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Measures and lacunary sets

by

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Abstract. We establish new connections between some classes of lacunary sets. The main tool is the use of (p,q)-summing or weakly compact operators (for Riesz sets). This point of view provides new properties of stationary sets and allows us to generalize to more general abelian groups than the torus some properties of p-Sidon sets. We also construct some new classes of Riesz sets.

1. Introduction, notations and definitions. In the first section, we make precise the framework of this paper.

In the second section, we give some inequalities satisfied by measures whose spectrum contains either a stationary set (introduced by G. Pisier in [P-1]) or a p-Sidon set, and the remaining part of this spectrum is a $\Lambda(1)$ set in a general setting or $\mathbb{N}$ in the setting of the torus. We obtain some Fourier multiplier properties for these measures.

The study of the behaviour of their Fourier coefficients leads to the construction of some new sets of continuity. This is done in the third section.

In the fourth part, we give a partial connection between stationary sets and $\Lambda(2)$ sets.

In the fifth part, we study a subclass of p-Sidon sets. This provides some Banach type condition for p-Sidon sets to contain products of two infinite sets.

In the sixth part, we construct new Riesz sets. We notice that CUC sets are Riesz sets. We extend results of F. Lust-Piquard ("$c_0 \not\subset C_\Lambda(G) \Rightarrow \Lambda$ is Riesz"), W. Rudin ("$E \subset \mathbb{N}$ is a $\Lambda(1)$ set $\Rightarrow \mathbb{Z}^- \cup E$ is Riesz") and Y. Meyer ("If E is the set of squares or the set of prime numbers, then $\mathbb{Z}^- \cup E$ is Riesz").

Let G be an infinite metrizable compact abelian group, equipped with its normalized Haar measure dx, and Γ its dual group (discrete and countable).

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G will mostly be the unit circle of the complex plane and then Γ will be identified with $\mathbb{Z}$ by $p \mapsto e_p$, where $e_p(x) = e^{2i\pi px}$.

We denote by $\mathcal{P}(G)$ the set of trigonometric polynomials over G, i.e. the set of all finite sums $\sum_{\gamma\in\Gamma} a_\gamma\gamma$, where $a_\gamma \in \mathbb{C}$.

We denote by $C(G)$ the space of complex continuous functions on G, with the norm $\|f\|_\infty = \sup_{x\in G}|f(x)|$. This is also the completion of $\mathcal{P}(G)$ for $\|\cdot\|_\infty$.

$M(G)$ denotes the space of complex regular Borel measures over G, equipped with the total variation norm. If $\mu \in M(G)$, its Fourier transform at the point γ is defined by $\widehat{\mu}(\gamma) = \int_G \gamma(-x)\,d\mu(x)$.

$L^p(G)$ denotes the Lebesgue space $L^p(G, dx)$ with the norm

$$\|f\|_p = \begin{cases} \Big(\int\limits_G |f(x)|^p\,dx\Big)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}|f(x)|, & p = \infty. \end{cases}$$

The map $f \mapsto f\,dx$ identifies $L^1(G)$ with a closed ideal of $M(G)$ equipped with the convolution.

If B is a normed space of functions on G which is continuously embedded in $M(G)$, and if Λ is a subset of Γ, we define

$$B_\Lambda = \{f \in B \mid \widehat{f}(\gamma) = 0,\ \forall\gamma \notin \Lambda\}.$$

This is the set of elements of B whose spectrum is contained in Λ.

$(\varepsilon_\gamma)_{\gamma\in\Gamma}$ denotes a *Bernoulli sequence* indexed by Γ, i.e. a sequence of independent random variables taking values $+1$ and -1 with probability $1/2$. Moreover, $(g_\gamma)_{\gamma\in\Gamma}$ stands for a sequence of centred independent complex gaussian random variables, normalized by $\mathbb{E}|g_\gamma|^2 = 1$, where $\mathbb{E}$ denotes expectation.

$|E|$ denotes the cardinality of the finite set E and E^c the complement (in Γ) of $E \subset \Gamma$.

We use notions which are standard in the framework of Banach space geometry, referring to [D-J-T] or [W] for the definitions of cotype, (p,q)-summing operators, etc. We nevertheless specify that we use the property (V) of Pełczyński in the following form (see [W] for the equivalent definition):

DEFINITION 1.1. A Banach space X has the *property* (V) of Pełczyński if, for every Banach space Y and every operator $T : X \to Y$ which is not weakly compact, there exists a subspace X_0 of X isomorphic to c_0 such that $T_{|X_0}$ is an isomorphic embedding.

We also use the property (V*) of Pełczyński in the following form:

DEFINITION 1.2. A Banach space X has the *property* (V*) of Pełczyński if, for every Banach space Y and every operator $T : Y \to X$ which is not weakly compact, there exists a complemented subspace Y_0 of Y and a complemented subspace X_0 of X isomorphic to ℓ^1 such that $T_{|Y_0} : Y_0 \to X_0$ is an isomorphism.

Let us now recall some classical definitions of lacunary subsets of Γ.

DEFINITION 1.3. Let $1 \le p < 2$. A subset Λ of Γ is a *p-Sidon set* if there exists a constant $C > 0$ such that for any f in $\mathcal{P}_\Lambda(G)$,

$$\Big(\sum_{\lambda\in\Lambda} |\widehat{f}(\lambda)|^p\Big)^{1/p} \le C\|f\|_\infty.$$

The best constant C is called the *p-Sidonicity constant* of Λ and is denoted by $S_p(\Lambda)$ (see for example [B] or [B-P]). Obviously, a p-Sidon set is a q-Sidon set for $q > p$. If Λ is a p-Sidon set and not a q-Sidon set for any $q < p$, then Λ is called a *true p-Sidon set.*

For $p = 1$, this is in fact the notion of Sidon set. For a deep study of such sets, we refer to [D-G], [L-R] or [P-2].

DEFINITION 1.4. A subset A of Γ is *dissociated* if for every sequence $(n_\gamma)_{\gamma\in A} \in \{-2, \ldots, 2\}^A$ with almost all n_γ equal to zero,

$$\prod_{\gamma\in A} \gamma^{n_\gamma} = 1 \Rightarrow \forall\gamma \in A : \gamma^{n_\gamma} = 1.$$

We recall that if A is dissociated, then it is a Sidon set.

DEFINITION 1.5. Let $(F_N)_{N\ge0}$ be an increasing sequence of finite subsets of Γ such that $\bigcup_{N=0}^\infty F_N = \Gamma$. A subset Λ of Γ is a *set of uniform convergence* relative to $(F_N)_{N\ge0}$ (for short, a *UC set*) if for all $f \in C_\Lambda(G)$, $(S_N f)_{N\ge0}$ converges to f in $C_\Lambda(G)$, where $S_N f = \sum_{\gamma\in F_N} \widehat{f}(\gamma)\gamma$.

For such a set, the *UC constant* $U(\Lambda)$ is

$$U(\Lambda) = \sup\{\|S_N(f)\|_\infty \mid f \in C_\Lambda(G),\ \|f\|_\infty = 1,\ N \ge 1\}.$$

REMARK 1.6. This notion depends on the choice of $(F_N)_{N\ge0}$. Here we shall be interested in the case $G = \mathbb{T}$, where the natural choice of $(F_N)_{N\ge0}$ is $F_N = \{-N, \ldots, N\}$. For a (non-exhaustive) survey on UC sets, one may read [L1].

DEFINITION 1.7. A subset Λ of $\mathbb{Z}$ is a *CUC set* if it is a UC set such that $\sup_{p\in\mathbb{Z}} U(p + \Lambda)$ is finite, i.e.

$$\sup\{\|S_{m,n}(f)\|_\infty \mid f \in C_\Lambda(G),\ \|f\|_\infty = 1,\ m, n \ge 1\} < \infty,$$

where $S_{m,n} f = \sum_{-m\le q\le n} \widehat{f}(q)e_q$.

DEFINITION 1.8. A subset Λ of Γ is a *set of continuity* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\mu \in M(G)$ with $\|\mu\| = 1$,

$$\varlimsup_{\Gamma\setminus\Lambda} |\widehat{\mu}(n)| < \delta \Rightarrow \varlimsup_{\Lambda} |\widehat{\mu}(n)| < \varepsilon.$$

The relations between sets of continuity and some other thin sets (in particular, UC, $\Lambda(1)$, p-Sidon) were studied in [F-P].

DEFINITION 1.9. A subset Λ of Γ is a *Riesz set* if $M_\Lambda(G) = L^1_\Lambda(G)$. More precisely, this means that, given any μ in $M_\Lambda(G)$, there is some h in $L^1(G)$ such that $\widehat{h}(\gamma) = \widehat{\mu}(\gamma)$ for all γ in Γ.

More about Riesz sets can be found in [L-P] or [Go].

DEFINITION 1.10. Let $0 < p < \infty$. A subset A of Γ is a $\Lambda(p)$ *set* if there exists $q \in]0,p[$ such that $L^p_A(G) = L^q_A(G)$.
This implies that $L^p_A(G) = L^r_A(G)$ for all $r \in]0,p[$.

REMARK 1.11. For $p > 1$, this is equivalent to: every measure $\mu \in M_A(G)$ actually lies in $L^p_A(G)$.
We recall that, if A is a $\Lambda(1)$ set, then there is some $p > 1$ such that A is a $\Lambda(p)$ set. Thus, A is a $\Lambda(1)$ set if and only if $L^1_A(G)$ is a reflexive space (see [B-E]).

Let us introduce the following norm on $\mathcal{P}(G)$, called the $C^{\text{a.s.}}$ *norm* ("almost surely continuous"):

$$[\![f]\!] = \int_\Omega \Big\| \sum_{\gamma\in\Gamma} \varepsilon_\gamma(\omega)\widehat{f}(\gamma)\gamma \Big\|_\infty d\mathbb{P}(\omega). \tag{1}$$

REMARK 1.12. By [M-P], replacing the Bernoulli sequence $(\varepsilon_\gamma)_{\gamma\in\Gamma}$ by a gaussian sequence $(g_\gamma)_{\gamma\in\Gamma}$ gives an equivalent norm on $\mathcal{P}(G)$.

$C^{\text{a.s.}}(G)$ is, by definition, the completion of $\mathcal{P}(G)$ for the norm $[\![\cdot]\!]$. This is also the set of functions in $L^2(G)$ such that the integral in (1) is finite, or the set of functions in $L^2(G)$ such that, almost surely, $\varepsilon_\gamma(\omega)\widehat{f}(\gamma) = \widehat{f^\omega}(\gamma)$ with f^ω in $C(G)$ (for the equivalence between the quantitative and the qualitative definition, we refer to [K]); $C^{\text{a.s.}}(G)$ is also called the space of almost surely continuous random Fourier series.

DEFINITION 1.13. A subset Λ of Γ is *stationary* (for short, $\Lambda \in \mathcal{S}$) if there exists $C > 0$ such that, for all $f \in \mathcal{P}_\Lambda(G)$,

$$[\![f]\!] \le C\|f\|_\infty.$$

The best constant C is called the *stationarity constant* of Λ and is denoted by $K_S(\Lambda)$.

$\mathcal{S}$ obviously contains the Sidon sets and G. Pisier showed that $\mathcal{S}$ contains all finite products of such sets. $\mathcal{S}$ is thus strictly larger than the class of Sidon sets because it is well known that a Sidon set cannot be the product of two infinite sets.

J. Bourgain [Bo] proved that if A_1 and A_2 are infinite then

$$A_1 \times A_2 \in \mathcal{S} \Leftrightarrow A_1, A_2 \in \mathcal{S} \cap \Lambda(2).$$

Some new properties of the class $\mathcal{S}$ are studied in [L2], and $\mathcal{S}$ is compared to some other classes of lacunary sets.

2. Measures with lacunary spectrum and Fourier multipliers. We now give some properties of measures with lacunary spectrum containing either a stationary or a p-Sidon set. This is done in terms of multipliers and we deduce some information on the behaviour of the Fourier–Stieltjes coefficients of such measures.

In the following, X denotes any Banach space of functions on G, with cotype 2, which admits the characters as an unconditional basis and has the following property:

There is some function $\psi_X : \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{x\to\infty} \psi_X(x) = \infty$ and

$$\text{for all finite } A \subset \Gamma, \qquad \Big\|\sum_{\gamma\in A}\gamma\Big\|_X \ge \psi_X(|A|)\Big\|\sum_{\gamma\in A}\gamma\Big\|_2. \tag{2}$$

An X-*set* is a subset Λ of Γ such that $C_\Lambda(G)$ is canonically and continuously embedded in X. Property (2) of X will be used in Lemma 2.5 and in Theorem 3.2.

An example is $X = C^{\text{a.s.}}(G)$ in which case an X-set is exactly a stationary set. $C^{\text{a.s.}}(G)$ has cotype 2 by [P-1] and (2) is a corollary of the following inequality for the $C^{\text{a.s.}}$ norm ([L2]).

Let $P \in \mathcal{P}(G)$; for $\delta > 0$, set $E_\delta = \{\gamma \in \Gamma \mid |\widehat{P}(\gamma)| \ge \delta\}$ and $N_\delta = |E_\delta|$. Then

$$[\![P]\!] \ge c\delta\sqrt{N_\delta \log N_\delta}$$

(where c depends neither on P nor on δ)

Another example is given by the closure of the polynomials for the ℓ^p norm ($p < 2$) of their coefficients: $\|P\| = \|\widehat{P}\|_p$. Then the X-sets are p-Sidon sets. We clearly have (2).

We now study some multiplier inequalities.

THEOREM 2.1. *Let Λ be an X-set and let $A \subset \Gamma$ be a $\Lambda(1)$ set. Then there exists $c > 0$ such that for every measure μ in $M_{\Lambda\cup A}(G)$ and every h in $L^2(G)$,*

$$\Big\|\sum_{\lambda\in\Lambda}\widehat{\mu}(\lambda)\widehat{h}(\lambda)\lambda\Big\|_X \le c\|\mu\|\cdot\|h\|_2. \tag{3}$$

This means that, for every measure $\mu \in M_{\Lambda\cup A}(G)$, the function $\widehat{\mu}1_\Lambda$ defines a bounded multiplier from $L^2(G)$ to X.

Proof. We use the following theorem due independently to Kislyakov and Pisier (cf. Th. 15.13, p. 316 of [D-J-T]), where S denotes a subspace of $C(G)$: "If the quotient $C(G)/S$ is reflexive and X has cotype 2 then any operator $T : S \to X$ is 2-summing".

This is applied to $S = C_{\Lambda^c}(G)$, observing that the dual of $C(G)/C_{\Lambda^c}(G)$ is $M_\Lambda(G) = L^1_\Lambda(G)$ (recall Remark 1.11), which is reflexive, hence $C(G)/C_{\Lambda^c}(G)$ is reflexive too.

Now fix μ in $M_{\Lambda\cup A}(G)$ and take $T = T_\mu$ defined by

$$C_{A^c}(G) \to X, \qquad f \mapsto \mu * f.$$

This operator is bounded because for all f in $C_{A^c}(G)$, $f*\mu$ lies in $C_\Lambda(G)$ and Λ is an X-set. Moreover, $\|T_\mu\| \le K\|\mu\|$, where K denotes the norm of the embedding of $C_\Lambda(G)$ into X.

T_μ is therefore 2-summing and the Pietsch factorization theorem provides us with a constant C' and a probability measure ν on G such that, for every h in $C_{A^c}(G)$,

$$\|T_\mu(h)\|_X \le C'\|\mu\| \Big(\int_G |h(x)|^2\, d\nu(x)\Big)^{1/2}.$$

Actually, we can choose $d\nu = dx$. Indeed, the unconditionality of the characters for the norm in X provides a constant k such that for every $g\in G$ and $h \in C_{A^c}(G)$,

$$\|T_\mu(h)\|_X^2 \le k\|T_\mu(h_g)\|_X^2 \le k(C'\|\mu\|)^2 \int_G |h(x+g)|^2\, d\nu(x).$$

We integrate the previous inequality over G to obtain, via Fubini's theorem and the unconditionality of the characters in $L^2(G)$,

$$\|T_\mu(h)\|_X^2 \le k(C'\|\mu\|)^2 \int_G \|h_x\|_2^2\, d\nu(x) = k(C'\|\mu\|)^2 \int_G \|h\|_2^2\, d\nu(x)$$
$$= k(C'\|\mu\|)^2 \|h\|_2^2.$$

We conclude by using the density of $C_{A^c}(G)$ in $L^2_{A^c}(G)$. ∎

In the case of the torus, we get a stronger result.

THEOREM 2.2. *Let Λ be an X-set included in $\mathbb{N}$. Then there exists $c>0$ such that for every measure μ in $M_{\Lambda\cup\mathbb{Z}^-}(\mathbb{T})$ and every h in $L^2(\mathbb{T})$,*

$$\Big\|\sum_{\lambda\in\Lambda} \widehat{\mu}(\lambda)\widehat{h}(\lambda)\lambda\Big\|_X \le c\|\mu\|\cdot\|h\|_2. \tag{4}$$

This means that, for each measure $\mu \in M_{\Lambda\cup\mathbb{Z}^-}$-$\mathbb{T}$, $\widehat{\mu}1_\Lambda$ defines a multiplier from $L^2(\mathbb{T})$ to X.

Proof. We give two proofs.

First method. We use the following generalization of the Kahane–Katznelson–de Leeuw theorem due to Kislyakov.

There exists $c>0$ such that for every $h\in \ell^2(\mathbb{N})$, there exists $f \in A(\mathbb{D})$ such that

$$\|f\|_\infty \le c\|h\|_2, \qquad |\widehat{f}(\gamma)| \ge |h_\gamma| \quad \text{for all } \gamma\in\mathbb{N}. \tag{5}$$

Fix h in ℓ^2 and μ in $M_{\Lambda\cup\mathbb{Z}^-}(\mathbb{T})$, and let $f\in C_{\mathbb{N}}(\mathbb{T}) = A(\mathbb{D})$ be as in (5).

We have $\mu * f \in C_\Lambda(\mathbb{T})$ and $\|\mu*f\|_\infty \le \|\mu\|\cdot\|f\|_\infty \le c\|\mu\|\cdot\|h\|_2$. As Λ is an X-set, this implies $\|\mu * f\|_X \le C\|\mu\|\cdot\|h\|_2$ for some constant $C>0$.

On the other hand, using the unconditionality of the characters in X, we have

$$2\|\mu*f\|_X \ge \Big\|\sum_{\lambda\in\Lambda}\widehat{\mu}(\lambda)h_\lambda\lambda\Big\|_X.$$

Hence

$$\Big\|\sum_{\lambda\in\Lambda}\widehat{\mu}(\lambda)h_\lambda\lambda\Big\|_X \le 2C\|\mu\|\cdot\|h\|_2. \ ∎$$

Second method. We use a theorem which is due to Bourgain ([W], p. 305): every bounded operator from $A(\mathbb{D})$ to a cotype 2 space is 2-summing.

Hence the operator T_μ, introduced in the proof of 2.1, is actually 2-summing. The proof ends as in Theorem 2.1. ∎

In the particular case of stationary sets, we immediately get the following two corollaries:

COROLLARY 2.3. *Let Λ be a stationary set and let A be either a $\Lambda(1)$ set (for any G) or $\mathbb{Z}^-$ (when $G=\mathbb{T}$). Then there exists $c>0$ such that for every measure μ in $M_{\Lambda\cup A}(G)$ and every h in $L^2(G)$,*

$$\Big[\!\!\Big[\sum_{\lambda\in\Lambda}\widehat{\mu}(\lambda)\widehat{h}(\lambda)\lambda\Big]\!\!\Big] \le c\|\mu\|\cdot\|h\|_2. \tag{6}$$

Concerning p-Sidon sets, in the case of the torus, we obtain a new proof of an inequality due to Fournier and Pigno [F-P], which was based on an inequality of Edwards and a multiplier inequality of Stechkin. In a more general setting, we get the following corollary:

COROLLARY 2.4. *Let Λ be a p-Sidon set and let A be a $\Lambda(1)$ set. Set $r = 2p/(2-p)$. Then, for every measure $\mu\in M_{\Lambda\cup A}(G)$,*

$$\Big(\sum_{\lambda\in\Lambda}|\widehat{\mu}(\lambda)|^r\Big)^{1/r} \le c\|\mu\| \tag{7}$$

where c depends only on A and Λ.

Proof. The previous results on X-sets applied to p-Sidon sets provide $c>0$ such that for every $\mu\in M_{\Lambda\cup A}(G)$ and every $h\in L^2(G)$,

$$\Big(\sum_{\lambda\in\Lambda}|\widehat{\mu}(\lambda)\widehat{h}(\lambda)|^p\Big)^{1/p} \le c\|\mu\|\cdot\|h\|_2.$$

Now, ℓ^r is exactly the space of multipliers from ℓ^2 to ℓ^p. ∎

LEMMA 2.5. *Let A and B be subsets of Γ such that there exists $c > 0$ such that for every $\mu \in M_{A\cup B}(G)$ and every $h \in L^2(G)$,*

$$\Big\| \sum_{\lambda\in A} \widehat{\mu}(\lambda)\widehat{h}(\lambda)\lambda\Big\|_X \le c\|\mu\|\cdot\|h\|_2. \tag{8}$$

Then for every $\mu \in M_{A\cup B}(G)$,

$$\psi_X(|\{\gamma\in A \mid |\widehat{\mu}(\gamma)|\ge\delta\}|) \le k\|\mu\|/\delta \tag{9}$$

where k depends only on the constant c in (8).

Proof. Let $\mu \in M_{A\cup B}(G)$ and $\delta > 0$.

Set $\Lambda_\delta = \{\gamma\in A \mid |\widehat{\mu}(\gamma)|\ge\delta\}$. Let Λ'_δ be a finite subset of Λ_δ. Observe that

$$f := \frac{1}{|\Lambda'_\delta|^{1/2}}\sum_{\gamma\in\Lambda'_\delta}\gamma \in L^2(G) \quad\text{and}\quad \|f\|_2 = 1.$$

The hypothesis leads to $\|\mu * f\|_X \le c\|\mu\|$. Since for every $\gamma \in \Lambda'_\delta$, $\widehat{\mu * f}(\gamma) = \frac{1}{|\Lambda'_\delta|^{1/2}}\widehat{\mu}(\gamma)$, as Λ is an X-set, we get the inequality (where c' depends only on c)

$$\|\mu * f\|_X \ge c'\delta\psi_X(|\Lambda'_\delta|).$$

Hence $k\|\mu\| \ge \delta\psi_X(|\Lambda'_\delta|)$. Taking the upper bound over all finite subsets Λ'_δ of Λ_δ, we conclude that Λ_δ itself is finite and

$$k\|\mu\| \ge \delta\psi_X(|\Lambda_\delta|).$$

This proves (9). ■

3. Applications to sets of continuity

DEFINITION 3.1. We say that an infinite metrizable abelian compact group G has the *property $\mathcal{D}$* if any infinite subset of $\Gamma = \widehat{G}$ contains an infinite dissociated subset.

This essentially means that we discard the case where the set $\{\gamma^2 \mid \gamma \in \Gamma\}$ is finite.

By [F-P], if Λ is either a UC, a p-Sidon or a $\Lambda(1)$ subset of $\mathbb{N}$, then $\mathbb{Z}^-\cup\Lambda$ is a set of continuity. We prove a similar result for stationary sets. We also study the case of an infinite metrizable abelian compact group having the property $\mathcal{D}$ and show that the union of a $\Lambda(1)$ and a stationary subset (resp. a p-Sidon subset) of Γ is again a set of continuity. This follows from part 2.

We state all results for p-Sidon sets or stationary sets but the statements and proofs actually remain valid for any X-set.

We use the following principle essentially contained in [F-P].

THEOREM 3.2. *Let A and B be subsets of Γ (where $\widehat{\Gamma} = G$ has the property $\mathcal{D}$) such that B is a set of continuity and there is a strictly increasing function ϕ from $\mathbb{N}$ to $\mathbb{R}^{+*}$, tending to ∞, such that*

$$\forall\mu\in M_{A\cup B}(G),\ \forall\delta>0:\quad \phi(|\{\gamma\in A \mid |\widehat{\mu}(\gamma)|\ge\delta\}|) \le \|\mu\|/\delta. \tag{10}$$

Then $A\cup B$ is a set of continuity.

We get some new classes of sets of continuity.

THEOREM 3.3. *Let A be a stationary subset of Γ (where $\widehat{\Gamma} = G$ has the property $\mathcal{D}$), and let B be either a $\Lambda(1)$ set or $\mathbb{Z}^-$. Then $A\cup B$ is a set of continuity.*

Proof. Corollary 2.3 and Lemma 2.5 lead to (9) with A a stationary set and B a $\Lambda(1)$ set or $\mathbb{Z}^-$. So, the hypothesis (10) of Theorem 3.2 is satisfied with $\phi(t) = C(\log t)^{1/2}$ where C depends only on A and B. As $\mathbb{Z}^-$ (by the classical Katznelson–de Leeuw theorem) and every $\Lambda(1)$ set are, in particular, sets of continuity, Theorem 3.2 concludes the proof. ■

For p-Sidon sets, we get:

THEOREM 3.4. *Let A be a p-Sidon subset of Γ (where $\widehat{\Gamma} = G$ has the property $\mathcal{D}$), and B a $\Lambda(1)$ set. Then $A\cup B$ is a set of continuity.*

Proof. Corollary 2.4 and Lemma 2.5 lead to (9) with A a p-Sidon set and B a $\Lambda(1)$ set. So, hypothesis (10) of Theorem 3.2 is satisfied with $\phi(t) = Ct^{1/p-1/2}$ where C depends only on A and B. As B is a $\Lambda(1)$ set, it is, in particular, a set of continuity. Theorem 3.2 concludes the proof. ■

In the special case of the torus, we get a result which is already contained in [F-P]: the union of a p-Sidon subset of $\mathbb{N}$ and $\mathbb{Z}^-$ is a set of continuity.

4. Spreadable stationary sets and $\Lambda(2)$ sets. These notions are inspired by the following one due to [M-P].

DEFINITION 4.1. Let Λ be a subset of Γ. A disjoint family $\{\Sigma_j\}$ of subsets of Λ is a *Sidon partition* of Λ if $\bigcup_j \Sigma_j = \Lambda$ and there is some constant $C > 0$ such that, for every choice of σ_j in Σ_j, the set $\{\sigma_j\}_j$ is Sidon with a Sidon constant bounded by C.

We recall the following:

THEOREM 4.2 ([M-P], p. 131). *If $\{\Sigma_j\}_j$ is a Sidon partition of $\Lambda\subset\Gamma$ then there exists $M > 0$ such that, for every $h \in C^{\mathrm{a.s.}}(G)$,*

$$\sum_j \|h_j\|_2 \le M\Big[\!\Big[\sum_j h_j\Big]\!\Big] \quad \textit{with} \quad h_j = \sum_{\lambda\in\Sigma_j}\widehat{h}(\lambda)\lambda.$$

We introduce a subclass of stationary sets.

DEFINITION 4.3. A subset Λ of Γ is *spreadable stationary* if there exists a constant C such that, for every family of finite subsets Λ_j of Λ, there exists a stationary set S, with stationarity constant less than C, and characters

γ_j such that $\{\gamma_j \Lambda_j\}_j$ is a Sidon partition with constant less than C and $\gamma_j \Lambda_j \subset S$.

REMARK 4.4. A sufficient condition for Λ to be a spreadable stationary set is that there exists an infinite subset H of Γ such that $H \cdot \Lambda$ is again stationary. Of course, any spreadable stationary set is stationary.

A natural problem arising from Bourgain's theorem ([Bo]) on the product of stationary sets is about the link between stationary sets and $\Lambda(2)$ sets. We are able to answer this question in the setting of spreadable stationary sets.

THEOREM 4.5. *Any spreadable stationary set is a $\Lambda(2)$ set.*

Proof. Consider a finite family of polynomials f_j with spectrum Λ_j included in Λ. By hypothesis, we get a constant C and characters γ_j such that $\{\gamma_j \Lambda_j\}_j$ is a Sidon partition. There exists a stationary set S with constant less than C containing the spectrum of each polynomial $\gamma_j f_j$. So, we get

$$\Big[\!\Big[\sum_j \gamma_j f_j\Big]\!\Big] \le C \sup_{|\varepsilon_j|=1} \Big\|\sum_j \varepsilon_j \gamma_j f_j\Big\|_\infty = C \sup_{x\in G} \sum_j |\gamma_j(x) f_j(x)|$$
$$\le C \sup_{x\in G} \sum_j |f_j(x)|.$$

Now Theorem 4.2 provides C' such that

$$\sum_j \|f_j\|_2 = \sum_j \|\gamma_j f_j\|_2 \le C' \Big[\!\Big[\sum_j \gamma_j f_j\Big]\!\Big].$$

So we conclude that

$$\sum_j \|f_j\|_2 \le CC' \sup_{x\in G} \sum_j |f_j(x)|.$$

This means that the formal inclusion of $C_\Lambda(G)$ in $L^2(G)$ is 1-summing. Using the Pietsch domination theorem and the translation invariance as in Theorem 2.1, we get the result. ∎

5. Spreadable p-Sidon sets.

We introduce the notion of spreadable p-Sidon set.

DEFINITION 5.1. A subset Λ of Γ is *spreadable p-Sidon* if there is some constant C such that for every family $(\Lambda_j)_j$ of finite subsets of Λ, there are characters γ_j and a p-Sidon set E, with $S_p(E)$ less than C, such that the sets $\gamma_j \Lambda_j$ are disjoint and included in E.

REMARK 5.2. A sufficient condition is the existence of an infinite set H such that $H \cdot \Lambda$ is a p-Sidon set. Obviously, any spreadable p-Sidon set is p-Sidon.

NOTATION. Let A be a p-Sidon set. We denote by j_A the operator

$$C_A(G) \to \ell^p, \qquad h \mapsto \widehat{h},$$

whose norm is $S_p(A)$ by definition.

THEOREM 5.3. *Let Λ be a spreadable p-Sidon set. Then j_Λ is $(p,1)$-summing.*

Proof. Let $(f_j)_{1\le j\le n}$ be a finite family of polynomials $\in C_\Lambda(G)$. There exist characters $h_1, \ldots, h_n$ in Γ and a p-Sidon set E such that the polynomials $h_j f_j$ have disjoint spectra (because of the spreadable character of Λ) included in E. The p-Sidonicity of E gives

$$\Big\|\Big(\sum_{1\le j\le n} h_j f_j\Big)^\wedge\Big\|_p \le S_p(E) \Big\|\sum_{j=1}^n h_j f_j\Big\|_\infty \le S_p(E) \sup_{x\in G} \sum_j |f_j(x)|.$$

As the spectra of the polynomials $h_j f_j$ are disjoint,

$$\Big(\sum_{1\le j\le n} \|\widehat{f_j}\|_p^p\Big)^{1/p} = \Big(\sum_{1\le j\le n} \|\widehat{h_j f_j}\|_p^p\Big)^{1/p} = \Big\|\Big(\sum_{1\le j\le n} h_j f_j\Big)^\wedge\Big\|_p.$$

So

$$\Big(\sum_{1\le j\le n} \|j_\Lambda(f_j)\|^p\Big)^{1/p} \le S_p(E) \sup_{x\in G} \sum_{1\le j\le n} |f_j(x)|.$$

This proves the result. ∎

REMARK 5.4. With Remark 5.2 and the previous theorem, we get a Banach type condition for a p-Sidon set Λ to contain the product of two infinite sets A and B: j_A and j_B are $(p,1)$-summing, with $(p,1)$-summing norm dominated independently of A and B (but obviously depending on Λ).

Using the $(p,1)$-summing point of view, we recover an old result on products of p-Sidon sets which is very easy to prove using Walsh matrices. Conversely, the matrix point of view does not seem to imply $(p,1)$-summing properties.

COROLLARY 5.5 ([L-R]). *Let Λ be a p-Sidon set, and let A and B be infinite subsets of Γ such that $A \cdot B \subset \Lambda$. Then $p \ge 4/3$.*

Proof. By Remark 5.4, the following lemma proves the claim.

LEMMA 5.6. *Let $1 \le p < 4/3$. There is no infinite set A such that j_A is $(p,1)$-summing.*

Proof. Suppose the existence of such a set A. We give two proofs.

(p,q)-summing point of view. As $p < 4/3$ implies $p/(2-p) < 2$, we may choose a real q such that $p/(2-p) < q < 2$.

The injection i from ℓ^p to ℓ^q is $((1/p+1/2-1/q)^{-1},1)$-summing (cf. Th. on p. 209 of [D-J-T]). By a theorem of König, Retherford and Tomczak-Jaegermann (p. 208 of [D-J-T]), the composition $i\circ j_A$ is 2-summing.

By the Pietsch domination theorem and the translation invariance of the operator $i\circ j_A$, we get $\|i\circ j_A(h)\|_q\le C\|h\|_2$ for every h in $C_A(G)$, which extends by density to every h in $L^2_A(G)$.

This is clearly impossible for $q<2$.

Elementary point of view: *Walsh matrices.* Given an arbitrary integer n, select n distinct characters $a_1,\dots,a_n$ in A. For $1\le j\le n$, we define

$$f_j=\sum_{k=1}^{n}\frac{1}{\sqrt{n}}\exp\left(\frac{2i\pi kj}{n}\right)a_k.$$

The polynomials f_j have spectra included in A. For every ε_j of modulus 1,

$$\Big\|\sum_{1\le j\le n}\varepsilon_j f_j\Big\|_\infty=\Big\|\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{\sqrt{n}}\exp\left(\frac{2i\pi kj}{n}\right)\varepsilon_j a_k\Big\|_\infty$$
$$\le\|\{\varepsilon_j\}\|_2\sup_{x\in G}\|\{a_k(x)\}\|_2=n.$$

As j_A is $(p,1)$-summing,

$$n^{2/p-1/2}=\Big(\sum_{j=1}^{n}n^{1-p/2}\Big)^{1/p}=\Big(\sum_{j=1}^{n}\|\widehat{f}_j\|_p^p\Big)^{1/p}\le Cn$$

for arbitrary n, hence $p\ge4/3$. ■

We add an immediate corollary of Theorem 5.3.

COROLLARY 5.7. *Let Λ be a spreadable p-Sidon set and $\mu\in M_\Lambda(G)$. Then μ defines a Fourier multiplier from the Lorentz space $L^{p,1}(G)$ to ℓ^p. In particular, μ defines a Fourier multiplier from $L^r(G)$ to ℓ^p, for any $r>p$.*

Proof. It suffices to compose j_Λ with the convolution operator defined by μ acting from $C(G)$ to $C_\Lambda(G)$ and use [D-J-T], Th. 10.9, p. 204. ■

6. Riesz sets. The proof of the following theorem is very easy but we have not been able to find any reference to it.

THEOREM 6.1. *Every CUC set in $\mathbb{Z}$ is a Riesz set.*

Proof. Let Λ be a CUC set. Then there exists a measure δ on $\mathbb{T}$ such that $\widehat{\delta}(n)=0$ for every $n\in\Lambda\cap\mathbb{Z}^-$, and $\widehat{\delta}(n)=1$ for every $n\in\Lambda\cap\mathbb{N}$ (cf. [S-T]).

Given μ in $M_\Lambda(\mathbb{T})$, the measure $\delta*\mu$ is in $M_{\mathbb{N}}(\mathbb{T})$. As $\mathbb{N}$ is a Riesz set, there is some f in L^1 such that $\widehat{f}(n)=\widehat{\delta*\mu}(n)$ for every $n\in\mathbb{Z}$. A fortiori, $\widehat{f}(n)=\widehat{\mu}(n)$ for every $n\in\mathbb{N}$.

Hence, $f-\mu$ lies in $M_{\mathbb{Z}^-}(\mathbb{T})$ and in the same way ($\mathbb{Z}^-$ is a Riesz set), there is some g lying in L^1 such that $\widehat{g}(n)=(\widehat{f-\mu})(n)$ for any n in $\mathbb{Z}$. Therefore $\widehat{\mu}=\widehat{f+g}$ and $f+g\in L^1$. ■

REMARK 6.2. When Λ is a CUC set, the previous proof actually shows that the Hilbert transform is bounded from $M_\Lambda(\mathbb{T})$ to H^1. Hence, for every measure μ with spectrum in Λ,

$$\mu^+=\sum_{n\in\Lambda\cap\mathbb{N}}\widehat{\mu}(n)e_n\quad\text{and}\quad\mu^-=\sum_{n\in\Lambda\cap\mathbb{Z}^-}\widehat{\mu}(n)e_{-n}\quad\text{are in }H^1.$$

DEFINITION 6.3. $X\subset L^\infty(G)$ has the property "$c_0\not\subset X$" if X does not contain any subspace isomorphic to c_0.

We shall exhibit some new Riesz sets. A theorem of Dressler and Pigno states that the union of a Riesz set and a Rosenthal set Λ (i.e. one with $L^\infty_\Lambda(G)=C_\Lambda(G)$) is again a Riesz set. We recall two longstanding open problems in the theory of lacunary sets:

(Q1) If "$c_0\not\subset C_\Lambda(G)$", does it follow that Λ is a Rosenthal set?

(Q2) If "$c_0\not\subset L^\infty_\Lambda(G)$", does it follow that Λ is a Rosenthal set?

(the converses of (Q1) and (Q2) are clearly true).

If we could find a Riesz set E and a set Λ such that $c_0\not\subset C_\Lambda(G)$ and $E\cup\Lambda$ is not Riesz, we could get a counterexample to (Q1). The following results, which generalize some classical ones on Riesz sets, show that we have to turn down some situations.

THEOREM 6.4. *Let $\Lambda\subset\Gamma$ have the property "$c_0\not\subset C_\Lambda(G)$" and let E be a Riesz set such that $C_{E^c}(G)$ has the property* (V) *of Pełczyński. Then $E\cup\Lambda$ is a Riesz set.*

Proof. Fix μ in $M_{E\cup\Lambda}(G)$ and f in $L^\infty_{E^c}(G)$.

Consider the convolution operator U associated with μ, acting from $C_{E^c}(G)$ to $C_\Lambda(G)$. Since $C_{E^c}(G)$ has the property (V) of Pełczyński by hypothesis, as $C_\Lambda(G)$ does not contain c_0, it follows that U is weakly compact.

A fortiori, $\mu*f$ defines a weakly compact operator $L^1\to C(G)$, hence $\mu*f$ belongs to $C(G)$; $f\in L^\infty_{E^c}(G)$ being arbitrary, μ is a Fourier multiplier from $L^\infty_{E^c}(G)$ to $C(G)$.

We then apply a theorem of Heard [He]: there exists g in $L^1(G)$ interpolating $\widehat{\mu}$ on E^c, that is, $\widehat{g}(n)=\widehat{\mu}(n)$ for any n in E^c. Therefore $\mu-g\in M_E(G)$. Since E is a Riesz set, $\mu-g\in L^1(G)$, and $\mu=g+(\mu-g)$. This concludes the proof. ■

We deduce the following:

COROLLARY 6.5. *Let $\Lambda \subset \Gamma$ have the property "$c_0 \not\subset C_\Lambda(G)$" and let E be a $\Lambda(1)$ subset of Γ. Then $E \cup \Lambda$ is a Riesz set.*

Proof. *First method.* It suffices to prove that E satisfies the hypothesis of Theorem 6.4. As the space $R = M_E(G)$ is reflexive, E is a Riesz set and as $X = C(G)$ has the property (V) of Pełczyński, a result of G. Godefroy and P. Saab (Th. III.4 of [G-S]) implies that $R^\perp = C_{E^c}(G)$ has the property (V) of Pełczyński.

Second method. We give a self-contained proof. If $E \cup \Lambda$ is not a Riesz set, then there exists a measure $\mu \in M_{E\cup\Lambda}(G)$ and a sequence (f_n) of continuous functions spanning a subspace X of $C(G)$ isomorphic to c_0 such that the sequence $(\mu * f_n)$ also spans such a subspace ([L-P], p. 71). We can choose (f_n) normalized.

The convolution operator defined by μ acting from $C(G)$ to the quotient space $C(G)/C_\Lambda(G)$ is weakly compact because it factors through the reflexive space $C(G)/C_{E^c}(G)$.

By a characterization of weakly compact operators from $C(G)$ ([D-J-T], Th. 15.2, p. 309), for every $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for every $f \in X$,

$$\|\overline{\mu * f}\| \le N(\varepsilon)\|f\|_1 + \varepsilon\|f\|_\infty.$$

Hence we get a sequence Δ_n in $C_\Lambda(G)$ such that $\|\mu * f_n - \Delta_n\|_\infty$ converges to 0. Indeed, the canonical injection from $C(G)$ to L^1 is 1-summing and the weakly unconditionally Cauchy sequence in $C(G)$, $\sum f_n$, is absolutely converging in $L^1(G)$. Then, for every $\varepsilon > 0$, there exists an integer m such that, for every $n \ge m$: $\|f_n\|_1 \le N(\varepsilon)^{-1}\varepsilon$ and $\|\overline{\mu * f_n}\| \le 2\varepsilon$. Hence we can choose a sequence Δ_n in $C_\Lambda(G)$ such that $\|\mu * f_n - \Delta_n\|_\infty \le 2\varepsilon$ for any $n \ge m$, which was the claim.

Hence, there exists a subsequence of $(\Delta_n)_n$ which is equivalent to a subsequence of $(\mu * f_n)$. We conclude that $C_\Lambda(G)$ contains a subspace isomorphic to c_0. This gives a contradiction. ∎

In the case of the torus, we have the following generalization of a theorem of F. Lust-Piquard.

COROLLARY 6.6. *Let $\Lambda \subset \mathbb{N}$ have the property "$c_0 \not\subset C_\Lambda(\mathbb{T})$". Then $\mathbb{Z}^- \cup \Lambda$ is a Riesz set.*

Proof. It is well known that $C_{\mathbb{N}}(\mathbb{T}) = A(\mathbb{D})$ has the property (V) of Pełczyński. On the other hand, the F. and M. Riesz theorem asserts that $\mathbb{N}$ is a Riesz set. Theorem 6.4 gives the claim. ∎

REMARK 6.7. The converse is false: it is known ([Me]) that the union of the set of squares and the negative integers is a Riesz set but $c_0 \subset C_{\{n^2\}}(\mathbb{T})$ ([L-P2]).

We deal with the question (Q2). We consider the union of sets Λ satisfying the stronger condition "$c_0 \not\subset L^\infty_\Lambda(G)$" with more general sets E than in Theorem 6.4.

DEFINITION 6.8. $E \subset \Gamma$ is *nicely placed* if the unit ball of $L^1_E(G)$ is closed for the topology of convergence in measure. $E \subset \Gamma$ is a *Shapiro set* if every subset of E is nicely placed.

We refer to [Go] for nicely placed and Shapiro sets. We recall that Shapiro sets are both Riesz and nicely placed ([Go]).

THEOREM 6.9. *Let $\Lambda \subset \Gamma$ have the property "$c_0 \not\subset L^\infty_\Lambda(G)$" and let E be a Riesz set such that $L^1(G)/L^1_E(G)$ has the property* (V*) *of Pełczyński. Then $E \cup \Lambda$ is a Riesz set.*

Proof. The idea is to dualize the proof of Theorem 6.4. Fix μ in $M_{E\cup\Lambda}(G)$.

If $f \in L^1(G)$ and $h \in L^1_{\Lambda^c}(G)$, then $\mu * (f + h) = \mu * f + \mu * h$ and $\mu * h \in L^1_E(G)$. Hence, the multiplier U defined by μ, acting from $L^1(G)/L^1_{\Lambda^c}(G)$ to $L^1(G)/L^1_E(G)$, is well defined.

The space $L^1(G)/L^1_{\Lambda^c}(G)$ does not contain any complemented subspace isomorphic to ℓ^1, because this is equivalent (see [D], p. 48) to the property "$c_0 \not\subset L^\infty_\Lambda(G)$".

By assumption, the operator U is weakly compact. Fix a function h in $L^\infty_{E^c}(G)$. The multiplier operator T defined by h acting from $L^1(G)/L^1_E(G)$ to $C(G)$ is bounded (by $\|h\|_\infty$). Denote by S the canonical surjection from $L^1(G)$ to $L^1(G)/L^1_{\Lambda^c}(G)$. The multiplier operator defined by $\mu * h$ from $L^1(G)$ to $C(G)$ is $T \circ U \circ S$, which is also weakly compact. We conclude that $\mu * h$ lies in $C(G)$, hence μ is a multiplier from $L^\infty_{E^c}(G)$ to $C(G)$.

The proof ends as in Theorem 6.4. ∎

COROLLARY 6.10. *Let $\Lambda \subset \Gamma$ be such that "$c_0 \not\subset L^\infty_\Lambda(G)$" and let E be a Riesz set which is nicely placed. Then $E \cup \Lambda$ is a Riesz set.*

Proof. As E is nicely placed, $L^1(G)/L^1_E(G)$ has the property (V*) of Pełczyński and we may use Theorem 6.9. Indeed, $L^1(G)$ and $L^1_E(G)$ are L-summands in their biduals (see [H-W-W] for precise definitions), hence $L^1(G)/L^1_E(G)$ is itself an L-summand in its bidual (Cor. 1.3, p. 160 of [H-W-W]). By a theorem of Pfitzner, every L-summand in its bidual has the property (V*) of Pełczyński (Th. 2.7, p. 173 of [H-W-W]). This proves the claim. ∎

The following result generalizes classical results of W. Rudin (for $\Lambda(1)$) and Y. Meyer [Me]:

COROLLARY 6.11. *Let $E \subset \mathbb{N}$ be either a $\Lambda(1)$ set or the set of sums of squares ($\{n^2+m^2 \mid n,m \in \mathbb{N}\}$) or the set of prime numbers. Let $\Lambda \subset \mathbb{N}$ have the property "$c_0 \not\subset L^\infty_\Lambda(\mathbb{T})$". Then $\mathbb{Z}^- \cup E \cup \Lambda$ is a Riesz set.*

Proof. By [Go], if E is either the set of sums of squares or the set of prime numbers, then $\mathbb{Z}^- \cup E$ is Riesz and nicely placed. When E is a $\Lambda(1)$ set, then $\mathbb{Z}^- \cup E$ is even Shapiro. ∎

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