k+B_0\ldots B_{N+1}R_N$$

where R_N is a remainder term. Since $\{B_0\ldots B_j : j=0,1,\ldots\}$ is an orthogonal set of inner functions, $\|f\|^2=\langle f,f\rangle=\sum_{k=0}^{N}|a_k|^2+\|B_0\ldots B_{N+1}R_N\|^2$. Hence $\sum_{k=0}^{\infty}|a_k|^2<\infty$. Let $R=f-\sum_{k=0}^{\infty}a_kB_0\ldots B_k$; thus

$$R-B_0\ldots B_{N+1}R_N=\sum_{k=N+1}^{\infty}a_kB_0\ldots B_k.$$

As N tends to ∞, $\|R-B_0\ldots B_{N+1}R_N\|^2=\sum_{k=N+1}^{\infty}|a_k|^2$ tends to 0. Hence $B_0\ldots B_{N+1}R_N$ converges to R in H^2. Since $B_0\ldots B_{N+1}R_N$ has an $(N+1)$-fold zero at 0, we conclude that $R=0$ and f has the desired form. ■

The first question of interest is: how big is the span of $\{B_0\ldots B_k : k=0,1,\ldots\}$? Since none of the $B_0\ldots B_k$ for any k is fixed by ϕ under composition, the span contains more than those functions that satisfy $f\circ\phi=f$. Denote by k_α the reproducing kernel function at α in D. Every function of the form $zk_{\phi^{[n]}(0)}$, where $n\neq 0$, is perpendicular to that span since such functions are perpendicular to constants and

$$\frac{B_0}{z}(\phi^{[n]}(0))=0.$$

Hence the span is not all of H^2.

COROLLARY 3. *Let ϕ be a hyperbolic disc automorphism with fixed points* 1 *and* -1 *where* 1 *is the Denjoy–Wolff point. If $f\in H^\infty$, and M_f commutes with C_ϕ, then $f=\sum_{n=0}^{\infty}a_nB_0\ldots B_n$ with $\sum_{n=0}^{\infty}|a_n|^2<\infty$.*

Second: could B_0 in Theorem 6 be replaced by another function to give a necessary and sufficient condition? By Theorem 7.21 in [7], p. 227, for each $\theta\in[0,2\pi]$ there is a function h_θ such that $h_\theta\circ\phi=e^{i\theta}h_\theta$ and, moreover, both h_θ and $1/h_\theta$ are in H^∞. In place of B_0, one could use the function $h_\pi B_0$, since this is fixed by ϕ. The problem with using this function is that it is not an inner function and the estimates for the powers of the norms become unbounded. There is something that h_θ does contribute to the problem, as follows:

If C_ϕ is a hyperbolic composition operator induced by a disc automorphism with fixed points 1 *and* -1 *with the Denjoy–Wolff point at* 1, *then $f\in H^2$ is an eigenvector associated with eigenvalue $e^{i\theta}$ if and only if $f=h_\theta g$ where g is an eigenvector associated with eigenvalue* 1.

If ϕ is a hyperbolic disc automorphism with fixed points other than -1 and 1, we may conjugate it by a disc automorphism so that it has fixed points 1 and -1, with 1 the Denjoy–Wolff point ([14], p. 332). Thus the above results can be translated to apply to any composition operator induced by a hyperbolic disc automorphism.

References

[1] D. F. Behan, *Commuting analytic functions without fixed points*, Proc. Amer. Math. Soc. 37 (1973), 114–120.
[2] R. B. Burckel, *Iterating analytic self-maps of discs*, Amer. Math. Monthly 88 (1981), 396–407.
[3] B. Cload, *Generating the commutant of a composition operator*, in: Contemp. Math. 213, Amer. Math. Soc., 1998, 11–15.
[4] —, *Composition operators: hyperinvariant subspaces, quasi-normals, and isometries*, Proc. Amer. Math. Soc., to appear.
[5] C. C. Cowen, *The commutant of an analytic Toeplitz operator*, Trans. Amer. Math. Soc. 239 (1978), 1–31.
[6] —, *Commuting analytic functions*, ibid. 265 (1981), 69–95.
[7] C. C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, 1995.
[8] —, —, *Some problems on composition operators*, in: Contemp. Math. 213, Amer. Math. Soc., 1998, 17–25.
[9] P. Duren, *Theory of H^p Spaces*, Academic Press, 1970.
[10] M. H. Heins, *A generalization of the Aumann–Carathéodory "Starrheitssatz"*, Duke Math. J. 8 (1941), 312–316.
[11] K. Hoffman, *Banach Spaces of Analytic Functions*, Dover, 1988.
[12] P. Hurst, *A model for the invertible composition operators on H^2*, Proc. Amer. Math. Soc. 124 (1996), 1847–1856.
[13] E. A. Nordgren, *Composition operators*, Canad. J. Math. 20 (1968), 442–449.
[14] E. A. Nordgren, P. Rosenthal and F. S. Wintrobe, *Invertible composition operators on H^p*, J. Funct. Anal. 73 (1987), 324–344.
[15] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, 1987.
[16] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, New York, 1993.
[17] A. L. Shields, *On fixed points of commuting analytic functions*, Proc. Amer. Math. Soc. 15 (1964), 703–706.
[18] A. L. Shields and L. J. Wallen, *The commutants of certain Hilbert space operators*, in: Indiana Univ. Math. Surveys 13, Amer. Math. Soc., Providence, 1974, 49–128.

Department of Mathematics
Brock University
St. Catharines, Ontario
Canada L2S 3A1
E-mail: bcload@spartan.ac.brocku.ca

Received June 1, 1998
Revised version August 28, 1998 (4116)

INFORMATION FOR AUTHORS

Manuscripts should be typed on one side only, with double or triple spacing and wide margins, and submitted in duplicate, including the original typewritten copy.

An **abstract** of not more than 200 words and the AMS Mathematics Subject Classification are required.

Figures must be prepared in a form suitable for direct reproduction. Sending EPS, PCX, TIF or CorelDraw files will be most helpful. The author should indicate on the margin of the manuscript where figures are to be inserted.

References should be arranged in alphabetical order, typed with double spacing, and styled and punctuated according to the examples given below. Abbreviations of journal names should follow Mathematical Reviews. Titles of papers in Russian should be translated into English.

Examples:

[6] D. Beck, *Introduction to Dynamical Systems*, Vol. 2, Progr. Math. 54, Birkhäuser, Basel, 1978.
[7] R. Hill and A. James, *An index formula*, J. Differential Equations 15 (1982), 197–211.
[8] J. Kowalski, *Some remarks on $J(X)$*, in: Algebra and Analysis (Edmonton, 1973), E. Brook (ed.), Lecture Notes in Math. 867, Springer, Berlin, 1974, 115–124.
[Nov] A. S. Novikov, *An existence theorem for planar graphs*, preprint, Moscow University, 1980 (in Russian).

Authors' **affiliation** should be given at the end of the manuscript.

Authors receive only page **proofs** (one copy). If the proofs are not returned promptly, the article will be printed in a later issue.

Authors receive 50 **reprints** of their articles. Additional reprints can be ordered.

The publisher strongly encourages submission of manuscripts written in TEX. **On acceptance of the paper,** authors will be asked to send discs (preferably PC) plus relevant details to the Editorial Committee, or transmit the file by electronic mail to:

STUDIA@IMPAN.GOV.PL

Recommended format of manuscripts:

\magnification=\magstep1	\documentstyle[12pt,leqno]{article}
\hsize=30truecc	\textwidth=30cc
\baselineskip=16truept	\baselineskip=16pt
(Plain TEX or AMS-TEX)	(LATEX)

Home page: http://www.impan.gov.pl/PUBL/sm.html