A7. Let $G = (V_G, E_G)$, $H = (V_H, E_H)$ be graphs. A map $\varphi : V_G \cup E_G \to V_H \cup E_H$ is called a *(graph) homomorphism* if $\varphi(V_G) \subset V_H$, $\varphi(E_G) \subset E_H$, and $\varphi(e)$ joins $\varphi(v)$ and $\varphi(w)$ whenever e joins v and w, $e \in E_G$, $v, w \in V_G$. Write $\varphi : G \to H$.

A8. If $\varphi : G \to H$ is a graph homomorphism and $G = (V_G, E_G)$ is connected then the image graph $(\varphi(V_G), \varphi(E_G))$ is connected.

A9. Let the graph $g = (V_g, E_g)$ be a simple loop (i.e. the graph formed by taking the vertices and edges of a simple loop). Let $\varphi : g \to h$ be a graph homomorphism, with $h = (\varphi(V_g), \varphi(E_g))$. If h has an endpoint $W = \varphi(V)$, then there exist 2 edges $e_1 \neq e_2 \in E_g$ such that V is an endpoint of both e_1 and e_2, and $\varphi(e_1) = \varphi(e_2) =$ the unique edge in h with W as one endpoint.

A10. A *simple path* from a vertex V to a vertex W is a path (see A3) from V to W such that all of its edges $e_1, \ldots, e_{n-1}$ are distinct. We include the one-point path V_1 (with no edge), and any simple loop ($V_1 = V_n$). A graph G is called a *tree* if for any vertices V, W of G there is a unique simple path from V to W. We have the following results:

(1) G is a tree if and only if G is connected and has no loops.

(2) If G is a tree, r is a vertex of G and e is an edge of G, then there is a unique simple path $((V_1, \ldots, V_n), (e_1, \ldots, e_{n-1}))$ with $V_1 = r$ and $e_{n-1} = e$.

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Commutants of certain multiplication operators on Hilbert spaces of analytic functions

by

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Abstract. This paper characterizes the commutant of certain multiplication operators on Hilbert spaces of analytic functions. Let $A = M_z$ be the operator of multiplication by z on the underlying Hilbert space. We give sufficient conditions for an operator essentially commuting with A and commuting with A^n for some $n > 1$ to be the operator of multiplication by an analytic symbol. This extends a result of Shields and Wallen.

1. Introduction. Let H be a Hilbert space of complex-valued analytic functions on the open unit disc $\mathbb{D}$ such that point evaluations are bounded linear functionals on H. Then for every $w \in \mathbb{D}$ there exists a function k_w in H such that $f(w) = \langle f, k_w \rangle$ for all $f \in H$. Now if we define $K : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ by $K(z, w) = k_w(z)$, then K is a positive definite function with the reproducing property $f(w) = \langle f(\cdot), K(\cdot, w)\rangle$ for every $w \in \mathbb{D}$ and $f \in H$. The function K is called the *reproducing kernel* for H.

Recall that a function $K : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ is *positive definite* (denoted $K \gg 0$) provided

$$\sum_{j,k=1}^{n} a_j \overline{a}_k K(w_j, w_k) \geq 0$$

for any finite set of complex numbers $a_1, \ldots, a_n$ and any finite subset $w_1, \ldots, w_n$ of $\mathbb{D}$. Conversely, if $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is positive definite then

$$\Big\{ \sum_{j=1}^{n} a_j K(\cdot, w_j) : a_1, \ldots, a_n \in \mathbb{C} \text{ and } w_1, \ldots, w_n \in \mathbb{D} \Big\}$$

has dense linear span in a Hilbert space $H(K)$ of functions with

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$$\Big\|\sum_{j=1}^{n} a_j K(\cdot, w_j)\Big\|^2 = \sum_{j,k=0}^{n} a_j \overline{a}_k K(w_j, w_k)$$

and $f(w) = \langle f(\cdot), K(\cdot, w)\rangle$ for every w in $\mathbb{D}$ and f in $H(K)$. Thus evaluation at w is a bounded linear functional for each w in $\mathbb{D}$. Note also that convergence in $H(K)$ implies uniform convergence on compact subsets of $\mathbb{D}$.

Now if K is a kernel on $\mathbb{D}\times\mathbb{D}$ which is analytic in the first variable and consequently coanalytic in the second variable, then $K(z,\overline{w})$ is an analytic function on $\mathbb{D}\times\mathbb{D}$ in the two variables z and w. Hence $K(z,w)$ can be represented by the double power series $\sum_{j,k=0}^{\infty} a_{jk} z^j \overline{w}^k$. If C denotes the matrix $[a_{jk}]$ then such a K can be written more compactly in the form

$$K(z,w) = \overline{\mathbf{Z}}^* C \overline{\mathbf{W}} = \langle C\overline{\mathbf{W}}, \overline{\mathbf{Z}}\rangle_{l^2_+}$$

where $\mathbf{Z}$ denotes the column vector whose transpose is $(1, z, z^2, \ldots)$. (Here l^2_+ denotes the usual space of all square summable sequences.) It is well known that $K \gg 0$ if and only if $C > 0$. Henceforth for positive matrices C, $H(C)$ will denote the space $H(K)$ where $K = \overline{\mathbf{Z}}^* C \overline{\mathbf{W}}$. For more information about reproducing kernels the reader is referred to [2]. Some good sources on spaces of analytic functions are [3; 5; 6; 9; 11].

G. Adams, P. McGuire and V. Paulsen [1] have proved the following basic theorem in which it was shown how to produce bases for $H(C)$ via factorizations of the form $C = B^*B$.

THEOREM 1.1. *If $C = B^*B$ for some bounded operator B on l^2_+, then the operator V from $(\ker B^*)^\perp$ into $H(C)$ defined by*

$$(Vf)(z) = \langle B^* f, \overline{\mathbf{Z}}\rangle_{l^2_+}$$

is unitary.

COROLLARY 1.2. *If $C = B^*B$ and $\{f_n\}$ is an orthonormal basis for $(\ker B^*)^\perp$, then $\{\langle B^* f_n, \overline{\mathbf{Z}}\rangle_{l^2_+}\}$ is an orthonormal basis for $H(C)$.*

We can construct a basis for $H(C)$ by using the Cholesky decomposition of the nonnegative matrix C into the product U^*U, where U is upper triangular. For more details the reader is referred to [8].

Let $B(H)$ denote the algebra of all bounded operators on H. If $\mathcal{F} \subset B(H)$, then $\mathcal{F}' = \{S \in B(H) : TS = ST$ for all $T \in \mathcal{F}\}$ is the *commutant* of $\mathcal{F}$. Shields and Wallen [10] studied the commutants of the operator of multiplication by z and introduced interesting function theoretic methods. Another interesting reference is [7]. Čučković [4] studied the commutants of certain Toeplitz operators with symbol z^n on the Bergman space by decomposing the space into the direct sum of n subspaces. In this paper we study the commutants of the operator of multiplication by z^n on Hilbert spaces of analytic functions.

NOTATIONS. Throughout this paper, A is the operator M_z of multiplication by z. Let H denote a Hilbert space of analytic functions on $\mathbb{D}$ such that $1 \in H$, $zH \subset H$, point evaluations are bounded for every $w \in \mathbb{D}$ and $\dim\ker(M_z - w)^* = 1$. Let $B_0(H)$ denote the set of all compact operators on H. Also if $T \in B(H)$ then $T_n = T \oplus \ldots \oplus T$ (n times) which acts on $H \oplus \ldots \oplus H$ (n times). A complex-valued function $\varphi : \mathbb{D} \to \mathbb{C}$ is called a *multiplier* on H if $\varphi H \subset H$, that is, $\varphi f \in H$ for every f in H. It is well known ([10]) that every multiplier is bounded and analytic on $\mathbb{D}$. Given a multiplier φ let M_φ denote the operator of multiplication by φ. Because $1 \in H$ the multipliers are contained in H. If the set $\{z^n : n \ge 0\}$ is an orthogonal basis for H and φ is a multiplier, then $\varphi = \sum_{n=0}^{\infty} a_n z^n$. Because point evaluations are bounded the power series expansion of φ can be written as $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$.

2. The commutant of A^n.

First we state a lemma which will be used in the proof of the main result.

LEMMA 2.1. *Let $H = H(K)$ be a Hilbert space with reproducing kernel K. If the matrix of K is diagonal with positive entries then $\{1, z, z^2, \ldots\}$ is an orthogonal basis for H.*

Proof. Since $\{e_n\}$ is an orthonormal basis for l^2_+, where e_n has 1 in the nth coordinate and 0 elsewhere, putting $B = C^{1/2}$ in Corollary 1.2 completes the proof.

THEOREM 2.2. *Let H have a reproducing kernel of the form*

$$K(z,w) = \sum_{i=0}^{\infty} a_i (z\overline{w})^i, \quad a_i > 0,$$

or (equivalently) let the set $\{z^k : k \ge 0\}$ be an orthogonal basis for H. Assume there are constants $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ such that for all $m > 1$,

$$(1) \qquad \alpha \le a_{km}/a_k \le \beta \quad \text{for all } k = 0, 1, \ldots$$

and

$$(2) \qquad a_{km+1}/a_{km} \le \gamma, \quad k = 0, 1, \ldots$$

If $MA - AM \in B_0(H)$ and $M \in \{A^n\}'$ for some $n > 1$, then $M = M_\varphi$ for some analytic function φ.

Proof. Since $K(z,w)$ has a diagonal matrix with positive entries $\{a_n\}$, by Lemma 2.1 the set $\{z^k : k \ge 0\}$ is indeed an orthogonal basis for H. For $i = 0, 1, \ldots, n-1$ let

$$H_i = \bigvee_{k=0}^{\infty} \{z^{kn+i}\}.$$

Then $H = H_0 \oplus \ldots \oplus H_{n-1}$ and $A^n H_i \subset H_i$ for $i = 0, 1, \ldots, n-1$. So we can define $V_i : H \to H_i$ by $V_i(z^k) = z^{kn+i}$, $i = 0, 1, \ldots, n$.

To see that V_i is bounded let $f = \sum_{k=0}^{\infty} a_k z^k$ be in H. Then

$$\|f\|^2 = \sum_{k=0}^{\infty} |a_k|^2 \|z^k\|^2 < \infty.$$

Because $a_k = \|z^k\|^{-2}$ we see that (1) is equivalent to the existence of constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\|z^k\| \le \|z^{kn}\| \le C_2\|z^k\|$, $k = 0, 1, \ldots$ We therefore obtain

$$\Big\|\sum_{k=0}^{\infty} a_k z^{kn+i}\Big\|^2 = \Big\|A^i\Big(\sum_{k=0}^{\infty} a_k z^{kn}\Big)\Big\|^2 \le \|A^i\|^2 \Big(\sum_{k=0}^{\infty} |a_k|^2 \|z^{kn}\|^2\Big)$$
$$\le C_2^2 \|A^i\|^2 \sum_{k=0}^{\infty} |a_k|^2 \|z^k\|^2.$$

The above computations show that $\|V_i f\| \le C_2\|A^i\| \cdot \|f\|$ and hence V_i is bounded.

This way we get an operator $V = V_0 \oplus \ldots \oplus V_{n-1}$, which maps $H \oplus \ldots \oplus H$ into $H_0 \oplus \ldots \oplus H_{n-1}$. We also have $AH_i \subset H_{i+1}$ for $i = 0, 1, \ldots, n-2$. Consider the operator $H_i \to H_0$, $i = 0, 1, \ldots, n-1$, given by $z^{kn+i} \mapsto z^{kn}$, that is, $A^{-i}|_{H_i}$. This operator is bounded if and only if $\|z^{kn}\| \le C\|z^{kn+1}\|, k = 0, 1, \ldots$, for some constant $C > 0$. This condition is equivalent to $a_{kn+1} \le M a_{kn}, k = 0, 1, \ldots$, for some $M > 0$, which is condition (2). It is clear that the operator $V_0 : H \to H_0$ is given by $V_0 f = f \circ z^n$ and $V_i f = z^i f \circ z^n = A^i V_0 f$, $f \in H$, $i = 1, \ldots, n$. We now show that V_0 has a bounded inverse. By the same technique $V_0^{-1} : H_0 \to H$ defined by $V_0^{-1}(\sum_{k=0}^{\infty} a_k z^{kn}) = \sum_{k=0}^{\infty} a_k z^k$ or simply $V_0^{-1} f = f \circ z^{-n}$ exists and is bounded.

Assume $\varphi = (\varphi_i)_{i=0}^{n-1} \in H$ where $\varphi_i \in H_i$. Writing $\varphi(z) = \sum_{k=0}^{\infty} a_k z^k$ we have

$$\varphi_0(z) = \sum_{k=0}^{\infty} a_{kn} z^{kn},$$

$$\varphi_1(z) = \sum_{k=0}^{\infty} a_{kn+1} z^{kn+1}, \quad \ldots, \quad \varphi_{n-1}(z) = \sum_{k=0}^{\infty} a_{kn+n-1} z^{kn+n-1}.$$

Now if we consider M_φ from $H_0 \oplus \ldots \oplus H_{n-1}$ into itself, its matrix has the form

$$M_\varphi = (M_{ij})_{i,j=0}^{n-1}$$

where

$$M_{ij} = \begin{cases} M_{\varphi_{i-j}} & \text{for } i \ge j, \\ M_{\varphi_{n-j+i}} & \text{for } i < j, \end{cases}$$

and so it is constant on its diagonal. Also if an operator M on $H_0 \oplus \ldots \oplus H_{n-1}$ has this form then it is a multiplication operator.

To show what M_φ looks like first suppose that $z^{kn+i} \in H_i$ and $z^{mn+j} \in H_j$. Then since z^{kn+i} is a multiplier, $z^{kn+i} \cdot z^{mn+j}$ is equal to $z^{(k+m)n+i+j} \in H_{i+j}$ if $i + j \le n - 1$ or to $z^{(k+m+1)n+i+j-n} \in H_{i+j-n}$ if $i + j > n - 1$. So if $f_i \in H_i$ and $f_j \in H_j$, then $f_i f_j$ is in H_{i+j} for $i + j \le n - 1$ and in H_{i+j-n} for $i + j > n - 1$ provided this product makes sense. Now suppose that $f \in H$. Then

$$f = f_0 + f_1 + \ldots + f_{n-1}, \quad f_i \in H_i, \ i = 0, \ldots, n-1,$$

and also for φ a multiplier we have $\varphi = \varphi_0 + \ldots + \varphi_{n-1}$ where $\varphi_i \in H_i$ because $\varphi \in H$. So

$$\begin{aligned}\varphi f &= (\varphi_0 + \ldots + \varphi_{n-1})(f_0 + \ldots + f_{n-1}) \\ &= (\varphi_0 f_0 + \varphi_0 f_1 + \ldots + \varphi_0 f_{n-1}) + (\varphi_1 f_0 + \varphi_1 f_1 + \ldots + \varphi_1 f_{n-1}) \\ &\quad + \ldots + (\varphi_{n-1} f_0 + \ldots + \varphi_{n-1} f_{n-1}).\end{aligned}$$

After rearrangement we have

$$\begin{aligned}\varphi f = {} & (\varphi_0 f_0 + \varphi_1 f_{n-1} + \varphi_2 f_{n-2} + \ldots + \varphi_{n-1} f_1) \\ & + (\varphi_0 f_1 + \varphi_1 f_{n-2} + \ldots + \varphi_{n-1} f_2) + \ldots \\ & + (\varphi_0 f_{n-1} + \varphi_1 f_{n-2} + \ldots + \varphi_{n-1} f_0)\end{aligned}$$

where the kth parenthesis is in H_{k-1}. In fact $\varphi_i f_j \in H_{i+j}$ for $i+j \le n-1$ and $\varphi_i f_j \in H_{i+j-n}$ for $i + j > n - 1$ because $A^{kn+i} f_j = A^{kn} A^i f_j$ is in H_{i+j} for $i + j \le n - 1$ and in H_{i+j-n} for $i + j > n$. Writing $\varphi_i(z) = \sum_{k=0}^{\infty} a_{kn+i} z^{kn+i}$ completes the proof of Theorem 2.2.

In particular A can be represented as

$$A = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & z \\ z & 0 & 0 & \ldots & 0 & 0 \\ 0 & z & 0 & \ldots & 0 & 0 \\ . & . & z & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & z & 0 \end{pmatrix}.$$

Now since $M_\varphi f = \varphi f$ we can consider M_φ as a matrix operator from $H_0 \oplus H_1 \oplus \ldots \oplus H_{n-1}$ into itself given in the proof of Theorem 2.2.

We now show that if $T \in \{A^n\}'$, then $TVA_n = A^n TV$, where $A_n = A \oplus \ldots \oplus A$ acts on $H \oplus \ldots \oplus H$ and $T : H_0 \oplus \ldots \oplus H_{n-1} \to H_0 \oplus \ldots \oplus H_{n-1}$. To show this, note that

$$A_j^n V_j(z^k) = A_j^n(z^{kn+j}) = z^{(k+1)n+j} = V_j(z^{k+1}) = V_j A(z^k),$$

where for $j = 0, \ldots, n-1$, the restriction of A^n to H_j is denoted by $A_j^n = A^n|_{H_j}$. Therefore $A_j^n V_j = V_j A$ for $j = 0, 1, \ldots, n-1$ so we have $A^n V = VA_n$.

Hence if $T \in \{A^n\}'$ then $TA^n = A^nT$ and
$$TVA_n = TA^nV = A^nTV.$$

Now suppose
$$M = (M_{ij})_{i,j=0}^{n-1} \in \{A^n\}'$$
where $M_{ij} : H_j \to H_i$. Then $MVA_n = A^nMV$. But $MV = (M_{ij}V_j)_{i,j=0}^{n-1}$. Hence
$$M_{ij}V_jA = A_i^nM_{ij}V_j, \quad i,j = 0,\dots,n-1.$$
For $i = 0,1,\dots,n-1$ let $X_i : H \to H_i$ be the operator $X_i = M_{ij}V_j$. Set $X_i1 = f_i$, $f_i \in H_i$. Write
$$f_i = \sum_m a_m z^{mn+i} = A^i\Big(\sum_{m=0} a_m z^{mn}\Big) = A^iV_0\Big(\sum_{m=0}^{\infty} a_m z^m\Big).$$
Using the equality $X_iA = A_i^nX_i$ we have $X_iA(z^k) = A_i^nX_i(z^k)$. Therefore, $X_i(z^{k+1}) = z^nX_i(z^k)$. Hence
$$X_iz = z^nf_i, \quad X_iz^2 = z^{2n}f_i, \quad \dots, \quad X_iz^k = z^{kn}f_i, \quad \dots$$
Let $\varphi_{ij} \in H$ be defined by $\varphi_{ij} = \sum_m a_m z^m = V_0^{-1}A^{-i}f_i$. Then $M_{ij}V_j = V_iL_{ij}$, where L_{ij} is the operator of multiplication by φ_{ij}. We show that $X_i = V_iL_{ij}$. In fact, $X_i(z^k) = z^{kn}f_i$ and
$$V_iL_{ij}(z^k) = V_i(\varphi_{ij}z^k) = V_i\Big(\sum_m a_m z^{k+m}\Big) = \sum_m a_m z^{(k+m)n+i}$$
$$= z^{kn}V_i(\varphi_{ij}) = \frac{V_i(\varphi_{ij})}{z^j}(z^{kn+j}).$$
Therefore,
$$V_iL_{ij}(z^k) = \frac{V_i(\varphi_{ij})}{z^j}(z^{kn+j}) = \frac{\psi_{ij}}{z^j}(z^{kn+j}) = N_{ij}V_j(z^k),$$
where $\psi_{ij} = V_i(\varphi_{ij})$ and N_{ij} is the operator of multiplication by ψ_{ij}/z^j. Hence $MV = (M_{ij}V_j)_{i,j=0}^{n-1} = (V_iL_{ij})_{i,j=0}^{n-1} = (N_{ij}V_j)_{i,j=0}^{n-1}$. In what follows for a set of vectors $h_1,\dots,h_n$ the matrix transpose of the vector $[h_1\dots h_n]$ is denoted by $[h_1\dots h_n]'$.

We now have
$$MV[z^{k_0},z^{k_1},\dots,z^{k_{n-1}}]' = (V_iL_{ij})_{i,j=0}^{n-1}[z^{k_0},z^{k_1},\dots,z^{k_{n-1}}]'$$
$$= \Big(\sum_{j=0}^{n-1} z^{k_jn+j}\frac{\psi_{ij}}{z^j}\Big)_{i=0}^{n-1}.$$

We compute $MA - AM$ by writing
$$MA - AM = (N_{ij}A - AN_{i-1,j-1})_{i,j=0}^{n-1}$$

where $N_{-1,j} = N_{n-1,j}$ and $N_{i,-1} = N_{i,n-1}$. The restriction of $MA - AM$ to H_0 is
$$(MA - AM)|_{H_0}(h_0)$$
$$= [(N_{01}A - AN_{n-1,0})h_0, (N_{11}A - AN_{00})h_0, \dots, (N_{n-1,1}A - AN_{n-2,0})h_0]'$$
$$= [(\psi_{01} - z\psi_{n-1,0})h_0, (\psi_{11} - z\psi_{00})h_0, \dots, (\psi_{n-1,1} - z\psi_{n-2,0})h_0]'.$$
Therefore $MA - AM|_{H_0} = M_f|_{H_0}$ where
$$f = [\psi_{01} - z\psi_{n-1,0}, \psi_{11} - z\psi_{00}, \dots, \psi_{n-1,1} - z\psi_{n-2,0}]'.$$
Since $MA - AM$ is compact it follows that $MA - AM|_{H_0} = M_f|_{H_0}$ is a compact operator.

Now we show that $M_f : H_j \to H_j$ is also compact. This is clear because $M_f|_{H_j} = M_{z^j}(M_f|_{H_0})M_{z^{-j}}|_{H_j}$. Therefore
$$M_f : H_0 \oplus \dots \oplus H_{n-1} \to H_0 \oplus \dots \oplus H_{n-1}$$
is compact and so $f = 0$. This implies that $\varphi_{i0} = \varphi_{i+1,0}/z$, $i = 0,1,\dots,n-1$. Similarly, $MA - AM|_{H_1} = M_g|_{H_1}$ and $g = (\psi_{i,2}/z - \psi_{i-1,1})_{i=0}^{n-1}$ where $\psi_{-1,1} = \psi_{n-1,1}$. But $M_g|_{H_1}$ is a compact operator on H_1 and so $M_gA : H_0 \to H_1$ is compact. On the other hand $M_gA|_{H_0} = M_h$ where $h = (\psi_{i,2} - z\psi_{i-1,1})_{i=0}^{n-1}$ and $\psi_{-1,1} = \psi_{n-1,1}$. Now since $M_h : H_0 \to H$ is compact as before, this means that $h = 0$ or equivalently $\psi_{i,2} = z\psi_{i-1,1}$, $i = 0,1,\dots, n-1$, and $\psi_{-1,1} = \psi_{n-1,1}$. If we continue this way we conclude that $M = M_\varphi$ where $\varphi = (\varphi_{i,0})_{i=0}^{n-1}$.

REMARK. Condition (1) states that what corresponds to the boundedness of the composition operator V_0 is the existence of a uniform (with respect to k) bound on the ratios a_{km}/a_k. Similarly, their lower bound being > 0 should suffice for bounded invertibility of V_0. With condition (2) the boundedness of $A^{-i} : H_i \to H$ is guaranteed. Fortunately, this kind of assumptions holds true in most interesting spaces (Hardy, Bergman, Dirichlet etc.). What one must do is to exclude "exotic" weights such as $a_k = k!$. Note that the composition operator then maps some element $f \in H$ outside the space: it may happen that $V_0f \notin H$.

EXAMPLE 2.3. Assume B is the diagonal matrix with positive entries $\{a_0,a_1,\dots\}$, $(\limsup a_n^{1/n})^{-1} < \infty$, $K(z,w) = \sum_{i=0}^{\infty} a_i(z\overline{w})^i$ and
$$H = \Big\{\sum_{n=0}^{\infty} b_nz^n : \sum_{n=0}^{\infty} |b_n|^2/a_n < \infty\Big\}.$$
In this case $\{1,z,z^2,\dots\}$ is an orthogonal basis for H, and H satisfies the condition of Theorem 2.2 provided (1) and (2) hold. If $a_i = 1$ then H is the Hardy space H^2 of the unit disc, if $a_i = i+1$ then H can be identified with the Bergman space of analytic functions on $\mathbb{D}$, and if $a_i = (i+1)^{-1}$ then H

can be identified with the Dirichlet space of analytic functions on $\mathbb{D}$ whose derivatives are in the Bergman space.

COROLLARY 2.4. *Let H be as in Theorem 2.2 and u be a one-to-one analytic map of $\mathbb{D}$ onto $\mathbb{D}$ such that $uf, f\circ u \in H$ for every $f\in H$. Let $S\in\{M_{u^n}\}'$ and $SM_u - M_uS \in B_0(H)$. Then $S = M_\varphi$ for some analytic function φ.*

Proof. We define $V : H\to H$ by $Vf = f\circ u^{-1}$. Clearly, V is a bounded linear operator with inverse $V^{-1}f = f\circ u$. Then $V^{-1}AV = M_u$. Now since $S\in\{M_{u^n}\}'$, $S\in\{V^{-1}A^nV\}'$ and so $VSV^{-1}\in\{A^n\}'$. But

$$VSV^{-1}A - AVSV^{-1} = V(SM_u - M_uS)V^{-1}$$

is compact, so by Theorem 2.2, $VSV^{-1} = M_\psi$ for some analytic function ψ. Therefore, $S = V^{-1}M_\psi V$, and since $V^{-1}M_\psi V = M_{\psi\circ u}$, we have $S = M_{\psi\circ u}$.

THEOREM 2.5. *Let H have the reproducing kernel*

$$K(z,w) = \frac{1-z\overline{w}}{(1-z)(1-\overline{w})}\sum_{i=0}^{\infty} a_i(z\overline{w})^i,$$

where $\{a_i\}$ is a nondecreasing sequence of positive numbers. Suppose there are constants $\gamma_1>0$, $\gamma_2>0$ and $\gamma_3>0$ such that for all $m>1$,

$$\gamma_1 \le b_{km}/b_k \le \gamma_2, \quad k=0,1,\ldots, \tag{3}$$

where $b_0^2 = a_0$, $b_k^2 = a_k - a_{k-1}$, $k>0$, and

$$b_{km+1}/b_{km} \le \gamma_3, \quad k = 0,1,\ldots \tag{4}$$

If $M\in\{A^n\}'$ for some $n>1$ and $MA - AM\in B_0(H)$, then M is a multiplication operator with an analytic symbol.

Proof. Note that

$$\begin{aligned}
K(z,w) &= \frac{1-z\overline{w}}{(1-z)(1-\overline{w})}\sum_{i=0}^{\infty} a_i(z\overline{w})^i = \frac{1-z+z-z\overline{w}}{(1-z)(1-\overline{w})}\sum_{i=0}^{\infty} a_i(z\overline{w})^i\\
&= \Big(\frac{z}{1-z}+\frac{1}{1-\overline{w}}\Big)\sum_{i=0}^{\infty} a_i(z\overline{w})^i = \Big(\sum_{m=1}^{\infty}\overline{w}^m + \sum_{n=0}^{\infty} z^n\Big)\sum_{i=0}^{\infty} a_i(z\overline{w})^i\\
&= \sum_{i=0}^{\infty}\sum_{m=1}^{\infty} a_i z^i\overline{w}^{i+m} + \sum_{i=0}^{\infty}\sum_{n=0}^{\infty} a_i z^{i+n}\overline{w}^i.
\end{aligned}$$

If we denote the matrix of $K(z,w)$ by $C = (a_{ij})_{i,j=0}^{\infty}$, then

$$\begin{aligned}
a_{i,i+m} &= a_i, \quad i=0,1,2,\ldots, \quad m=1,2,\ldots,\\
a_{i+n,i} &= a_i, \quad i=0,1,2,\ldots, \quad n=0,1,2,\ldots
\end{aligned}$$

Hence $a_{ij} = a_i$ for $j\ge i$ and C is symmetric. The matrix C has $\{a_0, a_1, a_2,\ldots\}$ on its main diagonal and the same thing as its subdiagonals and

superdiagonals. Now if we write $C = U^*U$ with upper triangular matrix U, then the nonzero entries in each row of U are equal. Suppose the entries in the ith row of U are b_i. In fact

$$b_0^2 = a_0, \quad b_1^2 = a_1 - a_0, \quad b_2^2 = a_2 - a_1, \quad \ldots$$

Now by Corollary 1.2 and considering the orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for l_+^2 the set

$$\Big\{\sum_{i=0}^{\infty} b_0 z^i, \ldots, \sum_{i=0}^{\infty} b_k z^{i+k}, \ldots\Big\}$$

is an orthonormal basis for H. For simplicity let $f_0 = \sum_{i=0}^{\infty} z^i$. Then the set $\{f_0, zf_0, z^2f_0,\ldots\}$ is an orthogonal basis for H. It is clear that $\|z^kf_0\|^{-1} = b_k$. For $i = 0,1,\ldots,n-1$ let

$$H_i = \bigvee_{k=0}^{\infty}\{z^{kn+i}f_0\}.$$

Then $A^nH_i\subset H_i$ for $i=0,1,\ldots,n-1$ and $H = H_0\oplus\ldots\oplus H_{n-1}$. We denote by $A_j^n = A^n|_{H_j}$ the restriction of A^n to H_j, $j=0,1,\ldots,n-1$. Now similar to the notations in Theorem 2.2 for $i=0,1,\ldots,n-1$ we define

$$V_i : H\to H_i \quad\text{by}\quad V_i(z^kf_0) = z^{kn+i}f_0.$$

The boundedness of V_i follows from (3). We now have

$$A_j^nV_j(z^kf_0) = A_j^n(z^{kn+j}f_0) = z^{(k+1)n+j}f_0 = V_j(z^{k+1}f_0) = V_jA(z^kf_0).$$

Therefore $A_j^nV_j = V_jA$ for $j=0,\ldots,n-1$. Let $V = V_0\oplus V_1\oplus\ldots V_{n-1}$. Then V maps $H\oplus\ldots\oplus H$ into $H_0\oplus\ldots\oplus H_{n-1}$ and we have $A^nV = VA_n$. Hence $T\in\{A^n\}'$ if and only if $TVA_n = A^nTV$. Now if $M = (M_{ij})_{i,j=0}^{n-1}\in\{A^n\}'$, where $M_{ij} : H_j\to H_i$, then after some calculation we have $M_{ij}V_j = V_iL_{ij}$, where L_{ij} is the operator of multiplication by φ_{ij} for some analytic function $\varphi_{ij} = \sum_m a_mz^m$. We now have

$$\begin{aligned}
V_iL_{ij}(z^kf_0) &= V_i(\varphi_{ij}z^kf_0) = V_i\Big(\sum_{m=0}^{\infty} a_mz^m\cdot\sum_{m=0}^{\infty} z^{k+m}\Big)\\
&= V_i\Big(\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_l z^{k+l+m}\Big) = \sum_{l=0}^{\infty} a_l\sum_{m=0}^{\infty} z^{(k+l)n+i+m}\\
&= \Big(\sum_{m=0}^{\infty} a_m z^{mn+i}\Big)\Big(\sum_{m=0}^{\infty} z^{kn+m}\Big)\\
&= \Big(z^{-j}\sum_{m=0}^{\infty} a_m z^{mn+i}\Big)\Big(\sum_{m=0}^{\infty} z^{kn+j+m}\Big)\\
&= \Big(z^{-j}\sum_{m=0}^{\infty} a_m z^{mn+i}\Big)(z^{kn+j}f_0) = \frac{\psi_{ij}}{z^j}(z^{kn+j}f_0),
\end{aligned}$$

where $\psi_{ij} = \sum_{m=0}^{\infty} a_m z^{mn+i}$. Therefore,

$$(V_i L_{ij})(f) = \frac{\psi_{ij}}{z^j}(V_j f) \quad \text{for } f \in H.$$

Now using the same idea as in Theorem 2.2 we can complete the proof.

EXAMPLE 2.6. If $a_i = i+1$ in Theorem 2.5 then

$$K(z,w) = \frac{1}{(1-z)(1-\overline{w})(1-\overline{w}z)}$$

and $U = (U_{ij})_{i,j=0}^{\infty}$, where $U_{ij} = 0$ for $i > j$ and $U_{ij} = 1$ for $i \le j$. So $\{z^n/(1-z) : n \ge 0\}$ is an orthogonal basis for H.

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A class of l_1-preduals which are isomorphic to quotients of $C(\omega^\omega)$

by

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Abstract. For every countable ordinal α, we construct an l_1-predual X_α which is isometric to a subspace of $C(\omega^{\omega^{\omega^\alpha+2}})$ and isomorphic to a quotient of $C(\omega^\omega)$. However, X_α is not isomorphic to a subspace of $C(\omega^{\omega^\alpha})$.

1. Introduction. The study of quotients of $C(\alpha)$, for α a countable ordinal, is closely related to the problem of the isomorphic classification of the complemented subspaces of $C[0,1]$. Indeed, every complemented subspace of $C[0,1]$ is either isomorphic to a quotient of $C(\alpha)$ for some $\alpha < \omega_1$ (see [4]), or isomorphic to $C[0,1]$ (see [11]).

According to a result of Johnson and Zippin [8], every quotient of $C(\omega)$ is isomorphic to a subspace of $C(\omega)$. A natural question which arises then is if such a phenomenon occurs in $C(\alpha)$ for every $\alpha < \omega_1$. Alspach [1] gave a negative answer to this question by exhibiting a quotient of $C(\omega^\omega)$ which is not isomorphic to a subspace of $C(\alpha)$ for any $\alpha < \omega_1$.

Alspach's example left open the following question: Suppose X is isomorphic to a quotient of $C(\omega^\omega)$ and that there exists $\alpha < \omega_1$ with X isomorphic to a subspace of $C(\alpha)$. Is X isomorphic to a subspace of $C(\omega^\omega)$?

In this article, we answer this question in the negative by proving the following:

THEOREM 1.1. *For every countable ordinal α, there exists an l_1-predual space X_α with the following properties*:

1. *X_α is isomorphic to a quotient of $C(\omega^\omega)$.*

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