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On Arens–Michael algebras which do not have non-zero injective $\widehat{\otimes}$-modules

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Abstract. A certain class of Arens–Michael algebras having no non-zero injective topological $\widehat{\otimes}$-modules is introduced. This class is rather wide and contains, in particular, algebras of holomorphic functions on polydomains in $\mathbb{C}^n$, algebras of smooth functions on domains in $\mathbb{R}^n$, algebras of formal power series, and, more generally, any nuclear Fréchet–Arens–Michael algebra which has a free bimodule Koszul resolution.

One of the most general questions of homological algebra is the following. Does a given category have sufficiently many injective and projective objects? In this paper this question is treated from the point of view of topological homology (a relative homological algebra in categories of topological modules over topological algebras). A certain "asymmetry" between projectivity and injectivity for locally convex modules over non-normable algebras was noticed by J. L. Taylor [12]. It is remarkable that the original motivation for Taylor's paper [12] was to develop homological tools, needed for the construction of a multi-operator analytic functional calculus (see also [13] and the recent book [3]).

This "asymmetry" can be expressed as follows. On the one hand, for any additive category of locally convex spaces which satisfies certain natural conditions (see [12], [5]), the corresponding category of locally convex modules has sufficiently many projective objects. On the other hand, the analogous question about injective objects is not so clear, and the reason for this is a possible absence of so-called cofree objects in categories of locally convex modules over non-normable algebras (see [9]). The situation is satisfactory only for Banach algebras: it is known that the category of all Banach modules over a given Banach algebra has sufficiently many both projective and injective objects (see [5]). In order to obtain sufficiently many injective modules over a non-normable algebra A, one has to impose certain rather specific (and rather disagreeable) conditions on the considered cate-

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gory of topological modules (see [12] and [1]). In particular, the action of A on some modules of this category fails to be jointly continuous, even though the multiplication in A is jointly continuous.

The above question about injective modules was posed by A. Ya. Helemskii in [7]. More precisely, the problem was as follows. Does the category of all left Fréchet modules over an arbitrary Fréchet algebra A have sufficiently many injective objects? In the same paper it was conjectured that there exist Fréchet algebras which do not have non-zero injective Fréchet modules at all. This conjecture was proved in [10] for Fréchet algebras $\mathbb{C}[[z_1,\dots,z_n]]$ of formal power series. However, the problem was open for semisimple Fréchet algebras. In Helemskii's paper [7], it was pointed out that the possible candidates for such algebras are Fréchet algebras of holomorphic functions on domains in $\mathbb{C}^n$.

In this paper we prove this hypothesis and describe a certain class of Fréchet algebras having no non-zero injective Fréchet modules. This class contains Fréchet algebras of smooth functions on domains in $\mathbb{R}^n$, Fréchet algebras of holomorphic functions on polydomains in $\mathbb{C}^n$, Fréchet algebras of formal power series, and certain non-commutative algebras.

1. Preliminaries. Let us recall some definitions and fix some notation. Following Taylor (see [12]), by a *$\widehat{\otimes}$-algebra* we mean a complete Hausdorff locally convex space (l.c.s.) A over $\mathbb{C}$ together with the structure of an associative algebra such that the multiplication map $A\times A\to A$ is jointly continuous. We will often assume that A is an *Arens–Michael algebra*, i.e., the topology on A can be defined by a family $\{\|\cdot\|_\nu : \nu\in\Lambda\}$ of seminorms having the property $\|ab\|_\nu\le\|a\|_\nu\|b\|_\nu$ for all $a,b\in A$. A left *$\widehat{\otimes}$-module* over a $\widehat{\otimes}$-algebra A is a complete Hausdorff locally convex space X endowed with the structure of a left A-module such that the map $A\times X\to X$, $(a,x)\mapsto a\cdot x$, is jointly continuous. By replacing joint continuity with separate continuity, one obtains the definitions of $\check{\otimes}$-algebra and $\check{\otimes}$-module, respectively. Throughout the paper, all algebras and modules are assumed to be unital, and all morphisms are assumed to be continuous.

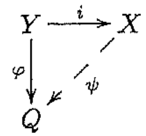
Let $\mathscr{C}$ be a complete additive subcategory of the category of all complete Hausdorff l.c.s.'s over $\mathbb{C}$ such that the following conditions are satisfied (see also [5]):

1) If $E\in\mathscr{C}$ and F is topologically isomorphic to E, then $F\in\mathscr{C}$;
2) If $E\in\mathscr{C}$ and E_0 is a closed subspace of E, then both E_0 and the completion of E/E_0 belong to $\mathscr{C}$;
3) If E and F belong to $\mathscr{C}$, then the completed projective tensor product $E\widehat{\otimes}F$ also belongs to $\mathscr{C}$.

In the sequel, we assume that such a category $\mathscr{C}$ is given. If A is a $\widehat{\otimes}$-algebra whose underlying space belongs to $\mathscr{C}$, then we denote by A-mod($\mathscr{C}$) the category of all left $\widehat{\otimes}$-modules X over A such that the underlying space of X belongs to $\mathscr{C}$.

We refer to [5] and [12] for the basic notions in topological homology. If X and Y are left $\widehat{\otimes}$-modules over a $\widehat{\otimes}$-algebra A, then we denote by $\mathbf{h}_A(X,Y)$ the space of all A-module morphisms of X to Y. This space is endowed with the strong topology, i.e., the topology of uniform convergence on bounded sets. If Z is a right $\widehat{\otimes}$-module over A, then $Z\widehat{\otimes}_A Y$ stands for the projective tensor product of Z and Y. By definition, $Z\widehat{\otimes}_A Y$ is the completion of the quotient space of $Z\widehat{\otimes}Y$ by the subspace generated by elements $z\cdot a\otimes y - z\otimes a\cdot y$ ($z\in Z$, $y\in Y$, $a\in A$). The nth projective derived functor of $\mathbf{h}_A(\cdot,Y)$ (resp., $(\cdot)\widehat{\otimes}_A Y$) is denoted by $\operatorname{Ext}^n_A(\cdot,Y)$ (resp., $\operatorname{Tor}^A_n(\cdot,Y)$). (For the general theory of derived functors in topological homology, see [5].) These functors take values in the category of all (not necessarily separated) locally convex spaces. If M is a $\widehat{\otimes}$-bimodule over A, then the space $\mathscr{H}^n(A,M)=\operatorname{Ext}^n_{A^e}(A,M)$ (resp., $\mathscr{H}_n(A,M)=\operatorname{Tor}^{A^e}_n(A,M)$) is called the *nth Hochschild cohomology* (resp., the *nth Hochschild homology*) of A with coefficients in M.

Recall also the definition of injective module and the definition of injective homological dimension. An A-module $Q\in A$-mod($\mathscr{C}$) is called *injective* (with respect to $\mathscr{C}$) if for each $X\in A$-mod($\mathscr{C}$) and a closed submodule $Y\subset X$ which is complemented as a subspace of X, any A-module morphism $\varphi: Y\to Q$ can be extended to an A-module morphism $\psi: X\to Q$, i.e., the following diagram is commutative:



Here i is the canonical embedding of Y into X.

The *injective homological dimension* of an A-module $Q\in A$-mod($\mathscr{C}$) is, by definition, the least number n such that $\operatorname{Ext}^p_A(X,Q)=0$ for all $X\in A$-mod($\mathscr{C}$) and all $p>n$, or, equivalently, $\operatorname{Ext}^{n+1}_A(X,Q)=0$ for all $X\in A$-mod($\mathscr{C}$); it is denoted by $\operatorname{injdh}_A Q$. If such an n does not exist, then it is convenient to put $\operatorname{injdh}_A Q=\infty$. It is known that Q is injective iff $\operatorname{injdh}_A Q=0$ (see [5]).

If A is a Banach algebra and $\mathscr{C}$ is the category of all Banach spaces, then the injective homological dimension of a module $Q\in A$-mod($\mathscr{C}$) can be equivalently defined as the least n such that Q has an injective resolution of length n. In the general case, however, the category A-mod($\mathscr{C}$) may not possess sufficiently many injective objects, so we have to use the former definition.

Suppose E is a left Banach module over a $\widehat{\otimes}$-algebra A, and F is an arbitrary left $\widehat{\otimes}$-module over A. Consider the space $\mathscr{L}(E,F)$ of all continuous linear maps of E to F. This is a complete locally convex space with respect to the topology of uniform convergence on the unit ball of E. This space has a natural structure of a $\widehat{\otimes}$-bimodule over A; namely, the action of A is defined by the formulas

$$(a\cdot\varphi)(x)=a\cdot\varphi(x),\ (\varphi\cdot a)(x)=\varphi(a\cdot x)\quad(\varphi\in\mathscr{L}(E,F),\ x\in E,\ a\in A).$$

We shall use the following result, which is a special case of Propositions 3.8 and 4.1 of [12].

PROPOSITION 1.1. *Let A be a $\widehat{\otimes}$-algebra and E,F left $\widehat{\otimes}$-modules over A. Suppose A is either a Fréchet space or a barreled (DF)-space, and E is a Banach space. Then there exist natural isomorphisms of vector spaces* $\operatorname{Ext}^n_A(E,F)\cong\mathscr{H}^n(A,\mathscr{L}(E,F))$.

Recall that an inverse system $\mathscr{X}=\{X_\nu,\tau^\mu_\nu,\Lambda\}$ of topological spaces is called *reduced* if for each $\nu\in\Lambda$ the image of the canonical map $\tau_\nu:\varprojlim\mathscr{X}\to X_\nu$ is dense in X_ν. We use the notation of [2, 2.5] in the following lemma.

LEMMA 1.2. *Suppose $\mathscr{X}=\{X_\nu,\tau^\mu_\nu,\Lambda\}$ and $\mathscr{Y}=\{Y_{\nu'},\tau^{\mu'}_{\nu'},\Lambda'\}$ are inverse systems of topological spaces, and $\mathscr{X}$ is reduced. Let $\{\varphi,f_{\nu'}\}:\mathscr{X}\to\mathscr{Y}$ be a map of inverse systems such that for each $\nu'\in\Lambda'$ the image of the map $f_{\nu'}:X_{\varphi(\nu')}\to Y_{\nu'}$ is dense in $Y_{\nu'}$. Then the image of the map $f=\varprojlim\{\varphi,f_{\nu'}\}$ is dense in $\varprojlim\mathscr{Y}$.*

Proof. Put $X=\varprojlim\mathscr{X}$ and $Y=\varprojlim\mathscr{Y}$. It is known that the family

$$\{\tau_{\nu'}^{-1}(U_{\nu'}):U_{\nu'}\subset Y_{\nu'}\text{ open},\ \nu'\in\Lambda'\}$$

of sets is a base of the topology on Y (see [2, 2.5.5]). Therefore, it suffices to check that $\tau_{\nu'}f(X)$ is dense in Y for all $\nu'\in\Lambda'$. Taking into account the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y\\ {\scriptstyle\tau_{\varphi(\nu')}}\downarrow & & \downarrow{\scriptstyle\tau_{\nu'}}\\ X_{\varphi(\nu')} & \xrightarrow[f_{\nu'}]{} & Y_{\nu'}\end{array}$$

we obtain

$$\overline{\tau_{\nu'}f(X)}=\overline{f_{\nu'}\tau_{\varphi(\nu')}(X)}=\overline{f_{\nu'}(\overline{\tau_{\varphi(\nu')}(X)})}=\overline{f_{\nu'}(X_{\varphi(\nu')})}=Y_{\nu'}.\ \blacksquare$$

Let E and F be complete locally convex spaces. Consider the space $\mathscr{L}_{\mathrm e}(E'_\tau,F)$ of all linear continuous maps from the dual space E', equipped with the Mackey topology, to F. It is a locally convex space with respect

to the topology of uniform convergence on equicontinuous sets. Recall that the *Grothendieck canonical mapping* $g:F\widehat{\otimes}E\to\mathscr{L}_{\mathrm e}(E'_\tau,F)$ is defined by $g(y\otimes x)(x')=\langle x',x\rangle y$. Since the strong topology on E' is finer than the Mackey topology, we have the natural embedding of l.c.s.'s

$$\mathscr{L}_{\mathrm e}(E'_\tau,F)\subset\mathscr{L}_{\mathrm e}(E'_\beta,F).$$

We also denote by g the composition of the above embedding with the Grothendieck map.

Recall some standard notation (see [11]). Suppose U is a convex circled 0-neighborhood in a l.c.s. E, and p_U is the Minkowski functional of U. By definition, put $E_U=E/p_U^{-1}(0)$. The completion of E_U with respect to the quotient norm of the seminorm p_U is denoted by $\widetilde{E}_U$. Let Φ_U denote the quotient map of E onto E_U. Further, if $B\subset E$ is a convex circled bounded subset, and p_B is the Minkowski functional of B, then put $E_B=\bigcup_n nB$. This is a normed space with respect to p_B. The canonical embedding $E_B\to E$ is denoted by Ψ_B.

Suppose $u:E\to F$ is a linear continuous map of l.c.s.'s E and F. For each l.c.s. G, one has the induced natural linear map

$$u_*:\mathscr{L}_{\mathrm e}(E'_\beta,G)\to\mathscr{L}_{\mathrm e}(F'_\beta,G)$$

defined by the rule $u_*(\varphi)=\varphi\circ u'$. Obviously, this map is continuous.

LEMMA 1.3. *Let $u:E\to F$ be a nuclear linear map of a l.c.s. E to a complete l.c.s. F. Then for each complete l.c.s. G there exists a natural linear continuous map $\sigma:\mathscr{L}_{\mathrm e}(E'_\beta,G)\to G\widehat{\otimes}F$ such that the following diagram is commutative*:

$$\begin{array}{ccc} \mathscr{L}_{\mathrm e}(E'_\beta,G) & \xrightarrow{\quad u_*\quad} & \mathscr{L}_{\mathrm e}(F'_\beta,G)\\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle g} & \\ & G\widehat{\otimes}F & \end{array}\tag{1}$$

Proof. Since u is nuclear, there exist a convex circled 0-neighborhood $U\subset E$ and a convex circled bounded set $B\subset F$ such that u belongs to the image of the canonical map

$$(\widetilde{E}_U)'\widehat{\otimes}F_B\to\mathscr{L}(\widetilde{E}_U,F_B)\to\mathscr{L}(E,F).$$

Take a preimage $\widetilde{v}\in(\widetilde{E}_U)'\widehat{\otimes}F_B$ of u under the above map, and put $v=(\Phi'_U\otimes\Psi_B)(\widetilde{v})\in E'_\beta\widehat{\otimes}F$. Define σ by the rule $\sigma(\varphi)=\varphi\otimes 1_F(v)$. Obviously, $\sigma:\mathscr{L}_{\mathrm e}(E'_\beta,G)\to G\widehat{\otimes}F$ is a linear map.

Take sequences $\{\lambda_n\}\subset\mathbb{C}$, $\{\widetilde{f}_n\}\subset(\widetilde{E}_U)'$ and $\{\widetilde{y}_n\}\subset F_B$ such that

$$\widetilde{v}=\sum_n\lambda_n\widetilde{f}_n\otimes\widetilde{y}_n,$$

and $\sum_n |\lambda_n| \le 1$, $\widetilde{f}_n \to 0$, $\widetilde{y}_n \to 0$. Then

$$u = \sum_n \lambda_n \langle f_n, \cdot \rangle y_n \quad \text{and} \quad u' = \sum_n \lambda_n \langle \cdot, y_n \rangle f_n.$$

Here $f_n = \widetilde{f}_n \circ \Phi_U$ and $y_n = \Psi_B(\widetilde{y}_n)$. Evidently, the sequence $\{f_n\} \subset E'$ is equicontinuous, and the sequence $\{y_n\} \subset F$ is bounded (cf. [11, 7.1]).

It follows from the definition of σ that

$$\sigma(\varphi) = \sum_n \lambda_n \varphi(f_n) \otimes y_n. \tag{2}$$

Hence,

$$u_*(\varphi)(h) = \varphi(u'(h)) = \sum_n \lambda_n \langle h, y_n \rangle \varphi(f_n) = g\sigma(\varphi)(h)$$

for each $h \in F'_\beta$. This proves that the diagram (1) is commutative.

It remains to prove that σ is continuous. The space $G \widehat{\otimes} F$ has a basis of 0-neighborhoods of the form $\overline{\Gamma(V \otimes W)}$, where V and W are convex circled 0-neighborhoods in G and F, respectively, and $\overline{\Gamma(V \otimes W)}$ is the closed convex circled hull of the set $V \otimes W = \{v \otimes w : v \in V,\ w \in W\}$. Take such a 0-neighborhood $\overline{\Gamma(V \otimes W)}$. Since the sequence $\{y_n\}$ is bounded, there exists $\lambda > 0$ such that $y_n \in \lambda W$ for all n. Put $D = \{f_n\}$. Then D is an equicontinuous set, and

$$M(D, \lambda^{-1}V) = \{\varphi \in \mathscr{L}(E'_\beta, G) : \varphi(D) \subset \lambda^{-1}V\}$$

is a 0-neighborhood in $\mathscr{L}_e(E'_\beta, G)$. We have $\varphi(f_n) \in \lambda^{-1}V$ for each $\varphi \in M(D, \lambda^{-1}V)$, hence, $\varphi(f_n) \otimes y_n \in V \otimes W$ for all n. It follows from (2) that $\sigma(\varphi) \in \overline{\Gamma(V \otimes W)}$. Therefore, $\sigma(M(D, \lambda^{-1}V)) \subset \overline{\Gamma(V \otimes W)}$, and σ is continuous. ∎

REMARK 1.1. Suppose that either G or F is isomorphic to a reduced inverse limit of Banach spaces having the approximation property. Then the Grothendieck map $g : G \widehat{\otimes} F \to \mathscr{L}_e(E'_\beta, F)$ is injective (see [8, 43.2]), and there exists a *unique* map σ such that the diagram (1) is commutative.

REMARK 1.2. Under the conditions of Lemma 1.3, suppose that E and F are right Banach modules over a $\widehat{\otimes}$-algebra A, G is a left $\widehat{\otimes}$-module over A, and u is an A-module morphism. Then the spaces $\mathscr{L}_e(E', G)$, $\mathscr{L}_e(F', G)$ and $G \widehat{\otimes} F$ are $\widehat{\otimes}$-bimodules over A (see [5, II.4.4, II.5.15]), and it is easy to check that u_* and g are A-bimodule morphisms. It is natural to ask whether σ is an A-bimodule morphism. Direct calculation shows that σ is a left A-module morphism. However, it is not clear if σ is a right A-module morphism. Indeed, one can easily check that the equality $\sigma(\varphi \cdot a) = \sigma(\varphi) \cdot a$ is equivalent to the following one:

$$\varphi \otimes \mathbf{1}_F\Big(\sum_n \lambda_n a \cdot f_n \otimes y_n\Big) = \varphi \otimes \mathbf{1}_F\Big(\sum_n \lambda_n f_n \otimes y_n \cdot a\Big)$$

($\varphi \in \mathscr{L}_e(E'_\beta, G)$, $a \in A$). Since u is an A-module morphism, it follows that

$$\sum_n \lambda_n \langle a \cdot f_n, \cdot \rangle y_n = \sum_n \lambda_n \langle f_n, \cdot \rangle y_n \cdot a$$

for all $a \in A$. However, this does not imply that the elements

$$\sum_n \lambda_n a \cdot f_n \otimes y_n \quad \text{and} \quad \sum_n \lambda_n f_n \otimes y_n \cdot a$$

of the space $E' \widehat{\otimes} F$ coincide. On the other hand, if the canonical map $E' \widehat{\otimes} F \to \mathscr{L}(E, F)$ is injective (this is true provided that either F or E' has the approximation property, see [8, 43.2]), then these elements do coincide, and σ is an A-bimodule morphism. Another sufficient condition for σ to be an A-bimodule morphism is the injectivity of the canonical map $g : G \widehat{\otimes} F \to \mathscr{L}_e(F'_\beta, G)$. This condition is satisfied whenever G is isomorphic to a reduced inverse limit of Banach spaces having the approximation property.

2. Main results.

Our main result is the following theorem.

THEOREM 2.1. *Let A be a nuclear Arens–Michael algebra such that the underlying space of A is either a Fréchet space or a (DF)-space. Suppose that there exists $n \in \mathbb{N}$ such that the following condition holds*:

(∗) *If M is a $\widehat{\otimes}$-bimodule over A such that $\mathscr{H}^n(A, M) = 0$, then the topology on the space $\mathscr{H}_0(A, M)$ is indiscrete.*

Then the category A-mod$(\mathscr{C})$ does not contain any non-zero module Q such that $\mathrm{Ext}^n_A(X, Q) = 0$ for each left Banach A-module X. If, moreover, $\mathscr{C}$ contains all Banach spaces, then $\mathrm{injdh}_A\, Q \ge n$ for each $Q \in A$-mod$(\mathscr{C})$.

Proof. Since A is an Arens–Michael algebra, it can be represented as a reduced inverse limit of Banach algebras: $A = \varprojlim \{A_\nu, \tau^\mu_\nu, \Lambda\}$ (see [6, 5.2.17]). Moreover, since A is nuclear, we see that for each $\nu \in \Lambda$ there exists $\mu \succ \nu$ such that the map τ^μ_ν is nuclear. We endow each A_ν with the structure of a right Banach A-module by putting $a_\nu \cdot a = a_\nu \tau_\nu(a)$, where $a_\nu \in A_\nu$, $a \in A$, and $\tau_\nu : A \to A_\nu$ is the canonical projection. Suppose $Q \in A$-mod$(\mathscr{C})$ is an A-module such that $\mathrm{Ext}^n_A(X, Q) = 0$ for each left Banach A-module X.

Since each complete nuclear (DF)-space is barreled (see [4]), we see that A satisfies the conditions of Proposition 1.1. Therefore,

$$\mathscr{H}^n(A, \mathscr{L}(A'_\nu, Q)) = \mathrm{Ext}^n_A(A'_\nu, Q) = 0$$

for all $\nu \in \Lambda$. This implies that the topology on $\mathscr{H}_0(A, \mathscr{L}(A'_\nu, Q))$ is indiscrete. By definition,

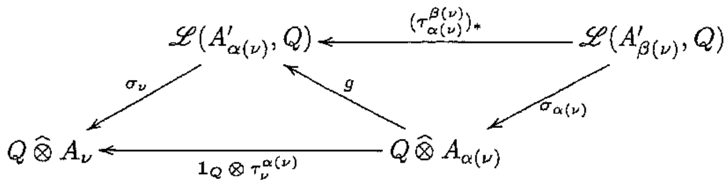
$$\mathscr{H}_0(A, \mathscr{L}(A'_\nu, Q)) = \mathrm{Coker}\, \varepsilon_\nu,$$

where the map

$$\varepsilon_\nu : A \mathbin{\widehat{\otimes}} \mathscr{L}(A'_\nu, Q) \to \mathscr{L}(A'_\nu, Q)$$

is defined by the rule $\varepsilon_\nu(a \otimes \varphi) = a \cdot \varphi - \varphi \cdot a$. Hence, the image of the map ε_ν is dense in $\mathscr{L}(A'_\nu, Q)$.

For each $\nu \in \Lambda$, take elements $\alpha(\nu) \succ \nu$ and $\beta(\nu) \succ \alpha(\nu)$ such that the maps $\tau_\nu^{\alpha(\nu)}$ and $\tau_{\alpha(\nu)}^{\beta(\nu)}$ are nuclear. Denote by σ_ν (resp., $\sigma_{\alpha(\nu)}$) the map constructed in Lemma 1.3 for $u = \tau_\nu^{\alpha(\nu)}$ (resp., $u = \tau_{\alpha(\nu)}^{\beta(\nu)}$). We obtain the following commutative diagram:



Taking into account Remark 1.2, we cannot claim that σ_ν or $\sigma_{\alpha(\nu)}$ is an A-bimodule morphism. However, $\varkappa_\nu = \sigma_\nu \circ (\tau_{\alpha(\nu)}^{\beta(\nu)})_*$ is an A-bimodule morphism. Indeed, since σ_ν is a left A-module morphism (see Remark 1.2), so is $\varkappa_\nu$. Further,

$$\begin{aligned}\varkappa_\nu(\varphi \cdot a) &= \sigma_\nu((\tau_{\alpha(\nu)}^{\beta(\nu)})_*(\varphi \cdot a)) = \sigma_\nu((\tau_{\alpha(\nu)}^{\beta(\nu)})_*(\varphi) \cdot a) = \sigma_\nu(g\sigma_{\alpha(\nu)}(\varphi) \cdot a)\\ &= \sigma_\nu(g(\sigma_{\alpha(\nu)}(\varphi) \cdot a)) = \mathbf{1}_Q \otimes \tau_\nu^{\alpha(\nu)}(\sigma_{\alpha(\nu)}(\varphi) \cdot a)\\ &= \mathbf{1}_Q \otimes \tau_\nu^{\alpha(\nu)}(\sigma_{\alpha(\nu)}(\varphi)) \cdot a = \varkappa_\nu(\varphi) \cdot a.\end{aligned}$$

Hence, $\varkappa_\nu$ is an A-bimodule morphism.

Note that the image of $\varkappa_\nu$ is dense in $Q \mathbin{\widehat{\otimes}} A_\nu$. Indeed, since $\tau_\nu^{\alpha(\nu)}$ has dense image, the same is true for $\mathbf{1}_Q \otimes \tau_\nu^{\alpha(\nu)}$. It follows from the left triangle of the above diagram that σ_ν also has dense image. Hence, the same is true for $\sigma_{\alpha(\nu)}$ and for the composition $\varkappa_\nu = (\mathbf{1}_Q \otimes \tau_\nu^{\alpha(\nu)}) \circ \sigma_{\alpha(\nu)}$.

For each $\nu \in \Lambda$ we obtain the following commutative diagram:

$$\begin{array}{ccc} A \mathbin{\widehat{\otimes}} \mathscr{L}(A'_{\beta(\nu)}, Q) & \xrightarrow{\varepsilon_{\beta(\nu)}} & \mathscr{L}(A'_{\beta(\nu)}, Q) \\ {\scriptstyle \mathbf{1}_A \otimes \varkappa_\nu}\downarrow & & \downarrow{\scriptstyle \varkappa_\nu} \\ A \mathbin{\widehat{\otimes}} Q \mathbin{\widehat{\otimes}} A_\nu & \xrightarrow[\widetilde{\varepsilon}_\nu]{} & Q \mathbin{\widehat{\otimes}} A_\nu \end{array}$$

Here the map $\widetilde{\varepsilon}_\nu$ is given by $a \otimes u \mapsto a \cdot u - u \cdot a$.

Since the image of $\varepsilon_{\beta(\nu)}$ is dense in $\mathscr{L}(A'_{\beta(\nu)}, Q)$, and the image of $\varkappa_\nu$ is dense in $Q \mathbin{\widehat{\otimes}} A_\nu$, we conclude that the image of $\widetilde{\varepsilon}_\nu$ is dense in $Q \mathbin{\widehat{\otimes}} A_\nu$. Using the fact that projective tensor product commutes with reduced inverse limits (see [8, 41.6]), we see that the map

$$\widetilde{\varepsilon} : A \mathbin{\widehat{\otimes}} Q \mathbin{\widehat{\otimes}} A \to Q \mathbin{\widehat{\otimes}} A, \quad a \otimes u \mapsto a \cdot u - u \cdot a,$$

is the inverse limit of the maps $\widetilde{\varepsilon}_\nu$. By Lemma 1.2, the image of $\widetilde{\varepsilon}$ is dense in $Q \mathbin{\widehat{\otimes}} A$. Therefore, the topology on the space

$$\mathrm{Coker}\, \widetilde{\varepsilon} = \mathscr{H}_0(A, Q \mathbin{\widehat{\otimes}} A) = \mathrm{Tor}_0^A(A, Q) = Q$$

is indiscrete. Since Q is Hausdorff, we conclude that $Q = 0$. ∎

Remark 2.1. It follows from the proof of Theorem 2.1 that one can require the condition $(*)$ only for A-bimodules of the form $M = \mathscr{L}(E, F)$, where E is a left Banach A-module and $F \in A\text{-mod}(\mathscr{C})$. In particular, if A is a Fréchet algebra and $\mathscr{C}$ is the category of Fréchet spaces, then it is sufficient to verify $(*)$ for Fréchet A-bimodules.

Suppose A is a $\widehat{\otimes}$-algebra, $A^e = A \mathbin{\widehat{\otimes}} A^{\mathrm{op}}$ is the enveloping algebra of A. Recall that the *diagonal ideal* $I_\Delta \subset A^e$ is the kernel of the canonical projection $\pi : A^e \to A$, $a \otimes b \mapsto ab$. We say that A *has a free bimodule Koszul resolution of length* n if there exist central elements $u_1, \ldots, u_n \in I_\Delta$ such that the Koszul complex $K_\bullet(A^e; u_1, \ldots, u_n)$ together with the canonical projection π gives a resolution of the A-bimodule A:

$$0 \leftarrow A \xleftarrow{\pi} K_\bullet(A^e; u_1, \ldots, u_n).$$

Corollary 2.2. *Let A be a nuclear Arens–Michael algebra such that the underlying space of A is either a Fréchet space or a (DF)-space. Suppose A has a free bimodule Koszul resolution of length n. Then $A\text{-mod}(\mathscr{C})$ does not contain any non-zero module Q such that $\mathrm{Ext}^n_A(X, Q) = 0$ for each left Banach A-module X. If, moreover, $\mathscr{C}$ contains all Banach spaces, then* $\mathrm{injdh}_A Q = n$ *for each non-zero* $Q \in A\text{-mod}(\mathscr{C})$.

Proof. Let us prove that A satisfies the conditions of Theorem 2.1. Suppose $K_\bullet = K_\bullet(A^e; u_1, \ldots, u_n)$ is a free bimodule Koszul resolution of A. Let M be an arbitrary $\widehat{\otimes}$-bimodule over A. Then

$$\mathscr{H}^p(A, M) = H^p(\mathbf{h}_{A^e}(K_\bullet, M)) \quad \text{and} \quad \mathscr{H}_p(A, M) = H_p(K_\bullet \mathbin{\widehat{\otimes}}_{A^e} M).$$

It is well known that the cochain complex $K^\bullet \mathbin{\widehat{\otimes}}_{A^e} M$ is topologically isomorphic to $\mathbf{h}_{A^e}(K_\bullet, M)[n]$, where the symbol $[n]$ means that the complex is translated to the left by n steps. Hence,

$$\mathscr{H}_p(A, M) = H^{-p}(K^\bullet \mathbin{\widehat{\otimes}}_{A^e} M) = H^{n-p}(\mathbf{h}_{A^e}(K_\bullet, M)) = \mathscr{H}^{n-p}(A, M)$$

for all p. By putting $p = 0$, we see that the conditions of Theorem 2.1 are satisfied. In particular, $\operatorname{injdh}_A Q \geq n$ provided that $\mathscr{C}$ contains all Banach spaces. Since A has a free bimodule resolution of length n, we see that $\operatorname{Ext}_A^p(E, F) = 0$ for all left $\widehat{\otimes}$-modules E, F over A and all $p > n$. This implies that $\operatorname{injdh}_A Q \leq n$ for each $Q \in A\text{-mod}(\mathscr{C})$. Finally, we get the equality $\operatorname{injdh}_A Q = n$. ∎

3. Examples. Consider the following Arens–Michael algebras:

1. The Fréchet algebra $\mathscr{O}(U)$ of holomorphic functions on a polydomain $U = U_1 \times \ldots \times U_n \subset \mathbb{C}^n$;
2. The Fréchet algebra $C^\infty(V)$ of smooth functions on a domain $V \subset \mathbb{R}^n$;
3. The Fréchet algebra $\mathbb{C}[[z_1, \ldots, z_n]]$ of formal power series;
4. The (DF)-algebra $\mathscr{O}(K)$ of germs of holomorphic functions on a compact set $K = K_1 \times \ldots \times K_n \subset \mathbb{C}^n$, where $K_i \subset \mathbb{C}$.

All these algebras have free bimodule Koszul resolutions. This was proved by Taylor [13] for cases 1 and 2. (For cases 3 and 4, the proof is analogous, so we omit it.) Hence, these algebras satisfy the conditions of Corollary 2.2.

5. Consider the algebra $\mathfrak{D}(\Omega)$ of "holomorphic functions in n free variables", constructed by Taylor [13]. This algebra is defined as follows. Consider the space M_k of $k \times k$-matrices over $\mathbb{C}$. Let $\Omega_k = (M_k)^n$ be the direct product of n copies of M_k, and let $\Omega = \bigcup_{k=1}^\infty \Omega_k$ be the disjoint union of all Ω_k. Consider the Fréchet algebra $\mathfrak{B}(\Omega) = \prod_{k=1}^\infty \mathscr{O}(\Omega_k, M_k)$. By definition, the algebra $\mathfrak{D}(\Omega)$ consists of all elements $f = (f_k) \in \mathfrak{B}(\Omega)$ that satisfy the condition $f_j(a)m = mf_k(b)$ whenever $a = (a_1, \ldots, a_n) \in \Omega_j$, $b = (b_1, \ldots, b_n) \in \Omega_k$ and $a_i m = m b_i$ for some $j \times k$-matrix m and all i. Obviously, $\mathfrak{D}(\Omega)$ is a nuclear Fréchet–Arens–Michael algebra. Moreover, it contains the free algebra $\mathbb{C}\langle\zeta_1, \ldots, \zeta_n\rangle$ on n generators, where $\zeta_i \in \mathfrak{D}(\Omega)$ is defined by $\zeta_i(a_1, \ldots, a_n) = a_i$. Taylor proved that $\mathfrak{D}(\Omega)$ has a free bimodule resolution of the form

$$0 \leftarrow \mathfrak{D}(\Omega) \xleftarrow{\pi} \mathfrak{D}(\Omega) \widehat{\otimes} \mathfrak{D}(\Omega) \xleftarrow{\delta} \mathfrak{D}(\Omega) \widehat{\otimes} \mathfrak{D}(\Omega) \widehat{\otimes} \mathbb{C}^n \leftarrow 0$$

(see [13, 6.15]); here $\pi(u \otimes v) = uv$ and $\delta(u \otimes v \otimes e_i) = u\zeta_i \otimes v - u \otimes \zeta_i v$. Therefore, for each $\widehat{\otimes}$-bimodule M over $\mathfrak{D}(\Omega)$ the space $\mathscr{H}^1(\mathfrak{D}(\Omega), M)$ is the cokernel of the map

$$M \to M \otimes \mathbb{C}^n, \quad x \mapsto \sum_{i=1}^n (x \cdot \zeta_i - \zeta_i \cdot x) \otimes e_i,$$

and the space $\mathscr{H}_0(\mathfrak{D}(\Omega), M)$ is the cokernel of the map

$$M \otimes \mathbb{C}^n \to M, \quad x \otimes e_i \mapsto x \cdot \zeta_i - \zeta_i \cdot x$$

(see [13, 6.3]). Evidently, if the former map is surjective, then so is the latter. Hence, $\mathscr{H}^1(\mathfrak{D}(\Omega), M) = 0$ implies that $\mathscr{H}_0(\mathfrak{D}(\Omega), M) = 0$. We see that the algebra $\mathfrak{D}(\Omega)$ satisfies the conditions of Theorem 2.1.

Now we can specify the category $\mathscr{C}$ in a suitable way depending on the algebra we consider. Namely, in examples 1, 2, 3, and 5 we take the category of Fréchet spaces, while in example 4 we use the category of complete barreled (DF)-spaces. Certainly, in all these cases we can use the category of all complete locally convex spaces as well. Then Theorem 2.1 and Corollary 2.2 imply the following.

COROLLARY 3.1. *There are no non-zero injective Fréchet modules and no non-zero injective $\widehat{\otimes}$-modules over the algebras $\mathscr{O}(U)$, $C^\infty(V)$, $\mathbb{C}[[z_1, \ldots, z_n]]$ and $\mathfrak{D}(\Omega)$.*

COROLLARY 3.2. *There are no non-zero injective barreled (DF)-modules and no non-zero injective $\widehat{\otimes}$-modules over the algebra $\mathscr{O}(K)$.*

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Ideals of finite rank operators, intersection properties of balls, and the approximation property

by

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Abstract. We characterize the approximation property of Banach spaces and their dual spaces by the position of finite rank operators in the space of compact operators. In particular, we show that a Banach space E has the approximation property if and only if for all closed subspaces F of c_0, the space $\mathcal{F}(F,E)$ of finite rank operators from F to E has the n-intersection property in the corresponding space $\mathcal{K}(F,E)$ of compact operators for all n, or equivalently, $\mathcal{F}(F,E)$ is an ideal in $\mathcal{K}(F,E)$.

1. Introduction. A subspace Y of a Banach space X is called an *ideal* if there exists a norm one projection P on X^* with $\ker P = Y^\perp$ (the annihilator of Y in X^*). A subspace Y of a Banach space X is said to have the *n-intersection property* for some natural number n if for all families $\{B_X(a_i,r_i)\}_{i=1}^n$ of closed balls in X whose centers a_i lie in Y,

$$\bigcap_{i=1}^n B_X(a_i,r_i) \neq \emptyset \quad \text{implies} \quad \bigcap_{i=1}^n B_Y(a_i,r_i+\varepsilon) \neq \emptyset \text{ for all } \varepsilon > 0.$$

Let E and F be Banach spaces. We denote by $\mathcal{L}(E,F)$ the Banach space of all bounded linear operators from E to F, and by $\mathcal{K}(E,F)$, $\mathcal{F}(E,F)$, and $\overline{\mathcal{F}}(E,F)$ its subspaces of compact, finite rank, and approximable operators (i.e. norm limits of finite rank operators).

In [12], Å. Lima characterized the metric approximation property of Banach spaces and their dual spaces by the position of finite rank operators in the space of bounded linear operators. It was proved e.g. that (cf. [12, Theorem 13]) for a Banach space E having the Radon–Nikodým property, the following assertions are equivalent:

(a) E has the metric approximation property,

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