[12] J. L. Taylor, *Homology and cohomology for topological algebras*, Adv. in Math. 9 (1972), 137–182.
[13] —, *A general framework for a multi-operator functional calculus*, ibid., 183–252.

Faculty of Mechanics and Mathematics
Moscow State University
Moscow 119899 GSP
Russia
E-mail: pirkosha@glasnet.ru

Received April 30, 1998 (4094)

STUDIA MATHEMATICA 133 (2) (1999)

Ideals of finite rank operators, intersection properties of balls, and the approximation property

by

ÅSVALD LIMA (Kristiansand) and EVE OJA (Tartu)

Abstract. We characterize the approximation property of Banach spaces and their dual spaces by the position of finite rank operators in the space of compact operators. In particular, we show that a Banach space E has the approximation property if and only if for all closed subspaces F of c_0, the space $\mathcal{F}(F,E)$ of finite rank operators from F to E has the n-intersection property in the corresponding space $\mathcal{K}(F,E)$ of compact operators for all n, or equivalently, $\mathcal{F}(F,E)$ is an ideal in $\mathcal{K}(F,E)$.

1. Introduction. A subspace Y of a Banach space X is called an *ideal* if there exists a norm one projection P on X^* with $\ker P = Y^\perp$ (the annihilator of Y in X^*). A subspace Y of a Banach space X is said to have the *n-intersection property* for some natural number n if for all families $\{B_X(a_i,r_i)\}_{i=1}^n$ of closed balls in X whose centers a_i lie in Y,

$$\bigcap_{i=1}^n B_X(a_i,r_i) \neq \emptyset \quad \text{implies} \quad \bigcap_{i=1}^n B_Y(a_i,r_i+\varepsilon) \neq \emptyset \text{ for all } \varepsilon > 0.$$

Let E and F be Banach spaces. We denote by $\mathcal{L}(E,F)$ the Banach space of all bounded linear operators from E to F, and by $\mathcal{K}(E,F)$, $\mathcal{F}(E,F)$, and $\overline{\mathcal{F}}(E,F)$ its subspaces of compact, finite rank, and approximable operators (i.e. norm limits of finite rank operators).

In [12], Å. Lima characterized the metric approximation property of Banach spaces and their dual spaces by the position of finite rank operators in the space of bounded linear operators. It was proved e.g. that (cf. [12, Theorem 13]) for a Banach space E having the Radon–Nikodým property, the following assertions are equivalent:

(a) E has the metric approximation property,

1991 *Mathematics Subject Classification*: Primary 46B20; Secondary 46B28.

(b) $\mathcal{F}(F, E)$ is an ideal in $\mathcal{L}(F, E)$ for all Banach spaces F,

(c) $\mathcal{F}(F, E)$ has the n-intersection property in $\mathcal{L}(F, E)$ for all n and all Banach spaces F.

The purpose of the present work is to characterize the approximation property of Banach spaces and their dual spaces by the position of finite rank operators in the space of compact operators. It will be shown e.g. that (cf. Theorem 5.1) for a Banach space E, the following assertions are equivalent:

(a) E has the approximation property,

(b) $\mathcal{F}(F, E)$ is an ideal in $\mathcal{K}(F, E)$ for all closed subspaces F of c_0,

(c) $\mathcal{F}(F, E)$ has the n-intersection property in $\mathcal{K}(F, E)$ for all n and all closed subspaces F of c_0.

The approximation property for E^* will be characterized similarly (cf. Theorem 5.2).

Our main Theorems 5.1 and 5.2 are formulated and proved in the final Section 5, where we also present an elementary (maybe the shortest) proof for the classical fact that E^* has the approximation property if and only if $\overline{\mathcal{F}}(E, F) = \mathcal{K}(E, F)$ for all Banach spaces F. Section 5 is based on results from Sections 2–4. In Section 2, we briefly deal with the notions of ideal and n-intersection property, and with the relation between them. In Sections 3 and 4, using different methods, we investigate when the n-intersection property of finite rank operators implies that compact operators are approximable. In particular, we show that the n-intersection property of $\mathcal{F}(E, F)$ in $\mathcal{K}(E, F)$ for all n implies the equality $\overline{\mathcal{F}}(E, F) = \mathcal{K}(E, F)$ whenever E^* is separable (Theorem 3.1), or one of the spaces E or F is reflexive (Corollary 4.6).

Let us fix some more notation. We consider normed spaces over the real field $\mathbb{R}$. (Note that the proofs hold with some modifications for the complex case as well.) In a normed space E, we denote the unit sphere by S_E, the closed unit ball by B_E, and the closed ball with center x and radius r by $B_E(x, r)$. For a set $A \subset E$, its norm closure is denoted by $\overline{A}$, its linear span by $\operatorname{span} A$, its convex hull by $\operatorname{conv} A$, and the set of its extreme points by $\operatorname{ext} A$. The set of all weak* strongly exposed points of B_{E^*} is denoted by $w^*\text{-sexp}\, B_{E^*}$ (for this notion, as well as the notion of weak* denting points, cf. e.g. [19]). For two Banach spaces E and F, we denote by $\mathcal{K}_{w^*}(E^*, F)$ the subspace of $\mathcal{K}(E^*, F)$ consisting of compact operators which are weak*-weakly continuous. It is easily seen that $\mathcal{K}_{w^*}(E^*, F) = \{T \in \mathcal{K}(E^*, F) : \operatorname{ran} T^* \subset E\}$. The tensor product $E \otimes F$ is always canonically identified with a subspace of $\mathcal{F}(E^*, F)$; in particular, $E^* \otimes F = \mathcal{F}(E, F)$. Some further notation will be given in subsequent sections before use.

2. Ideals and the n-intersection property for all n. In this section, we briefly discuss a general situation related to the notions of ideals and subspaces having the n-intersection property for all n.

Let X be a Banach space, and let Y be a subspace of X. Recall that Y is an *ideal* in X if there exists a norm one projection P on X^* with $\ker P = Y^\perp$. This notion was introduced by G. Godefroy, N. J. Kalton and P. D. Saphar [6] in 1993. Their paper contains an important study of so-called unconditional ideals. Note that there is an extensive literature on a special subclass of (unconditional) ideals, namely on M-ideals (cf. the monograph [8] by P. Harmand, D. Werner and W. Werner). It can be shown (cf. e.g. [12, Theorem 1]) that Y is an ideal in X if and only if it is *locally 1-complemented*, i.e. for every finite-dimensional subspace E of X and for every $\varepsilon > 0$, there exists an operator $T : E \to \overline{Y}$ with $\|T\| \le 1 + \varepsilon$ and $Tx = x$ for all $x \in E \cap \overline{Y}$.

Recall also that Y has the *n-intersection property* (n-IP) in X for all n if for all finite families $\{B_X(a_i, r_i)\}_{i=1}^n$ of balls in X whose centers belong to Y,

$$\bigcap_{i=1}^n B_X(a_i, r_i) \ne \emptyset \quad \text{implies} \quad \bigcap_{i=1}^n B_Y(a_i, r_i + \varepsilon) \ne \emptyset \text{ for all } \varepsilon > 0.$$

This notion was studied by Å. Lima [11] in 1983. But already in 1956, N. Aronszajn and P. Panitchpakdi [1, p. 423] showed that Y has the n-IP in X for all n whenever there is a norm one projection from X onto Y. And in 1964, it was proved by J. Lindenstrauss [15, p. 59] that every Banach space has the n-IP in its bidual for all n.

The following result clarifies the relation between the two notions we are interested in. Its proof will be postponed until the end of the section.

PROPOSITION 2.1. *Let Y be a subspace of a Banach space X.*

(a) *Y has the n-IP in X for all n if and only if Y is an ideal in every closed subspace $Z \subset X$ with $Y \subset Z$ and $\dim Z/Y = 1$.*

(b) *Y is an ideal in X if and only if Y is an ideal in every closed subspace $Z \subset X$ with $Y \subset Z$ and $\dim Z/Y < \infty$.*

Clearly, every ideal has the n-IP for all n. The converse is not true. J. Lindenstrauss [14, p. 78] has given an example of a 2-dimensional subspace Y of a 4-dimensional Banach space X which is not an ideal in X, but which is an ideal in every 3-dimensional subspace Z of X containing Y. (The authors are grateful to M. I. Kadec for pointing out this example.) An interesting example of an infinite-dimensional ideal Y in a Banach space X with $\dim X/Y = 2$ is also due to J. Lindenstrauss (cf. [16]): for this Y, there is no norm one projection from X onto Y, but there is a norm one projection onto Y from every Z with $X \supset Z \supset Y$ and $\dim Z/Y = 1$.

The next lemma, which should be essentially known (its proof uses simple arguments from [15, pp. 60, 51]), will be used in the proof of Proposition 2.1 as well as in Section 4.

Recall that a *Hahn–Banach extension operator* is a linear operator $\Phi : Y^* \to X^*$ such that Φy^* is a norm-preserving extension of y^* for all $y^* \in Y^*$.

LEMMA 2.2. *Let X be a Banach space, and let Y be a subspace of X with $\dim X/Y = 1$. If Y has the n-IP in X for all n, and $x \in B_X \setminus Y$, then there exists $y^{**} \in B_{Y^{**}}$ such that $\Phi : Y^* \to X^*$ defined by*

$$(\Phi y^*)(y + \alpha x) = y^*(y) + \alpha y^{**}(y^*), \quad y \in Y, \ \alpha \in \mathbb{R},$$

is a Hahn–Banach extension operator.

Proof. The n-IP for all n implies that for all $\varepsilon > 0$, and all finite families $\{y_i\}_{i=1}^n$ in Y containing 0,

$$B_Y(0, 1+\varepsilon) \cap \bigcap_{i=1}^n B_X(y_i, \|x - y_i\| + \varepsilon) \neq \emptyset.$$

Hence clearly

$$j^{**}(B_{Y^{**}}(0, 1+\varepsilon)) \cap \bigcap_{i=1}^n B_{X^{**}}(y_i, \|x - y_i\| + \varepsilon) \neq \emptyset,$$

where $j : Y \to X$ is the inclusion mapping. By the weak* compactness of $B_{X^{**}}(0,2)$ (for example), there exists $y^{**} \in B_{Y^{**}}$ such that

$$j^{**}(y^{**}) \in \bigcap_{y \in Y} B_{X^{**}}(y, \|x - y\|).$$

A straightforward verification shows that Φ given by y^{**} is a Hahn–Banach extension operator. ■

Proof of Proposition 2.1. (a) It is obvious from the definition that Y has the n-IP in X for all n if and only if Y has the n-IP in every $Z \subset X$ with $Y \subset Z$ and $\dim Z/Y = 1$. In the last case, the Hahn–Banach extension operator $\Phi : Y^* \to Z^*$ from Lemma 2.2 defines a norm one projection $P = \Phi j^*$ on Z^*, where $j : Y \to Z$ is the inclusion mapping, with $\ker P = Y^\perp$. This means that Y is an ideal in Z.

(b) Clearly, Y is locally 1-complemented in X if and only if Y is locally 1-complemented in every $Z \subset X$ with $Y \subset Z$ and $\dim Z/Y < \infty$. ■

3. Intersection properties of balls for finite rank operators in the space of compact operators.

The following theorem, which is a key result for the final Section 5, shows that if F^* is separable, then $\overline{\mathcal{F}}(F,E)$ cannot be a proper subspace of $\mathcal{K}(F,E)$ having the n-IP for all n.

THEOREM 3.1. *Let E and F be Banach spaces. If F^* is separable and $\mathcal{F}(F,E)$ has the n-IP in $\mathcal{K}(F,E)$ for all n, then $\overline{\mathcal{F}}(F,E) = \mathcal{K}(F,E)$.*

Before proceeding to the proof, we introduce the following notation. Let X be a Banach space, and let Y be a subspace of X. For $y^* \in Y^*$, denote by $\mathrm{HB}(y^*)$ the set of norm-preserving extensions of y^* to X. A straightforward verification gives the following well-known characterization for extending functionals defined on Y with $\dim X/Y = 1$ (cf. e.g. [15, p. 51]).

LEMMA 3.2. *Let X be a Banach space, and let Y be a subspace of X with $\dim X/Y = 1$. If $y^* \in S_{Y^*}$ and $x \in X \setminus Y$, then*

$$\mathrm{HB}(y^*) = \Big\{ x_t^* : t \in \bigcap_{y \in Y} B_{\mathbb{R}}(y^*(y), \|x - y\|) \Big\}$$

where

$$x_t^*(y + \alpha x) = y^*(y) + \alpha t, \quad y \in Y, \ \alpha \in \mathbb{R}.$$

Moreover, $x_t^* = x_s^*$ if and only if $t = s$.

Proof of Theorem 3.1. Suppose that $T \in \mathcal{K}(F,E) \setminus \overline{\mathcal{F}}(F,E)$. We shall construct a separable subspace $G \subset E$ and a sequence $(T_n)_{n=1}^\infty \subset F^* \otimes G$ such that $T \in \mathcal{K}(F,G)$, and $T_n \to T$ weakly. This will imply that $T \in \overline{\mathrm{conv}}\{T_1, T_2, \ldots\} \subset \overline{\mathcal{F}}(F,G) \subset \overline{\mathcal{F}}(F,E)$, a contradiction.

Put $E_1 = \overline{T(F)}$. Then $T \in \mathcal{K}(F, E_1)$, and E_1 is separable. Let $(S_n^1)_{n=1}^\infty$ be a dense sequence in $F^* \otimes E_1$. Assume that subspaces $E_1, \ldots, E_k \subset E$ and operators $(S_n^k)_{n=1}^\infty \subset F^* \otimes E_k$ are defined. Since $F^* \otimes E$ has the n-IP for all n, there exists

$$U_n^k \in \bigcap_{i=1}^n B_{F^* \otimes E}(S_i^k, \|T - S_i^k\| + 1/n), \quad n \in \mathbb{N}.$$

Put $E_{k+1} = \overline{\mathrm{span}}(E_k \cup \bigcup_{n=1}^\infty U_n^k(F))$. Then E_{k+1} is separable, $E_k \subset E_{k+1}$, and $U_n^k(F) \subset E_{k+1}$ for all n. Define

$$S_1^{k+1} = S_1^1, \ \ldots, \ S_k^{k+1} = S_k^1;$$
$$S_{k+1}^{k+1} = S_2^2, \ \ldots, \ S_{2k-1}^{k+1} = S_k^2;$$
$$S_{2k}^{k+1} = S_3^3, \ \ldots, \ S_{3k-3}^{k+1} = S_k^3;$$
$$\ldots$$
$$S_{k(k+1)/2}^{k+1} = S_k^k.$$

Further, choose $S_{k(k+1)/2+1}^{k+1}, S_{k(k+1)/2+2}^{k+1}, \ldots \in F^* \otimes E_{k+1}$ so that $(S_n^{k+1})_{n=1}^\infty$ is a dense sequence in $F^* \otimes E_{k+1}$. Note that, by the construction above, for all S_j^m, there is an n_0 such that

$$\forall n \geq n_0, \ \exists i \leq n(n+1)/2, \quad S_j^m = S_i^{n+1}.$$

Let $G = \overline{\bigcup_{k=1}^{\infty} E_k}$. Then G is a separable subspace of E, and $T \in \mathcal{K}(F,G)$. Let $T_n = U_{n(n+1)/2}^{n+1}$, $n \in \mathbb{N}$. Since $T_n(F) \subset E_{n+2} \subset G$,

$$(3.1)\quad T_n \in \bigcap_{i=1}^{n(n+1)/2} B_{F^*\otimes G}(S_i^{n+1}, \|T - S_i^{n+1}\| + 2/(n(n+1))), \qquad n \in \mathbb{N}.$$

Now we show that $\psi(T_n) \to \phi(T)$ for all $\psi \in S_{(F^*\otimes G)^*}$ and $\phi \in \mathcal{K}(F,G)^*$ satisfying $\mathrm{HB}(\psi) = \{\phi\}$. By Lemma 3.2,

$$\bigcap_{S \in F^*\otimes G} B_{\mathbb{R}}(\psi(S), \|T - S\|) = \{\phi(T)\}.$$

This implies that, for given $\varepsilon > 0$, there are $S, S' \in F^* \otimes G$ such that

$$|\phi(T) + \varepsilon - \psi(S)| > \|T - S\|, \qquad |\phi(T) - \varepsilon - \psi(S')| > \|T - S'\|.$$

Since $G = \overline{\bigcup_{n=1}^{\infty} E_n}$ and $F^* \otimes E_n = \overline{\{S_1^n, S_2^n, \ldots\}}$, there are S_j^{n+1} and S_k^{n+1} such that

$$(3.2)\qquad |\phi(T) + \varepsilon - \psi(S_j^{n+1})| > \|T - S_j^{n+1}\| \geq |\phi(T) - \psi(S_j^{n+1})|,$$

$$(3.3)\qquad |\phi(T) - \varepsilon - \psi(S_k^{n+1})| > \|T - S_k^{n+1}\| \geq |\phi(T) - \psi(S_k^{n+1})|.$$

As was mentioned above, we can assume that there is some n_0 so that (3.2) and (3.3) are valid for all $n \geq n_0$ and some $j = j(n)$, $k = k(n)$ satisfying $j, k \leq n(n+1)/2$. If moreover $2/(n(n+1)) \leq \varepsilon$, then by (3.1),

$$(3.4)\qquad |\psi(T_n) - \psi(S_j^{n+1})| \leq \|T - S_j^{n+1}\| + \varepsilon,$$

$$(3.5)\qquad |\psi(T_n) - \psi(S_k^{n+1})| \leq \|T - S_k^{n+1}\| + \varepsilon.$$

Inequalities (3.2)–(3.5) yield that

$$|\psi(T_n) - \phi(T)| < 2\varepsilon.$$

Hence, $\psi(T_n) \to \phi(T)$.

It remains to show that $T_n \to T$ weakly in $\mathcal{K}(F,G)$. As (T_n) is bounded (cf. (3.1) and recall that $S_1^{n+1} = S_1^1$), by Rainwater's theorem, it is sufficient that $\phi(T_n) \to \phi(T)$ for all $\phi \in \operatorname{ext} B_{\mathcal{K}(F,G)^*}$. But $\operatorname{ext} B_{\mathcal{K}(F,G)^*} = \{f^{**} \otimes g^* : f^{**} \in \operatorname{ext} B_{F^{**}},\ g^* \in \operatorname{ext} B_{G^*}\}$ by a result of W. M. Ruess and C. P. Stegall [20] (cf. also [8, p. 266]), and $\phi|_{F^*\otimes G} = f^{**} \otimes g^* \in \operatorname{ext} B_{(F^*\otimes G)^*}$ for $\phi = f^{**} \otimes g^* \in \mathcal{K}(F,G)^*$ with $f^{**} \in \operatorname{ext} B_{F^{**}}$, $g^* \in \operatorname{ext} B_{G^*}$. Since every $\psi \in \operatorname{ext} B_{(F^*\otimes G)^*}$ has a unique norm-preserving extension to $\mathcal{K}(F,G)$ (cf. [12, Lemma 10]), we have $\mathrm{HB}(\phi|_{F^*\otimes G}) = \{\phi\}$ for all $\phi \in \operatorname{ext} B_{\mathcal{K}(F,G)^*}$. As was proved above, this implies $\phi(T_n) \to \phi(T)$. ∎

4. On cases when proper closed subspaces cannot have the n-IP for all n. One of such cases was demonstrated by Theorem 3.1 of the previous section. For a more general context, the proof of Theorem 3.1 essentially yields the following result (from which it seems to be impossible to derive Theorem 3.1).

PROPOSITION 4.1. *Let X be a Banach space, and let Y be a separable subspace of X having the n-IP for all n. Then, for all $x \in X$, there exists a bounded sequence (y_n) in Y such that $y^*(y_n) \to x^*(x)$ for all $y^* \in Y^*$ and $x^* \in X^*$ satisfying $\mathrm{HB}(y^*) = \{x^*\}$. If moreover*

$$\operatorname{ext} B_{X^*} = \{x^* \in X^* : \exists y^* \in \operatorname{ext} B_{Y^*},\ \mathrm{HB}(y^*) = \{x^*\}\},$$

then $\overline{Y} = X$.

For non-separable subspaces $Y \subset X$, the next result can be proved. It follows from [6, Lemma 2.2]. For the sake of completeness, we present below an independent proof.

PROPOSITION 4.2. *Let X be a Banach space, and let Y be a subspace of X having the n-IP for all n. Then for all $x \in B_X$, there exists a net (y_λ) in B_Y such that $y^*(y_\lambda) \to x^*(x)$ for all $y^* \in Y^*$ and $x^* \in X^*$ satisfying $\mathrm{HB}(y^*) = \{x^*\}$. If moreover*

$$X^* = \overline{\operatorname{span}}\{x^* \in X^* : \exists y^* \in Y^*,\ \mathrm{HB}(y^*) = \{x^*\}\},$$

then $\overline{Y} = X$.

Proof. It is sufficient to consider $x \in B_X \setminus Y$. By Lemma 2.2, there exists $y^{**} \in B_{Y^{**}}$ such that $\Phi : Y^* \to (Y + \mathbb{R}\{x\})^*$, defined by

$$(\Phi y^*)(y + \alpha x) = y^*(y) + \alpha y^{**}(y^*),$$

is a Hahn–Banach extension operator. Using Goldstine's theorem, pick a net $(y_\lambda) \subset B_Y$ weak* convergent to y^{**}. Then $y^*(y_\lambda) \to y^{**}(y^*) = (\Phi y^*)(x) = x^*(x)$. The second assertion now follows immediately because then $y_\lambda \to x$ weakly in X. ∎

Proposition 4.2 as well as Lemma 4.3 will be used to prove Theorem 4.4 below, which is a key result for our final Section 5.

LEMMA 4.3. *Let E and F be Banach spaces. Consider $E \otimes F$ as a subspace of $\mathcal{L}(E^*, F)$. If e^* is a weak* denting point of B_{E^*}, and $f^* \in F^*$, then $f^* \otimes e^* \in (E \otimes F)^*$ has a unique norm-preserving extension to $\mathcal{L}(E^*, F)$ (which equals $f^* \otimes e^* \in \mathcal{L}(E^*, F)^*$).*

Proof. The proof is similar to that of [13, Lemma 3.4(a)]. We can assume $\|f^*\| = 1$. Consider any norm-preserving extension ψ of $f^* \otimes e^* \in (E \otimes F)^*$ to $\mathcal{L}(E^*, F)$ and $T \in \mathcal{L}(E^*, F)$. The equality $\psi(T) = f^*(Te^*)$ can be shown essentially as in the above-mentioned proof if the operators S and U it uses are defined in the following manner. Let $\varepsilon > 0$, and choose $\delta \in (0, \varepsilon)$ and $e \in E$ such that $e^*(e) = 1$, $\|e\| \leq 1 + \delta\varepsilon$, and $\|e^* - x^*\| \leq \varepsilon$ whenever

$x^* \in B_{E^*}$ and $x^*(e) > 1-\delta$. Then choose $f \in F$ such that $f^*(f) = 1$ and $\|f\| \le 1+\delta\varepsilon$, and define $S = e \otimes f$ and $U = e \otimes Te^*$. ■

REMARK. Applying Lemma 4.3 to F^* instead of F (respectively, to E^* instead of E), and using the canonical isometric isomorphism between $\mathcal{L}(E^*, F^*)$ and $\mathcal{L}(F, E^{**})$, one arrives at the following improvement of [13, Lemma 3.4(a) (respectively (b))].

(a) If e^* is a weak* denting point of B_{E^*}, and $f^{**} \in F^{**}$, then $f^{**} \otimes e^* \in \mathcal{F}(F,E)^*$ has a unique norm-preserving extension to $\mathcal{L}(F, E^{**})$.

(b) If e^{**} is a weak* denting point of $B_{E^{**}}$, and $f^* \in F^*$, then $e^{**} \otimes f^* \in \mathcal{F}(E,F)^*$ has a unique norm-preserving extension to $\mathcal{L}(E^{**}, F)$.

Recall for the following that a Banach space E is said to have *property U in its bidual* E^{**} if every $x^* \in E^*$ has a unique norm-preserving extension to E^{**}. (For a recent study of such spaces cf. [18].)

THEOREM 4.4. *If one of Banach spaces E or F has property U in its bidual, and $E \otimes F$ has the n-IP in $\mathcal{K}_{w^*}(E^*, F)$ for all n, then $\overline{E \otimes F} = \mathcal{K}_{w^*}(E^*, F)$.*

Proof. Suppose that E has property U in E^{**}. Then E is an Asplund space (equivalently, E^* has the Radon–Nikodým property), and $B_{E^*} = \overline{\mathrm{conv}}(w^*\text{-sexp}\, B_{E^*})$ (cf. e.g. [8, pp. 125–127] for a proof). Since E is an Asplund space, it can be proved similarly to [4, Theorem 1] that the transformation V from the completed projective tensor product $F^* \widehat{\otimes}_\pi E^*$ to $(\mathcal{K}_{w^*}(E^*, F))^*$ defined by $V(u)(T) = \sum f_i^*(Te_i^*)$ (for $u = \sum f_i^* \otimes e_i^* \in F^* \widehat{\otimes}_\pi E^*$ and $T \in \mathcal{K}_{w^*}(E^*, F)$) is a quotient map. Since $Vu = u$ for all $u \in F^* \otimes E^*$, this easily implies that $(\mathcal{K}_{w^*}(E^*, F))^* = \overline{F^* \otimes E^*}$. Set $A = \{f^* \otimes e^* : f^* \in F^*,\ e^* \in w^*\text{-sexp}\, B_{E^*}\}$. Then $(\mathcal{K}_{w^*}(E^*, F))^* = \overline{\mathrm{span}}\, A$. Consider now any $\phi \in A$ as an element of $(E \otimes F)^*$. By Lemma 4.3, such a ϕ has a unique norm-preserving extension (to the whole of $\mathcal{L}(E^*, F)$). Hence, $\overline{E \otimes F} = \mathcal{K}_{w^*}(E^*, F)$ by Proposition 4.2.

In the case when F has property U in F^{**}, the claim is an immediate consequence of the proved one, and the canonical isometric isomorphisms $E \otimes F \cong F \otimes E$ and $\mathcal{K}_{w^*}(E^*, F) \cong \mathcal{K}_{w^*}(F^*, E)$. ■

Using the canonical isometric isomorphism between $\mathcal{K}(E,F)$ and $\mathcal{K}_{w^*}(E^{**}, F)$ (defined by the map $T \to T^{**}$), one immediately obtains the following two corollaries.

COROLLARY 4.5. *Let E and F be Banach spaces. If F has property U in F^{**}, and $\mathcal{F}(E,F)$ has the n-IP in $\mathcal{K}(E,F)$ for all n, then $\overline{\mathcal{F}}(E,F) = \mathcal{K}(E,F)$.*

COROLLARY 4.6. *If one of Banach spaces E or F is reflexive, and $\mathcal{F}(E,F)$ has the n-IP in $\mathcal{K}(E,F)$ for all n, then $\overline{\mathcal{F}}(E,F) = \mathcal{K}(E,F)$.*

REMARK. Corollary 4.6 improves [12, Theorem 15] where both spaces E and F were assumed to be reflexive.

5. The approximation property. Recall that a Banach space E is said to have the *approximation property* (AP) if for every compact set $K \subset E$ and every $\varepsilon > 0$, there is an operator $T \in \mathcal{F}(E,E)$ such that $\|Tx - x\| \le \varepsilon$ for all $x \in K$. The next two theorems, describing the relation between the approximation property and the question of approximating compact operators, are classical. They are due to A. Grothendieck [7, Chap. I, pp. 164–165, 167] (in [9, p. 400], (a)⇔(c) of Theorem (AP) is formulated and proved using ε-products of L. Schwartz). For the sake of completeness, we shall also indicate an elementary (probably the shortest) proof of them.

THEOREM (AP). *For a Banach space E, the following assertions are equivalent:*

(a) *E has the approximation property.*
(b) *$\overline{\mathcal{F}}(F,E) = \mathcal{K}(F,E)$ for all Banach spaces F.*
(c) *$\overline{E \otimes F} = \mathcal{K}_{w^*}(E^*, F)$ for all Banach spaces F.*

Proof. (a)⇒(c). The proof follows by doing some modifications to the proof in [17, pp. 32–33] and is omitted.

(c)⇒(b). Clearly, $\overline{F^* \otimes E} = \mathcal{K}_{w^*}(F^{**}, E)$ for all Banach spaces F. But $\mathcal{K}_{w^*}(F^{**}, E)$ canonically identifies with $\mathcal{K}(F,E)$.

(b)⇒(a). See the proof in [17, pp. 32–33]. ■

THEOREM (AP*). *For a Banach space E, the following assertions are equivalent:*

(a) *E^* has the approximation property.*
(b) *$\overline{\mathcal{F}}(E,F) = \mathcal{K}(E,F)$ for all Banach spaces F.*

Proof. Apply the equivalence (a)⇔(c) from the previous theorem to E^*, and use the canonical identification $\mathcal{K}_{w^*}(E^{**}, F) \cong \mathcal{K}(E,F)$. ■

REMARK. Note that the (elementary) proof of Theorem (AP*) given here does not use the principle of local reflexivity (cf. e.g. the proof of the same theorem in [17, pp. 33–34]).

If, in the definition of the AP, operators $T \in \mathcal{F}(E,E)$ can be chosen so that $\|T\| \le 1$, then E is said to have the *metric approximation property* (MAP). In [12], Å. Lima characterized the MAP of Banach spaces and their dual spaces by the position of the finite rank operators in the space of bounded linear operators. He proved, among other things, that a Banach space E with the Radon–Nikodým property has the MAP if and only if $\mathcal{F}(F,E)$ is an ideal in $\mathcal{L}(F,E)$ (or has the n-IP in $\mathcal{L}(F,E)$ for all n) for all Banach spaces F (cf. also [2] for related results). The next two theorems, which are the main results of the present work, describe the AP of Banach spaces and their dual spaces through the position of finite rank operators in the space of compact operators.

THEOREM 5.1. *For a Banach space E, the following assertions are equivalent*:

(a) *E has the approximation property.*
(b_1) *$\mathcal{F}(F,E)$ is an ideal in $\mathcal{K}(F,E)$ for all Banach spaces F.*
(b_2) *$\mathcal{F}(F,E)$ has the n-IP in $\mathcal{K}(F,E)$ for all n and all separable reflexive Banach spaces F.*
(b_3) *$\mathcal{F}(F,E)$ has the n-IP in $\mathcal{K}(F,E)$ for all n and all closed subspaces $F\subset c_0$.*
(c_1) *$E\otimes F$ is an ideal in $\mathcal{K}_{w^*}(E^*,F)$ for all Banach spaces F.*
(c_2) *$E\otimes F$ has the n-IP in $\mathcal{K}_{w^*}(E^*,F)$ for all n and all separable reflexive Banach spaces F.*
(c_3) *$E\otimes F$ has the n-IP in $\mathcal{K}_{w^*}(E^*,F)$ for all n and all closed subspaces $F\subset c_0$.*

Proof. It is clear from the definitions that a subspace Y of a Banach space X is an ideal in X or has the n-IP for all n in X if and only if its closure $\overline{Y}$ does. Hence, by Theorem (AP), (a)$\Rightarrow$(b_1) and (a)$\Rightarrow$(c_1). The implications (b_1)$\Rightarrow$(b_2)&(b_3) and (c_1)$\Rightarrow$(c_2)&(c_3) are obvious.

(b_2)$\Rightarrow$(a). By Theorem 3.1, $\overline{\mathcal{F}}(F,E)=\mathcal{K}(F,E)$ for all separable reflexive Banach spaces F. Let G be an arbitrary Banach space, and $T\in\mathcal{K}(G,E)$. By a theorem of T. Figiel and W. B. Johnson ([5] and [10]), T factors compactly through some separable reflexive Banach space F. (This fact can also be deduced from the Davis–Figiel–Johnson–Pełczyński factorization theorem (cf. e.g. [9, p. 374])). That is, $T=BA$ where $A\in\mathcal{K}(G,F)$ and $B\in\mathcal{K}(F,E)$. Now, $B\in\overline{\mathcal{F}}(F,E)$, and therefore $T=BA\in\overline{\mathcal{F}}(G,E)$. We have shown that $\overline{\mathcal{F}}(G,E)=\mathcal{K}(G,E)$. By Theorem (AP), this implies (a).

(b_3)$\Rightarrow$(a). The proof is the same as for the previous implication with the only difference that it uses a compact factorization of $T\in\mathcal{K}(G,E)$ through a closed subspace F of c_0. (This is clear from a version of the Figiel–Johnson factorization theorem—cf. [5, Corollary 6.2]; cf. also e.g. [3, p. 15] or [9, pp. 369–370].)

(c_2)$\Rightarrow$(a) is clear because (c_2) is just an equivalent restatement of (b_2) (recall that, canonically, $E\otimes F\cong F^{**}\otimes E=\mathcal{F}(F^*,E)$ and $\mathcal{K}_{w^*}(E^*,F)\cong\mathcal{K}_{w^*}(F^*,E)=\mathcal{K}_{w^*}(F^{***},E)\cong\mathcal{K}(F^*,E)$ for reflexive F).

(c_3)$\Rightarrow$(a). Since closed subspaces of c_0 have property U in their biduals (this is well known cf. e.g. [8, pp. 105, 111, 11]), by Theorem 4.4, (c_3) implies that $\overline{E\otimes F}=\mathcal{K}_{w^*}(E^*,F)$ for all closed subspaces F of c_0. The following claim shows that this condition implies (c) of Theorem (AP), and therefore (a) is true.

CLAIM. *Let E and G be Banach spaces, and $T\in\mathcal{K}_{w^*}(E^*,G)$. Then there is a closed subspace F of c_0 so that $T=BA$ for some operators $A\in\mathcal{K}_{w^*}(E^*,F)$ and $B\in\mathcal{K}(F,G)$.*

To establish the claim, we follow the proof of the Figiel–Johnson factorization theorem in [10] and [5]. Similarly to [10, Proposition 1], we show that the subspace of $\overline{E\otimes G}$ consisting of operators $S:E^*\to G$ which admit a factorization $T=BA$ for some operators $A\in\overline{E\otimes C_\infty}$ and $B\in\overline{\mathcal{F}}(C_\infty,G)$, where C_∞ denotes the c_0-sum of finite-dimensional Banach spaces introduced by W. B. Johnson [10, p. 341] (cf. also [21, p. 425]), is a Banach space under the norm

$$\|S\|=\inf\{\|B\|\|A\|:S=BA,\ A\in\overline{E\otimes C_\infty},\ B\in\overline{\mathcal{F}}(C_\infty,G)\}.$$

By the proof of [10, Theorem 1], this subspace is equal to the whole space $\overline{E\otimes G}$.

Let $T\in\mathcal{K}_{w^*}(E^*,G)$. It is clearly enough to prove the factorization of T for an F which is isomorphic to a closed subspace of c_0. We now continue similarly to the proof of [5, Proposition 3.1]. Let $j:G\to l_\infty(\Gamma)$ be the canonical embedding (for $\Gamma=B_{G^*}$). Since $l_\infty(\Gamma)$ has the AP,

$$jT\in\mathcal{K}_{w^*}(E^*,l_\infty(\Gamma))=\overline{E\otimes l_\infty(\Gamma)}$$

by Theorem (AP). By the above, $jT=BA$ for some $A\in\mathcal{K}_{w^*}(E^*,C_\infty)$ and $B\in\mathcal{K}(C_\infty,l_\infty(\Gamma))$. Considering $F=B^{-1}(\operatorname{ran} j)$ instead of C_∞ and taking $j^{-1}B$ instead of B yield the desired factorization since C_∞ (and so F as well) is isomorphic to a closed subspace of c_0. The claim and hence Theorem 5.1 are proved. ■

THEOREM 5.2. *For a Banach space E, the following assertions are equivalent*:

(a) *E^* has the approximation property.*
(b_1) *$\mathcal{F}(E,F)$ is an ideal in $\mathcal{K}(E,F)$ for all Banach spaces F.*
(b_2) *$\mathcal{F}(E,F)$ has the n-IP in $\mathcal{K}(E,F)$ for all n and all separable reflexive Banach spaces F.*
(b_3) *$\mathcal{F}(E,F)$ has the n-IP in $\mathcal{K}(E,F)$ for all n and all closed subspaces F of c_0.*

Proof. Applying the equivalences (a)$\Leftrightarrow$(c_1)$\Leftrightarrow$(c_2)$\Leftrightarrow$(c_3) from Theorem 5.1 to E^*, and using the canonical identification $\mathcal{K}_{w^*}(E^{**},F)\cong\mathcal{K}(E,F)$, one immediately gets (a)$\Leftrightarrow$(b_1)$\Leftrightarrow$(b_2)$\Leftrightarrow$(b_3). ■

Acknowledgements. The second-named author wishes to acknowledge the warm hospitality provided by Å. Lima and his colleagues at Agder College, where a part of this work was done in October 1996. Her visit was supported by the Norwegian Academy of Science and Letters, and by Estonian Science Foundation Grant 1820.

References

[1] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. 6 (1956), 405–439.
[2] C.-M. Cho, *The metric approximation property and intersection properties of balls*, J. Korean Math. Soc. 31 (1994), 467-475.
[3] J. Diestel, *Sequences and Series in Banach Spaces*, Springer, New York, 1984.
[4] M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. 21 (1975), 38–49.
[5] T. Figiel, *Factorization of compact operators and applications to the approximation problem*, Studia Math. 45 (1973), 191–210.
[6] G. Godefroy, N. J. Kalton and P. D. Saphar, *Unconditional ideals in Banach spaces*, ibid. 104 (1993), 13–59.
[7] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[8] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.
[9] H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.
[10] W. B. Johnson, *Factoring compact operators*, Israel J. Math. 9 (1971), 337–345.
[11] Å. Lima, *Uniqueness of Hahn-Banach extensions and liftings of linear dependences*, Math. Scand. 53 (1983), 97–113.
[12] —, *The metric approximation property, norm-one projections and intersection properties of balls*, Israel J. Math. 84 (1993), 451–475.
[13] —, *Property* (wM^*) *and the unconditional metric compact approximation property*, Studia Math. 113 (1995), 249–263.
[14] J. Lindenstrauss, *On a problem of Nachbin concerning extension of operators*, Israel J. Math. 1 (1963), 75–84.
[15] —, *Extension of compact operators*, Mem. Amer. Math. Soc. 48 (1964).
[16] —, *On projections with norm 1—an example*, Proc. Amer. Math. Soc. 15 (1964), 403–406.
[17] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Ergeb. Math. Grenzgeb. 92, Springer, Berlin, 1977.
[18] E. Oja and M. Põldvere, *On subspaces of Banach spaces where every functional has a unique norm-preserving extension*, Studia Math. 117 (1996), 289–306.
[19] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. 1364, Springer, Berlin, 1993.
[20] W. M. Ruess and C. P. Stegall, *Extreme points in duals of operator spaces*, Math. Ann. 261 (1982), 535–546.
[21] I. Singer, *Bases in Banach Spaces II*, Springer, Berlin, 1981.

Department of Mathematics
Agder College
Tordenskjoldsgate 65
N-4604 Kristiansand, Norway
E-mail: Asvald.Lima@hia.no

Faculty of Mathematics
Tartu University
Vanemuise 46
EE-2400 Tartu, Estonia
E-mail: eveoja@math.ut.ee

Received May 11, 1998
Revised version August 24, 1998 (4101)

STUDIA MATHEMATICA 133 (2) (1999)

Toeplitz operators in the commutant of a composition operator

by

BRUCE CLOAD (St. Catharines, Ont.)

Abstract. If ϕ is an analytic self-mapping of the unit disc D and if $H^2(D)$ is the Hardy–Hilbert space on D, the composition operator C_ϕ on $H^2(D)$ is defined by $C_\phi(f) = f \circ \phi$. In this article, we consider which Toeplitz operators T_f satisfy $T_f C_\phi = C_\phi T_f$.

1. Introduction. Denote the unit disc in the complex plane by D and the boundary of the unit disc by ∂D. We will be working with the following spaces: $H^2(D)$, the Hardy–Hilbert space which is defined as the set of analytic functions on D with square summable power series coefficients, $H^\infty(D)$, the set of bounded analytic functions on D, $L^2(\partial D)$, the set of L^2 functions on the unit circle with respect to normalized Lebesgue measure, and $L^\infty(\partial D)$, the corresponding L^∞ space on the unit circle. For notational convenience, throughout the paper, we may abbreviate the symbols for these spaces as H^2, H^∞, L^2, and L^∞. For more information on these spaces see [9] and [11].

If ϕ is an analytic function on D, we may define a bounded operator on $H^2(D)$, for all f in H^2, by $C_\phi(f) = f \circ \phi$. These composition operators provide a large class of examples of operators on Hilbert spaces and, moreover, there are excellent single variable complex analysis techniques with which to analyze them. For more information on composition operators see [7] and [16].

The commutant of a composition operator, C_ϕ, is the weakly closed algebra generated by all operators that commute with C_ϕ. For a brief discussion on the commutant of a composition operator see [8]. In [3], we consider the question of how to generate the commutant of C_ϕ for some specific functions ϕ. In [4], we examine other aspects of the commutant of a composition operator. In [5], Carl Cowen showed that if f is a covering map of D onto a bounded domain in the complex plane, then the commutant of the Toeplitz

1991 *Mathematics Subject Classification*: Primary 47B38; Secondary 47B35, 47A05.
The author would like to thank Peter Rosenthal, Joe Repka, and Ruben Martinez for their helpful comments and suggestions on this article.