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Examples of $\Lambda(4)$ sets E and a graph structure in $E \times E$

by

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Abstract. We construct examples of $\Lambda(4)$ sets $E \subset \mathbb{Z}$. The construction uses certain families of thin intervals $\{I_k\} = \mathcal{L} \approx E$. The $\Lambda(4)$ property for E is obtained from the stronger result that $\|f\|_4 \le c\|(\sum |f_{I_k}|^2)^{1/2}\|_4$ where $\widehat{f}$ is supported on $(\bigcup I_k) \cap \mathbb{Z}$ and f_{I_k} is defined by $\widehat{f}_{I_k} = \chi_{I_k}\widehat{f}$. The proof of the latter involves a graph structure defined in terms of $\mathcal{L} \times \mathcal{L}$ (which is essentially $E \times E$).

0. Introduction. A set of integers E is called a $\Lambda(4)$ *set* if there exists a constant $c > 0$ such that for any trigonometric polynomial f on the circle with $\widehat{f}$ supported on E we have

$$\|f\|_4 \le c\|f\|_2. \tag{0.1}$$

In this paper we construct examples of such sets E for which the proof of inequality (0.1) seems to require a new combinatorial method and therefore may be of interest. This topic and the examples are motivated by previous work of K. E. Hare and the author on lacunary intervals in [H1] and [H2]. The main new feature here is the following. First let us recall a well-known sufficient condition for a set E to be $\Lambda(4)$, which was given by W. Rudin in [R]:

(0.2) *If there is a constant $M \in \mathbb{N}$ such that every nonzero $n \in \mathbb{Z}$ has at most M representations as $n = n_1 - n_2$, where $(n_1, n_2) \in E \times E$, then E is a $\Lambda(4)$ set.*

It is a fact that any examples of $\Lambda(4)$ sets E which can be recovered from the results of [H1] or [H2] are essentially of type (0.2). This is because (0.2) was simply the backbone for that work, which, as we should point out, is concerned with the more difficult problem of Littlewood–Paley inequalities and arbitrarily supported $\widehat{f}$, instead of merely the basic $\Lambda(p)$ property.

On the other hand, the present $\Lambda(4)$ examples E will be such that condition (0.2) does not always hold. Moreover, the version of (0.2) us-

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ing $n = n_1 + n_2$ also does not hold for our E, and there does not appear to be any easy way to decompose E via finite unions, or via other methods such as the Marcinkiewicz–Zygmund Theorem, so as to reduce the problem to (0.2). Therefore, the examples may possibly provide a backbone for new Littlewood–Paley inequalities in the same sense as (0.2) was a backbone for the new Littlewood–Paley inequalities of [H1] and [H2]. In any case, the proof of (0.1) for our examples already demonstrates a combinatorial method more general than (0.2), at the basic level of the $\Lambda(4)$ problem.

Let us not dismiss (0.2) though. In fact, we do conjecture that our present examples E (to be constructed in Section 1 below) can be decomposed as a finite (uniformly bounded to be precise; we actually work with a family of examples E of arbitrary but finite cardinality) union of sets, each satisfying (0.2) (with a uniform constant M; again for the same reason). Nevertheless, we shall prove (0.1) for our E's without this decomposition, by using a different combinatorial method. To give at least some hint of this different method here, we could roughly say that instead of trying to decompose E, it is sufficient to decompose $E \times E$ in a certain sense.

At this point we should recall a simple remark about finite $\Lambda(4)$ sets in comparison with infinite ones. Any finite set $E \subset \mathbb{Z}$ is of course $\Lambda(4)$. But a "nontrivial" finite $\Lambda(4)$ set E is one such that the cardinality of E is large while the constant c in (0.1) is fixed. For example, our constant will turn out to be $c = 4^{1/4}$ for all of our E's, which shall however have arbitrarily large finite cardinalities. It is well known that given any infinite family of finite E's for which a fixed constant c holds in (0.1), there exists one big $\Lambda(4)$ set F which contains all of the E's as translates. To prove this, one can use standard Littlewood–Paley theory to paste together translates of E's in lacunary intervals. We shall not perform the formal step of translating and pasting together our E's.

In the next section we define the new examples of $\Lambda(4)$ sets E. We first construct some tree-like families of dyadic intervals on the real line $\mathbb{R}$, because these actually play the important parts in the proof of (0.1). The only reason why we do this on $\mathbb{R}$ is for convenience—at any time we can exhibit examples of sets E in $\mathbb{Z}$ by choosing integer elements from various real intervals, if they contain any integers. (If not, then one can either magnify the whole picture until the intervals on a desired scale do contain integers, or one can simply remember to begin the construction process with a sufficiently large interval.)

1. Definition of the examples. Consider any sequence of integers n_j, $j = 0, 1, \ldots$, satisfying $n_{j+1} \geq n_j + 3$. (We are usually interested in the case where n_0 is large and *negative*). For each j, let $l_j = 2^{-n_j}$ and let $\mathcal{M}^j$ be the family of intervals in $\mathbb{R}$ given by

$$\mathcal{M}^j = \{[0, l_j] + 4kl_j : k \in \mathbb{Z}\},$$

where $[a, b] + t$, $t \in \mathbb{R}$ denotes the translated interval $[a + t, b + t]$. (Note that if n_0 is large negative then l_0 is a large positive integer power of 2, and $l_{j+1} \leq l_j/8$. Also, if we fix j then the intervals in $\mathcal{M}^j$ are not merely disjoint—they are in fact separated by gaps of 3 times their length l_j. Not surprisingly, our choice of the specific base 2 and the specific gap length 3 times l_j is only one of many other choices which would serve our purposes equally well.) Let us call the sequence $\{\mathcal{M}^j\}_{j=0}^{\infty}$ the *frame* determined by the sequence $\{n_j\}_{j=0}^{\infty}$.

Next, given a frame $\{\mathcal{M}^j\}$, define a *forest* in $\{\mathcal{M}^j\}$ to be a sequence $\{\mathcal{L}^j\}_{j=0}^{\infty}$ where each $\mathcal{L}^j$ is a family of real intervals and the properties 1.1 to 1.4 below are satisfied:

1.1. $\mathcal{L}^j \subset \mathcal{M}^j$ for all $j \geq 0$.

1.2. If $I \in \mathcal{L}^{j+1}$ then $I \subset J$ for some $J \in \mathcal{L}^j$.

1.3. For all $j \geq 0$ and for all $I \in \mathcal{L}^j$, I satisfies exactly one of the following two alternatives, 1.3.1 or 1.3.2:

1.3.1. (Alternative 1) There is exactly 1 element $J \in \mathcal{L}^{j+1}$ such that $J \subset I$. We say I is of *Type* 1.

1.3.2. (Alternative 2) There are exactly 2 elements $J, K \in \mathcal{L}^{j+1}$ ($J \neq K$) such that $J \subset I$ and $K \subset I$. We say I is of *Type* 2.

1.4. (The Sparseness Property) For each $j \geq 0$, the number of Type 2 intervals $I \in \mathcal{L}^j$ is at most 2.

For each $j \geq 0$, we call $\mathcal{L}^j$ the *level* j of the forest $\{\mathcal{L}^j\}$. An interval $I \in \mathcal{L}^j$ will be called an interval I *on level* j. Furthermore, various objects which we shall construct later, using only intervals on a fixed level j, will be called objects *on level* j.

Notice that the definition of a frame $\{\mathcal{M}^j\}$ guarantees that each $I \in \mathcal{M}^j$ contains at least 2 elements of $\mathcal{M}^{j+1}$. It follows that one can construct many kinds of forests $\{\mathcal{L}^j\}$ inductively. We can take any nonempty subset of $\mathcal{M}^0$ and call it $\mathcal{L}^0$. Proceeding inductively, given $\mathcal{L}^j$, construct $\mathcal{L}^{j+1}$ by considering each $I \in \mathcal{L}^j$. We can decide arbitrarily which I will be of Type 1 or of Type 2, except that by 1.4 we may only designate at most 2 of the I to be of Type 2; the others will all have to be of Type 1. Then, if I was to be of Type 1, choose any one of the available intervals $J \in \mathcal{M}^{j+1}$ such that $J \subset I$. If I was to be of Type 2, choose any two such intervals $J, K \in \mathcal{M}^{j+1} (J \neq K)$ such that $J, K \subset I$. Put all of these chosen intervals J and K into $\mathcal{L}^{j+1}$, thus completing one induction step. Notice also that the choice of the sequence $\{n_j\}_{j=0}^{\infty}$ (which determined the frame $\{\mathcal{M}^j\}$) need not have been specified at the start; it can be chosen inductively, during the above induction steps. In fact, at the start of the above induction step we can assume that n_j already exists, and we can choose n_{j+1} to be any integer satisfying $n_{j+1} \geq n_j + 3$, as we like.

To define our examples E, let $\{\mathcal{L}^j\}$ be any forest such that $\mathcal{L}^0$ consists of exactly one interval I_0, so that $\mathcal{L}^0 = \{I_0\}$. We may think of such a forest $\{\mathcal{L}^j\}$ as being a "forest consisting of exactly one tree". Then fix any index $j \geq 0$ and let E be any set such that

$$E \subset \mathbb{Z} \cap \Big(\bigcup_{I \in \mathcal{L}^j} I \Big),$$

and such that $E \cap I$ has at most one element for each $I \in \mathcal{L}^j$. In other words, we choose one integer (if any) from each interval I of some fixed $\mathcal{L}^j$ and call the resulting set E.

THEOREM A. *There exists a constant $c > 0$ such that for any of the examples E just defined we have* (0.1) *with this constant c.*

Theorem A is an immediate consequence of the following result, Theorem 6.2, which we reproduce here from Section 6 of this paper. The notation is fully explained in Section 6.

6.2. THEOREM. *Let $\{\mathcal{L}^i\}_{i=0}^{\infty}$ be a forest such that $\mathcal{L}^0$ consists of exactly one interval I_0, i.e. $\mathcal{L}^0 = \{I_0\}$. Fix $j \geq 0$ and suppose that the trigonometric polynomial f satisfies*

$$\operatorname{Spec}(f) \subset \bigcup_{I \in \mathcal{L}^j} I, \quad i.e. \quad f = \sum_{I \in \mathcal{L}^j} f_I.$$

Then

$$\int_{\mathbb{T}} |f|^4 \leq 4 \int_{\mathbb{T}} \Big(\sum_{I \in \mathcal{L}^j} |f_I|^2 \Big)^2,$$

where the integrals are with respect to normalized Lebesgue measure on $\mathbb{T}$.

To conclude this section we remark that in [H2], a very similar inductive process was used, but with one crucial limitation. This was that at most one Type 2 interval I was permitted in the Sparseness Property 1.4. Here we permit two Type 2 intervals. This is crucial because having only one Type 2 interval actually implies the difference condition (0.2) for the set E, whereas permitting two Type 2 intervals allows examples of sets E which fail to satisfy the difference condition (0.2) (i.e. with a uniform constant M for all such E). We leave these assertions as an exercise for the interested reader—they can be seen geometrically by drawing a few examples of one-tree forests, beginning with one (big) interval. In particular, to see how (0.2) can fail, the main point is that if $n = n_1 - n_2$ (where n_1 and n_2 are, say, the left endpoints of some two distinct intervals I and J at some level $\mathcal{L}^k$ of the inductive construction process), then we can create an additional pair (m_1, m_2) such that $n = m_1 - m_2$ on the next level $\mathcal{L}^{k+1}$ by using 2 copies (translates) of the same Type 2 subdivision, one inside a neighbourhood of n_1 (namely inside I) and the other inside a neighbourhood of n_2 (namely inside J). It follows that we can create as many such pairs as we wish, all having the same difference n. Moreover, we can do this for any number of n's that we like. Thus (0.2) fails in a very strong sense. We can also do a similar construction regarding the equation $n = n_1 + n_2$, with some other values of the variables, to obtain many pairs in E which have the same sum.

2. The definition of some structures in $\mathbb{R} \times \mathbb{R}$. Our one and only aim now is to prove Theorem 6.2. Why then have we bothered to define general forests instead of just forests with exactly one interval $I_0 \in \mathcal{L}^0$? The answer is convenience and uniformity of notation: If $\{\mathcal{L}^j\}_{j=0}^{\infty}$ is a forest in the frame $\{\mathcal{M}^j\}_{j=0}^{\infty}$, then $\{\mathcal{L}^{k+j}\}_{j=0}^{\infty}$ is a forest in the frame $\{\mathcal{M}^{k+j}\}_{j=0}^{\infty}$ for any $k \geq 0$. That is, it is easy to cut the forest at some level k and still use the same notation for what remains beyond level k. In fact, our main technical Lemma 6.1 will concern an arbitrary level $k \geq 0$, an arbitrary pair of disjoint intervals $A, B \in \mathcal{L}^k$, and some intervals inside A and B. Thus only levels k and beyond k are relevant there. Therefore it suffices to prove any such lemma in the special case $k = 0$. (It is the same as re-labelling indexes so that they start at 0 instead of at the given k). From this point on we are preparing for the proof of Lemma 6.1 in the special case $k = 0$.

2.1. Assume $\mathcal{L}^0$ has at least two elements, A, B, $A \neq B$ (and thus $A \cap B = \emptyset$) and fix such A and B, from this point until Lemma 6.1. Define

$$\mathcal{L}_A^j = \{I \in \mathcal{L}^j : I \subset A\}, \quad \mathcal{L}_B^j = \{I \in \mathcal{L}^j : I \subset B\}.$$

A Cartesian product b of the form

$$b = I \times J, \quad I \in \mathcal{L}_A^j, \ J \in \mathcal{L}_B^j,$$

will be called a *block* on level j. Further, $\mathcal{L}_A^j \otimes \mathcal{L}_B^j$ will denote the set of all blocks b on level j.

2.2. Let $I_1, I_2 \in \mathcal{L}_A^j$ and $J_1, J_2 \in \mathcal{L}_B^j$ and consider the set of blocks, s, given by

$$s = \{I_1 \times J_1, I_1 \times J_2, I_2 \times J_1, I_2 \times J_2\}.$$

We call s a *square* on level j if one of the following two conditions holds:

2.2.1. $I_1 = I_2$ and $J_1 = J_2$. (In this case s reduces to the one-block set $s = \{I_1 \times J_1\}$.)

2.2.2. There exists a real number $t \neq 0$ such that $I_1 + t = I_2$ and $J_1 + t = J_2$. (In this case s consists of 4 congruent, disjoint blocks whose centres form an ordinary square in $\mathbb{R}^2$.)

Also, if I_1, I_2, J_1, J_2 satisfy 2.2.1 or 2.2.2, we write $s = s(I_1, I_2, J_1, J_2)$. We denote by S^j the set of all squares s on level j.

2.3. Consider a block $b = I \times J$ on some level j. We define sets $\beta(b)$ and $\gamma(b)$ by

$$\beta(b) = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x - y \in I - J\},$$
$$\gamma(b) = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x + y \in I + J\},$$

where $I - J = \{r - s : r \in I,\ s \in J\}$ and $I + J = \{r + s : r \in I,\ s \in J\}$, and we call $\beta(b)$ the $\oplus$ *band generated by* b and $\gamma(b)$ the $\ominus$ *band generated by* b. (The choice of signs is due to the fact that $\beta(b)$ consists of the lines of slope $+1$ passing through b, and $\gamma(b)$ consists of the lines of slope -1 through b.) Also, let β^j (resp. γ^j) denote the set of all $\oplus$ bands (resp. $\ominus$ bands) generated by all blocks b on level j:

$$\beta^j = \{\beta(b) : b = I \times J,\ I \in \mathcal{L}_A^j,\ J \in \mathcal{L}_B^j\},$$
$$\gamma^j = \{\gamma(b) : b = I \times J,\ I \in \mathcal{L}_A^j,\ J \in \mathcal{L}_B^j\}.$$

2.4. Most of the time we want to consider only the blocks contained in a given band, rather than the band itself. So if $b \in \mathcal{L}_A^j \otimes \mathcal{L}_B^j$, we define

$$\lambda(b) = \{c \in \mathcal{L}_A^j \otimes \mathcal{L}_B^j : \beta(c) = \beta(b)\},$$
$$\mu(b) = \{c \in \mathcal{L}_A^j \otimes \mathcal{L}_B^j : \gamma(c) = \gamma(b)\},$$

and we call $\lambda(b)$ the $\oplus$ *line generated by* b, and $\mu(b)$ the $\ominus$ *line generated by* b. Let λ^j (resp. μ^j) denote the set of all $\lambda(b)$ (resp. $\mu(b)$) for $b \in \mathcal{L}_A^j \otimes \mathcal{L}_B^j$.

2.5. Our definition of $\mathcal{M}^i$ guarantees that $\oplus$ bands on the same level are either equal or disjoint, as are $\oplus$ lines, $\ominus$ bands, and $\ominus$ lines. This is shown by the following.

2.6. PROPOSITION (Disjointness of parallel bands)

2.6.1. *Let $I_1, I_2, J_1, J_2 \in \mathcal{M}^j$. Then there exist unique $a, b \in \mathbb{R}$ such that $I_1 = J_1 + a$ and $I_2 = J_2 + b$. We have: $(I_1 - J_1) \cap (I_2 - J_2) \neq \emptyset$ iff $a = b$ iff $I_1 - J_1 = I_2 - J_2$ iff $\beta(I_1 \times J_1) \cap \beta(I_2 \times J_2) \neq \emptyset$.*

2.6.2. *For any $\oplus$ bands $\beta_1, \beta_2 \in \beta^j$ we have either $\beta_1 = \beta_2$ or $\beta_1 \cap \beta_2 = \emptyset$. For any $\oplus$ lines $\lambda_1, \lambda_2 \in \lambda^j$ we have $\lambda_1 = \lambda_2$ or $\beta(\lambda_1) \cap \beta(\lambda_2) = \emptyset$, where $\beta(\lambda_i)$ denotes the (unique) $\oplus$ band containing λ_i on level j.*

2.6.3. *Analogous results hold for $\ominus$ bands and $\ominus$ lines.*

Proof. 1. Looking at the definition of $\mathcal{M}^j$, we see that $a = 4ml_j$ and $b = 4nl_j$ for unique $m, n \in \mathbb{Z}$. Thus.

$$I_1 - J_1 = [-l_j, l_j] + a = [-l_j, l_j] + 4ml_j,$$
$$I_2 - J_2 = [-l_j, l_j] + b = [-l_j, l_j] + 4nl_j.$$

If $a \neq b$ then $|a - b| \geq 4l_j$ so clearly $(I_1 - J_1) \cap (I_2 - J_2) = \emptyset$. If $(I_1 - J_1) \cap (I_2 - J_2) = \emptyset$ then $I_1 - J_1 \neq I_2 - J_2$. If $I_1 - J_1 \neq I_2 - J_2$ then $a \neq b$. The definition of $\beta(\ )$ immediately gives $(I_1 - J_1) \cap (I_2 - J_2) \neq \emptyset$ iff $\beta(I_1 \times J_1) \cap \beta(I_2 \times J_2) \neq \emptyset$. This proves 2.6.1.

2. Let $I_1, I_2 \in \mathcal{L}_A^j$ and $J_1, J_2 \in \mathcal{L}_B^j$. Consider $\beta_1 := \beta(I_1 \times J_1)$ and $\beta_2 := \beta(I_2 \times J_2)$. If $\beta_1 \cap \beta_2 \neq \emptyset$ then $(I_1 - J_1) \cap (I_2 - J_2) \neq \emptyset$, by definition of $\beta(\ )$. Thus $I_1 - J_1 = I_2 - J_2$ by 2.6.1, and so $\beta_1 = \beta_2$ by definition of $\beta(\ )$. Next, let $\lambda_i = \lambda(I_i \times J_i)$, $i = 1, 2$. Clearly, the $\oplus$ band $\beta(I_i \times J_i)$ contains λ_i, and it is unique by the above. Thus $\beta(\lambda_i)$ is a well-defined notation. Now suppose $\beta(\lambda_1) \cap \beta(\lambda_2) \neq \emptyset$. Then $\beta(\lambda_1) = \beta(\lambda_2)$ by the above, and thus $\beta(I_1 \times J_1) = \beta(I_2 \times J_2)$, implying $\lambda(I_1 \times J_1) = \lambda(I_2 \times J_2)$ by definition of $\lambda(\ )$. Thus $\lambda_1 = \lambda_2$. This proves 2.6.2.

3. For the analogous statements with γ replacing β and μ replacing λ, we first consider the analogue of 2.6.1: Define $p, q \in \mathbb{R}$ by $I_1 = -J_1 + p$ and $I_2 = -J_2 + q$. We claim that

$$(I_1 + J_1) \cap (I_2 + J_2) \neq \emptyset \quad \text{iff} \quad p = q \quad \text{iff} \quad I_1 + J_1 = I_2 + J_2.$$

We leave the rest of the proof as an exercise.

2.7. Let s be a square on level j, and let it be written the same way as in 2.2, i.e.

$$s = s(I_1, I_2, J_1, J_2) = \{I_1 \times J_1, I_1 \times J_2, I_2 \times J_1, I_2 \times J_2\}$$

where $I_2 = I_1 + t$ and $J_2 = J_1 + t$, $t \in \mathbb{R}$. Consider the $\oplus$ line $\lambda(I_1 \times J_1) = \lambda(I_2 \times J_2) =: \lambda$ and the $\ominus$ line $\mu(I_1 \times J_2) = \mu(I_2 \times J_1) =: \mu$. We say that s *joins* λ *and* μ (or μ *and* λ; the order is irrelevant). Notice that this definition includes the case when s contains only one block ($t = 0$), and that λ and μ are uniquely determined by s. At this point, let us extend the notation for lines and bands and define $\lambda(s), \mu(s), \beta(s)$ and $\gamma(s)$ by

$$\lambda(s) = \lambda(I_1 \times J_1), \quad \mu(s) = \mu(I_2 \times J_1),$$
$$\beta(s) = \beta(I_1 \times J_1), \quad \gamma(s) = \gamma(I_2 \times J_1).$$

We call these the respective $\oplus/\ominus$ lines/bands *generated by the square* s. They are essentially the two "diagonals" of the square, viewed on level j. Given $\lambda \in \lambda^j$ and $\mu \in \mu^j$, it is conceivable, a priori, that there can be any number of squares s on level j such that s joins λ and μ (or none at all). However, Lemma 3.2 below implies (in particular) that there can be at most one such square s.

2.8. Define *graphs* G^j (the *graph on level* j) by

$$G^j = (V^j, E^j),$$

where the *vertex* set V^j is

$$V^j = \lambda^j \cup \mu^j,$$

the set of *edges* E^j is

$$E^j = S^j = \{\text{squares on level } j\},$$

and where the convention is that an edge $s \in E^j$ *joins* the two vertices $\lambda(s), \mu(s) \in V^j$, as described above in 2.7. In particular, the edges are not directed. Notice that there is no edge with both of its endpoints in λ^j, or both in μ^j.

3. The main combinatorial result. To state the result we need to define a *loop* in an undirected graph. The Appendix on graphs contains additional material. The following definition of a loop or closed path is by no means standard or all-inclusive. It does not include a loop traversed twice in the same direction for example, but it does permit *some* travelling over the same paths twice. This definition happens to be convenient in our induction hypothesis.

3.1. DEFINITION. Given an undirected graph G, a *loop* in G is a pair of sequences as follows. First, a sequence $V_1, \ldots, V_n$ of vertices of G, $n \geq 2$, with $V_1 = V_n$, and second, a sequence $e_1, \ldots, e_{n-1}$ of edges of G such that at least one of the e_i occurs only once in the sequence, and for all $1 \leq i \leq n-1$, e_i joins V_i and V_{i+1}.

The main result is the following.

3.2. LEMMA. *For each $j \geq 0$, there is no loop in G^j.*

The proof will proceed by induction on j (see 3.10). The approximate strategy is as follows. By the sparseness hypothesis 1.4, there is at most one block b on level j such that b splits into a square with 4 distinct blocks on level $j+1$. This puts a restriction on the relation between the graphs G^j and G^{j+1}. We will exploit this restriction by considering a (minimal) loop g in G^{j+1}, and lifting it to its "predecessor" $\widetilde{g}$ in G^j. We assume by induction that G^j has no loops, and so in particular $\widetilde{g}$ has no loops. This implies that $\widetilde{g}$ has at least 2 "endpoints". Then, examining $\widetilde{g}$ and g at these endpoints, we eventually get a contradiction to the sparseness hypothesis. We begin the proof with the definition of "predecessors".

3.3. *Definition of* $\widetilde{\ }$ *for intervals and blocks*

3.3.1. For an interval $I \in \mathcal{L}^j$, $j \geq 1$, let $\widetilde{I}$ denote the unique interval in $\mathcal{L}^{j-1}$ such that $\widetilde{I} \supset I$. $\widetilde{I}$ is well defined by 1.1 and 1.2.

3.3.2. For a block $b = I \times J \in \mathcal{L}_A^j \times \mathcal{L}_B^j$, $j \geq 1$, define $\widetilde{b} = \widetilde{I} \times \widetilde{J}$. Clearly, $\widetilde{b} \supset b$ and $\widetilde{b} \in \mathcal{L}_A^{j-1} \otimes \mathcal{L}_B^{j-1}$ is the unique block containing b.

Before defining $\widetilde{\ }$ for squares, we need the following.

3.4. PROPOSITION. *Let $I_1, I_2, J_1, J_2 \in \mathcal{L}^j$. If $I_1 = J_1 + t$ and $I_2 = J_2 + t$ for some $t \in \mathbb{R}$, then $\widetilde{I}_1 = \widetilde{J}_1 + a$ and $\widetilde{I}_2 = \widetilde{J}_2 + a$ for some $a \in \mathbb{R}$.*

Proof. Given t, we have $I_1 - J_1 = I_2 - J_2$ by Proposition 2.6.1. Clearly, $\widetilde{I}_1 - \widetilde{J}_1 \supset I_1 - J_1$ and $\widetilde{I}_2 - \widetilde{J}_2 \supset I_2 - J_2$. Hence $(\widetilde{I}_1 - \widetilde{J}_1) \cap (\widetilde{I}_2 - \widetilde{J}_2) \neq \emptyset$, so $\widetilde{I}_1 - \widetilde{J}_1 = \widetilde{I}_2 - \widetilde{J}_2$, and we obtain the required $a \in \mathbb{R}$, by Proposition 2.6.1 again.

3.5. *Definition of* $\widetilde{\ }$ *for bands, lines and squares*

3.5.1. For a $\oplus$ band $\beta \in \beta^j$, $j \geq 1$, we have $\beta = \beta(I_1 \times J_1)$ for some block $I_1 \times J_1$. If $\beta = \beta(I_2 \times J_2)$ for some other block $I_2 \times J_2$, then for both $i = 1, 2$ we have $\beta(\widetilde{I}_i \times \widetilde{J}_i) \supset \beta(I_i \times J_i) = \beta$, so $\beta(\widetilde{I}_1 \times \widetilde{J}_1)$ and $\beta(\widetilde{I}_2 \times \widetilde{J}_2)$ are not disjoint, and hence they are equal. This allows us to define $\widetilde{\beta} = \beta(\widetilde{I}_1 \times \widetilde{J}_1)$. Similarly, for a $\ominus$ band $\gamma = \gamma(I_1 \times J_1)$ we may define $\widetilde{\gamma}$ by $\widetilde{\gamma} = \gamma(\widetilde{I}_1 \times \widetilde{J}_1)$.

3.5.2. Similar reasoning shows that for a $\oplus$ line $\lambda_1 = \lambda(I_1 \times J_1)$ and a $\ominus$ line $\mu_1 = \mu(I_1 \times J_1)$ we may define $\widetilde{\lambda}_1 = \lambda(\widetilde{I}_1 \times \widetilde{J}_1)$ and $\widetilde{\mu}_1 = \mu(\widetilde{I}_1 \times \widetilde{J}_1)$.

3.5.3. Consider a square $s \in S^j$, $j \geq 1$. Let $s = s(I_1, I_2, J_1, J_2)$ with $I_2 = I_1 + t$ and $J_2 = J_1 + t$, $t \in \mathbb{R}$. Then $\widetilde{I}_2 = \widetilde{I}_1 + a$ and $\widetilde{J}_2 = \widetilde{J}_1 + a$ for some $a \in \mathbb{R}$, by Proposition 3.4. We may define

$$\widetilde{s} = s(\widetilde{I}_1, \widetilde{I}_2, \widetilde{J}_1, \widetilde{J}_2) \in S^{j-1},$$

since s can be written in only two ways as above (the other being $s = s(I_2, I_1, J_2, J_1)$ with t replaced by $-t$) and both ways give the same $\widetilde{s}$. We have $\widetilde{s} \supset s$, but $\widetilde{s}$ is not necessarily the only square in S^{j-1} containing s. (In fact, $\widetilde{s}$ is the smallest square containing s on level $j-1$).

The following proposition is immediate from the above definitions.

3.6. PROPOSITION. *If $j \geq 1$, $s \in S^j$, $\lambda \in \lambda^j$, $\mu \in \mu^j$ and s joins λ and μ, then $s \subset \widetilde{s} \in S^{j-1}$, $\lambda \subset \widetilde{\lambda} \in \lambda^{j-1}, \mu \subset \widetilde{\mu} \in \mu^{j-1}$, and $\widetilde{s}$ joins $\widetilde{\lambda}$ and $\widetilde{\mu}$.*

Thus, we may think of $\widetilde{\ }$ as a "graph homomorphism" from G^j to G^{j-1}. In particular, the $\widetilde{\ }$ image of a "connected" subgraph of G^j is a connected subgraph of G^{j-1} (see the Appendix).

3.7. PROPOSITION. *Let $j \geq 0$. There is at most one block $b \in \mathcal{L}_A^j \otimes \mathcal{L}_B^j$ such that b contains 4 distinct blocks on level $j+1$.*

Proof. Let $b = I \times J$ be such a block. Suppose $b_i = I_i \times J_i$, $i = 1, 2, 3, 4$, are distinct blocks on level $j+1$ contained in b. We claim that at least 2 of the I_i are distinct, for if not then $I_1 = I_2 = I_3 = I_4$, forcing all the J_i to be distinct. This means that the interval $J \in \mathcal{L}^i$ contains the 4 distinct intervals $J_i \in \mathcal{L}^{j+1}$, which contradicts 1.3. Thus 2 of the I_i are distinct, which means that I is of Type 2 (see 1.3). Similarly, J is also of Type 2. Recall that $I \subset A$, $J \subset B$ and $A \cap B = \emptyset$. Thus $I \neq J$. By the Sparseness

Property (see 1.4) we find that I, J are the only two Type 2 intervals in $\mathcal{L}^j$. Thus, b is uniquely determined.

3.8. DEFINITION. If the square $s \in S^j$, $j \geq 1$, consists of 4 distinct blocks and if $\widetilde{s}$ consists of just 1 block, then we call s a *basic square* on level j. The one block b such that $\widetilde{s} = \{b\}$ will be called a *basic block* on level $j-1$.

REMARKS. By Proposition 3.7, there is at most one basic block and at most one basic square on any given level. If $I \in \mathcal{L}_A^j$ and $J \in \mathcal{L}_B^j$ are both of Type 2, then $I \times J =: b$ is a basic block iff in addition the 4 subintervals of I and J in $\mathcal{L}^{j+1}$, which we can write as $I_1, I_2 = I_1 + x \subset I$ and $J_1, J_2 = J_1 + y \subset J$ for some $x, y > 0$, also satisfy $x = y$. In other words, the distance between I_1 and I_2 must equal the distance between J_1 and J_2. In the latter case $s := \{I_1 \times J_1, I_1 \times J_2, I_2 \times J_1, I_2 \times J_2\}$ is a basic square on level $j+1$.

3.9. PROPOSITION.

3.9.1. *Let $j \geq 1$. If the squares $S, T \in S^j$ satisfy* (i) $S \neq T$, (ii) $\widetilde{S} = \widetilde{T}$, *and* (iii) $\lambda(S) = \lambda(T) =: \lambda_0$, *then $\lambda_0 = \lambda(s_0)$ for some basic square $s_0 \in S^j$, and $\widetilde{\lambda}_0 = \lambda(\widetilde{S}) = \lambda(\widetilde{T}) = \lambda(b_0)$, where b_0 is the basic block such that $\widetilde{s}_0 = \{b_0\}$.*

3.9.2. *Similarly, if we replace* (iii) *by $\mu(S) = \mu(T) =: \mu_0$, we get $\mu_0 = \mu(s_0)$ for some basic square $s_0 \in S^j$, and $\widetilde{\mu}_0 = \mu(\widetilde{S}) = \mu(\widetilde{T}) = \mu(b_0)$, where b_0 is the basic block such that $\widetilde{s}_0 = \{b_0\}$.*

Proof of 3.9.1. Consider the square $r := \widetilde{S} = \widetilde{T} \in S^{j-1}$. There are 2 cases:

(I) $r = \{I \times J\}$, $I \in \mathcal{L}_A^{j-1}$, $J \in \mathcal{L}_B^{j-1}$, or

(II) $r = \{K_1 \times L_1, K_1 \times L_2, K_2 \times L_1, K_2 \times L_2\}$ where $K_1, K_2 \in \mathcal{L}_A^{j-1}$, $L_1, L_2 \in \mathcal{L}_B^{j-1}$ and for some $a \neq 0$, $K_2 = K_1 + a$, $L_2 = L_1 + a$.

CASE (I). We see that S, T are inside $I \times J$ (i.e. their blocks are subsets of $I \times J$). We claim that both I and J are of Type 2. Suppose not, and suppose I is of Type 1 and J is of Type 1. Then there is a unique $I_1 \subset I$ and a unique $J_1 \subset J$ on level j. So $I_1 \times J_1$ is the only block on level j contained in $I \times J$. Thus $S = T = \{I_1 \times J_1\}$, contradicting (i). Now suppose I is of Type 1 and J is of Type 2. Let $I_1 \subset I$ and $J_1, J_2 \subset J$, on level j, with $J_1 \neq J_2$. Then $I_1 - J_1 \neq I_1 - J_2$ so that $\lambda(I_1 \times J_1) \neq \lambda(I_1 \times J_2)$ but S, T are the two squares $\{I_1 \times J_1\}$, $\{I_1 \times J_2\}$ in some order, since $I_1 \times J_1$, $I_1 \times J_2$ are the only blocks in $I \times J$ on level j. Thus $\lambda(S) \neq \lambda(T)$, contradicting (iii). Similarly, we cannot have I of Type 2 and J of Type 1. Thus I and J are both of Type 2. Let their subintervals on level j be

$$I_1, I_2 = I_1 + t \subset I, \qquad J_1, J_2 = J_1 + u \subset J,$$

where $t, u \neq 0$. We show that $t = u$. If $t \neq u$, it is easy to see that the four $\oplus$ bands $\beta(I_1 \times J_1)$, $\beta(I_1 \times J_2)$, $\beta(I_2 \times J_1)$, $\beta(I_2 \times J_2)$ are disjoint, and so are the corresponding $\oplus$ lines $\lambda(I_1 \times J_1)$, $\lambda(I_1 \times J_2)$, $\lambda(I_2 \times J_1)$, $\lambda(I_2 \times J_2)$. Moreover, the only squares inside $I \times J$ on level j are then $\{I_1 \times J_1\}$, $\{I_1 \times J_2\}$, $\{I_2 \times J_1\}$, $\{I_2 \times J_2\}$. But $S \neq T$ are both inside $I \times J$. Thus S, T are some two of the $\{I_k \times J_l\}$, $k, l = 1, 2$, and hence $\lambda(S) \neq \lambda(T)$, contradicting (iii). We have shown that $t = u$. Hence $s(I_1, I_2, J_1, J_2)$ is a basic square, and we call it s_0. Now, S, T are some two of the squares s_0, $\{I_k \times J_l\}$, $k, l = 1, 2$, such that $\lambda(S) = \lambda(T)$, since these are all the squares inside $I \times J$ on level j. The distinct $\oplus$ lines of these squares are clearly $\lambda(s_0) = \lambda(\{I_1 \times J_1\}) = \lambda(\{I_2 \times J_2\})$, $\lambda(\{I_1 \times J_2\})$, $\lambda(\{I_2 \times J_1\})$. Hence S, T must be some two of the squares s_0, $\{I_1 \times J_1\}$, $\{I_2 \times J_2\}$, for otherwise $\lambda(S) \neq \lambda(T)$. This shows that $\lambda(s_0) = \lambda_0$. Also, $\widetilde{s}_0 = \{I \times J\} = \widetilde{S} = \widetilde{T}$, hence $b_0 = I \times J$ and we get $\widetilde{\lambda}_0 = \lambda(b_0) = \lambda(\widetilde{S}) = \lambda(\widetilde{T})$ as required. This completes the proof in Case (I).

CASE (II). Since r has 4 distinct blocks and $r = \widetilde{S} = \widetilde{T}$, we see that S must have 1 block in each of the 4 blocks of r, and so must T. Thus, S and T may be written as

$$S = \{E_1 \times F_1, E_1 \times F_2, E_2 \times F_1, E_2 \times F_2\},$$
$$T = \{G_1 \times H_1, G_1 \times H_2, G_2 \times H_1, G_2 \times H_2\},$$

where $E_i, G_i \subset K_i$ and $F_i, H_i \subset L_i$, $i = 1, 2$, are intervals on level j. Since S is a square, it follows that $E_2 = E_1 + p$ and $F_2 = F_1 + p$ for the same $p \in \mathbb{R}$ ($p \neq 0$). To prove this, recall that a square with 4 blocks always has 2 blocks $b_1 \neq b_2$ with $\lambda(b_1) = \lambda(b_2)$. Looking at S, the only possibility is $\lambda(E_1 \times F_1) = \lambda(E_2 \times F_2)$, from which the existence of p follows by Proposition 2.6.1. Similarly, $G_2 = G_1 + q$, $H_2 = H_1 + q$ for some $q \neq 0$. Next, we claim that either $E_1 \times F_1 \neq G_1 \times H_1$ or $E_2 \times F_2 \neq G_2 \times H_2$. For, if both are equalities, we get $E_i = G_i$, $F_i = H_i$, $i = 1, 2$, whence $S = T$, contradicting (i). Thus we may assume that, say (without loss of generality), $E_1 \times F_1 \neq G_1 \times H_1$. Now $\lambda(E_1 \times F_1) = \lambda(S) = \lambda(T) = \lambda(G_1 \times H_1)$, by (iii). Hence, there is a $t \neq 0$ such that $G_1 = E_1 + t$, $H_1 = F_1 + t$. Thus, the set

$$s_0 := \{E_1 \times F_1, E_1 \times H_1, G_1 \times F_1, G_1 \times H_1\}$$

is a square on level j with 4 distinct blocks. Also, $\widetilde{s}_0 = \{K_1 \times L_1\} =: \{b_0\}$, so that s_0 is a basic square. Finally, $\lambda(s_0) = \lambda(E_1 \times F_1) = \lambda(S) = \lambda_0$ and $\widetilde{\lambda}_0 = \lambda(\widetilde{E}_1 \times \widetilde{F}_1) = \lambda(K_1 \times L_1) = \lambda(r) = \lambda(\widetilde{S}) = \lambda(\widetilde{T})$ as required. This completes the proof of Case (II), and hence of 3.9.1.

Proof of 3.9.2. We leave this as an exercise. It is almost identical to the proof of 3.9.1. Alternatively, note that the mapping $x \mapsto -x$ applied to A will change all μ's to λ's and vice versa (in $\mathcal{L}_A^j \otimes \mathcal{L}_B^j$), and hence the proof of 3.9.2 reduces to 3.9.1.

We are almost ready to prove Lemma 3.2, except for some ideas related to (finite) graphs which, however, can be found in the Appendix.

3.10. *Proof of Lemma 3.2.* For $j=0$ we have by definition $\mathcal{L}_A^0=\{A\}$ and $\mathcal{L}_B^0=\{B\}$. Thus $\{A\times B\}$ is the only square (i.e. edge in G^0), and $\lambda(A\times B),\mu(A\times B)$ are the only $\oplus$ and $\ominus$ lines. Formally, $G^0=(V^0,E^0)$, $V^0=\lambda^0\cup\mu^0=\{\lambda(A\times B),\mu(A\times B)\}$, $E^0=\{\{A\times B\}\}$. Thus G^0 is the graph with one edge $e=\{A\times B\}$ and its 2 distinct endpoints $V_1=\lambda(A\times B)$ and $V_2=\mu(A\times B)$, so clearly G^0 has no loops.

We proceed by induction on j. Let $j\ge 0$ and assume that G^j has no loops. Suppose G^{j+1} has a loop. Then G^{j+1} has a simple loop by A2. Let g be a simple loop in G^{j+1}, and consider g as a subgraph of G^{j+1}, $g=(V_g,E_g)$. Define $\widetilde{g}=(\widetilde{V}_g,\widetilde{E}_g)$ where $\widetilde{V}_g=\{\widetilde{V}:V\in V_g\}$ and $\widetilde{E}_g=\{\widetilde{e}:e\in E_g\}$. Since $\widetilde{\ }$ is a graph homomorphism (Proposition 3.6) we see that $\widetilde{g}$ is a subgraph of G^j. Thus by the induction hypothesis $\widetilde{g}$ has no loops. Also, $\widetilde{g}$ has at least 2 distinct vertices (since $\lambda^j\cap\mu^j=\emptyset$) and is connected (since g is). Hence $\widetilde{g}$ has at least 2 distinct endpoints W_1,W_2, by A5 of the Appendix.

Consider W_1. We know that $W_1\in\lambda^j\cup\mu^j$, and without loss of generality we may assume $W_1\in\lambda^j$ (i.e. a similar argument would apply if $W_1\in\mu^j$). Then $W_1=\widetilde{\lambda}_0$ for some $\lambda_0\in V_g$, $\lambda_0\in\lambda^{j+1}$, and by A9 there exist edges $S_1\ne T_1\in E_g\subset S^{j+1}$ such that λ_0 is an endpoint of both S_1 and T_1 and $\widetilde{S}_1=\widetilde{T}_1$. Since $\lambda_0\in\lambda^{j+1}$, this translates to

$$\lambda(S_1)=\lambda(T_1)=\lambda_0.$$

Applying Proposition 3.9.1 to S_1 and T_1 we see that $\lambda_0=\lambda(s_0)$ where $s_0\in S^{j+1}$ is a basic square and

$$W_1=\lambda(\widetilde{S}_1)=\lambda(\widetilde{T}_1)=\lambda(b_0)$$

where b_0 is the basic block on level j such that $\widetilde{s}_0=\{b_0\}$.

Now consider $W_2\in\lambda^j\cup\mu^j$. We claim that if $W_1\in\lambda^j$ then $W_2\in\mu^j$. For suppose $W_2\in\lambda^j$. Then the above argument for W_1 applies equally to W_2 and gives $W_2=\widetilde{\lambda}_0'$, $\lambda_0'=\lambda(s_0')$ where $s_0'\in S^{j+1}$ is a basic square. But there is at most 1 basic square on any level, so $s_0'=s_0$, whence $W_2=W_1$, a contradiction. So we must have $W_2\in\mu_j$. Moreover, the argument for W_1 still applies to W_2 (via the symmetric Proposition 3.9.2) and gives us $W_2=\widetilde{\mu}_0$, $\mu_0\in V_g\cap\mu^{j+1}$, $S_2\ne T_2\in E_g\subset S^{j+1}$, $\widetilde{S}_2=\widetilde{T}_2$,

$$\mu(S_2)=\mu(T_2)=\mu_0$$

and $\mu_0=\mu(s_0)$ where $s_0\in S^{j+1}$ is the basic square on level $j+1$ (the same s_0 as above for W_1, since there is only one!), and

$$W_2=\mu(\widetilde{S}_2)=\mu(\widetilde{T}_2)=\mu(b_0).$$

This has proved that in the graph G^j, the square $\{b_0\}$ joins W_1 and W_2, since $W_1=\lambda(b_0)$ and $W_2=\mu(b_0)$. Now recall that $\widetilde{g}$ is a connected subgraph of G^j containing W_1 and W_2. If $\widetilde{g}$ does not contain $\{b_0\}$ as an edge, then we can construct a loop in G^j by simply adding $\{b_0\}$ to the edges of $\widetilde{g}$. I.e. let $W_1=V_1,\ldots,V_n=W_2$ together with the edges $e_1,\ldots,e_{n-1}$ be a path in $\widetilde{g}$ from W_1 to W_2, let $V_{n+1}=W_1$ and $e_n=\{b_0\}$. Then e_n appears only once, so we have a loop (Definition 3.1). This contradicts the assumption that G^j has no loops. Therefore $\{b_0\}$ is an edge in $\widetilde{g}$. It follows that $\widetilde{g}$ is the graph with one edge, $\{b_0\}$, and its two endpoints, W_1,W_2 (by A6). We shall now look at all possible g and check that none of them is a simple loop, thus obtaining a contradiction:

Let $b_0=I\times J$. Let $I_1,I_2\subset I$ and $J_1,J_2\subset J$ be the subintervals on level $j+1$ such that

$$s_0=\{I_1\times J_1,I_1\times J_2,I_2\times J_1,I_2\times J_2\}$$

and $I_2=I_1+t,\ J_2=J_1+t,\ t\ne 0$ (see Definition 3.8). Then $E_g\subset\{s_0,\{I_1\times J_1\},\{I_1\times J_2\},\{I_2\times J_1\},\{I_2\times J_2\}\}=:E_0$ and $V_g\subset\{\lambda(s_0)=\lambda(I_1\times J_1)=\lambda(I_2\times J_2),\lambda(I_1\times J_2),\lambda(I_2\times J_1)\}\cup\{\mu(s_0)=\mu(I_1\times J_2)=\mu(I_2\times J_1),\mu(I_1\times J_1),\mu(I_2\times J_2)\}=:V_0$. Observe that $\lambda(I_1\times J_2)$, $\lambda(I_2\times J_1)$, $\mu(I_1\times J_1)$, $\mu(I_2\times J_2)$ are all endpoints of the graph (V_0,E_0). So none of them can belong to V_g, since g has no endpoint, being a simple loop. It follows that $V_g=\{\lambda(s_0),\mu(s_0)\}$. Also, none of the one-block squares $\{I_1\times J_1\}$, $\{I_1\times J_2\}$, $\{I_2\times J_1\}$, $\{I_2\times J_2\}$ can belong to E_g, because each has one endpoint among those that were just eliminated. This forces $E_g=\{s_0\}$. Thus g is the graph with one edge and two distinct endpoints and hence has no loops, a contradiction. We have thus proved that G^{j+1} has no loops, and so the proof of Lemma 3.2 is complete.

4. Trees and edge colouring.

We will estimate certain summations in Section 5 by using the following corollary of Lemma 3.2.

4.1. Corollary. *For each $j\ge 0$, S^j can be written in the form $S^j=S_\lambda^j\cup S_\mu^j$ such that*

(i) $S_\lambda^j\cap S_\mu^j=\emptyset$,
(ii) *if $s_1,s_2\in S_\lambda^j$ and $s_1\ne s_2$, then $\lambda(s_1)\ne\lambda(s_2)$,*
(iii) *if $s_1,s_2\in S_\mu^j$ and $s_1\ne s_2$, then $\mu(s_1)\ne\mu(s_2)$.*

Recall that S^j is the set of edges of the graph G^j. The above splitting $S^j=S_\lambda^j\cup S_\mu^j$ may be viewed as colouring all edges either blue ($=S_\lambda^j$) or green ($=S_\mu^j$) such that there is at most one blue edge at every vertex in λ^j and at most one green edge at every vertex in μ^j (recall that the vertex set of G^j is $\lambda^j\cup\mu^j$). To achieve this we first discuss trees. Let a *simple path* be a path (see A3) such that all of its edges $e_1,\ldots,e_{n-1}$ are distinct. Define a *tree* to be a graph such that for any 2 vertices V, W there is a unique simple path from V to W. Two basic results (which we leave as an exercise in A10) are that

(1) a graph G is a tree if and only if G is connected and has no loops, and

(2) if r is a vertex of the tree G, then for any edge e of G there is a unique simple path whose first vertex is r and last edge is e.

Proof of Corollary 4.1. We first note (and leave the details as an exercise) that any graph can be written as a disjoint union of connected subgraphs (components). Since G^j has no loops (Lemma 3.2), each connected component has no loops and is thus a tree. Let T be such a connected component of G^j and colour its edges as follows. Fix a vertex $r_T \in \mu^j$ of T and call it the *root* of T. For any simple path starting from the root, colour the edges in order alternately blue, green, blue, green, blue, ... That is, if $r_T = V_1, \ldots, V_n$ and $e_1, \ldots, e_{n-1}$ are the vertices and edges of a simple path (where e_i joins V_i and V_{i+1}), let $e_1, e_3, e_5, \ldots$ be blue and let $e_2, e_4, e_6, \ldots$ be green. By the remark (2) preceding this proof, every edge e of T is thus assigned a well-defined and unique colour. After colouring every component T, define $S^j_\lambda = \{e \in S^j : e \text{ is blue}\}$ and $S^j_\mu = \{e \in S^j : e \text{ is green}\}$.

We now verify (ii). Let $s_1, s_2 \in S^j_\lambda$ and $s_1 \neq s_2$. Suppose that $\lambda(s_1) = \lambda(s_2) = V$. Then s_1 and s_2 are in the same component T. Let the unique simple path from r_T to V have vertices $r_T = V_1, \ldots, V_n = V$ and edges $e_1, \ldots, e_{n-1}$. Clearly, $r_T \neq V$ since $r_T \in \mu^i$ and $V \in \lambda^j$. Also, $e_{n-1} \neq s_1$, because otherwise the (unique) simple path from r_T with last edge s_2 would be $V_1, \ldots, V_n, V_{n+1} = \mu(s_2)$, with edges $e_1, \ldots, e_{n-1} = s_1, e_n = s_2$, so that s_1 and s_2 would have different colours. Similarly, $e_{n-1} \neq s_2$. It follows that $e_1, \ldots, e_{n-1}, s_1$ and $e_1, \ldots, e_{n-1}, s_2$ are the edge sequences of the simple paths from r_T with last edge s_1 and s_2 respectively. Now recall that our graphs have the extra property that every edge e has one vertex $\lambda(e) \in \lambda^j$ and one vertex $\mu(e) \in \mu^j$. But $V_1 = r_T \in \mu^j$ by definition. Therefore $V_2 \in \lambda^j$, $V_3 \in \mu^j$, $V_4 \in \lambda^j, \ldots$ etc. Since $V_n = V \in \lambda^j$, we see that n must be even. Therefore s_1 and s_2 are both green, contradiction. The proof of (iii) is similar.

An easy way to visualize the colouring of T and the above proof is as follows. We imagine the edges of T to be made of strings of equal length. Then we dangle T in the air, by holding it by some arbitrary node, say $r_T \in \mu^j$. Thus r_T is at the top of the tree (the root node). Finally, we colour the strings alternately blue, green, blue, etc., starting from the top and moving down. Moreover, the vertices on the same horizontal level are either all of type λ or all of type μ, and the type alternates, from the top down, in the manner $\mu, \lambda, \mu, \lambda, \ldots$ This observation immediately gives (ii) and (iii).

5. Choosing diagonals with bounded overlapping. Consider a square $s \in S^j$. There are at most 2 expressions for s of the form $s = s(A_1, A_2, B_1, B_2) = \{A_1 \times B_1, A_1 \times B_2, A_2 \times B_1, A_2 \times B_2\}$ where for $t \in \mathbb{R}$, $A_2 = A_1 + t$, $B_2 = B_1 + t$ and $A_1, A_2 \in \mathcal{L}^j_A$, $B_1, B_2 \in \mathcal{L}^j_B$, the second expression being $s = s(A_2, A_1, B_2, B_1)$ if $t \neq 0$. Note that $\{A_1 \times B_1, A_2 \times B_2\} = s \cap \lambda(s)$ is independent of the choice of expression, and so is $\{A_1 \times B_2, A_2 \times B_1\} = s \cap \mu(s)$. Let $\mathcal{L}^j_A \times \mathcal{L}^j_B$ denote, as usual, the Cartesian product $\mathcal{L}^j_A \times \mathcal{L}^j_B = \{(I, J) : I \in \mathcal{L}^j_A,\ J \in \mathcal{L}^j_B\}$.

5.1. DEFINITION. For each $s \in S^j$ define two functions $\chi_{s,\lambda}, \chi_{s,\mu} : \mathcal{L}^j_A \times \mathcal{L}^j_B \to \mathbb{R}$ by

$$\chi_{s,\lambda}(I, J) = \begin{cases} 1 & \text{if } I \times J \in s \cap \lambda(s), \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi_{s,\mu}(I, J) = \begin{cases} 1 & \text{if } I \times J \in s \cap \mu(s), \\ 0 & \text{otherwise.} \end{cases}$$

Note that this can also be written as

$$\chi_{s,\lambda}(I, J) = \chi_{s\cap\lambda(s)}(I \times J), \quad \chi_{s,\mu}(I, J) = \chi_{s\cap\mu(s)}(I \times J)$$

where $\chi_{s\cap\lambda(s)}, \chi_{s\cap\mu(s)} : \mathcal{L}^j_A \otimes \mathcal{L}^j_B \to \mathbb{R}$ are the usual indicator functions of $s \cap \lambda(s)$ and $s \cap \mu(s)$ respectively.

5.2. PROPOSITION. *Let $j \geq 0$ and let $S^j = S^j_\lambda \cup S^j_\mu$ be a decomposition as in Corollary 4.1. Then for all $(I, J) \in \mathcal{L}^j_A \times \mathcal{L}^j_B$,*

$$\sum_{s\in S^j_\lambda} \chi_{s,\lambda}(I, J) + \sum_{s\in S^j_\mu} \chi_{s,\mu}(I, J) \leq 2.$$

In other words, every block $I \times J \in \mathcal{L}^j_A \otimes \mathcal{L}^j_B$ is an element of at most 2 sets in the indexed collection $\{s \cap \lambda(s)\}_{s\in S^j_\lambda} \cup \{s \cap \mu(s)\}_{s\in S^j_\mu}$.

Proof. We simply verify the even stronger assertion that every fixed block $I \times J \in \mathcal{L}^j_A \otimes \mathcal{L}^j_B$ is an element of $\lambda(s)$ for at most one $s \in S^j_\lambda$, and that it is an element of $\mu(s)$ for at most one $s \in S^j_\mu$. Fix $I \times J \in \mathcal{L}^j_A \otimes \mathcal{L}^j_B$. If $I \times J \in \lambda(s)$ then $\lambda(I \times J) = \lambda(s)$. Thus part (ii) of Corollary 4.1 implies that there is at most one $s \in S^j_\lambda$ such that $I \times J \in \lambda(s)$. Similarly, if $I \times J \in \mu(s)$ then $\mu(I \times J) = \mu(s)$, and part (iii) implies the desired result.

6. Littlewood–Paley inequalities for functions with spectrum in $\mathcal{L}^j$. Let f be a trigonometric polynomial on the circle $\mathbb{T}$. We define $\mathrm{Spec}(f) = \{n \in \mathbb{Z} : \widehat{f}(n) \neq 0\}$. For $E \subset \mathbb{R}$, f_E is the function such that $\widehat{f_E} = \chi_E \widehat{f}$, i.e. $f_E(t) = \sum_{n\in\mathbb{Z}\cap E} \widehat{f}(n)e^{int}$, $t \in \mathbb{T}$. Given any forest $\{\mathcal{L}^i\}_{i=0}^\infty$ as in Section 1, we have the following result.

6.1. LEMMA. *Let $k \ge 0$ and suppose that $A, B \in \mathcal{L}^k$ are disjoint intervals. Fix $j \ge k$ and put $\mathcal{L}_A^j = \{I \in \mathcal{L}^j : I \subset A\}$ and $\mathcal{L}_B^j = \{I \in \mathcal{L}^j : I \subset B\}$. Suppose that f is a trigonometric polynomial such that*

$$\operatorname{Spec}(f) \subset \bigcup_{I \in \mathcal{L}^j} I, \quad \text{i.e.} \quad f = \sum_{I \in \mathcal{L}^j} f_I.$$

Then

$$\int_{\mathbb{T}} |f_A|^2 |f_B|^2 \le 2 \int_{\mathbb{T}} \Big(\sum_{I \in \mathcal{L}_A^j} |f_I|^2 \Big) \Big(\sum_{J \in \mathcal{L}_B^j} |f_J|^2 \Big),$$

where the integrals are taken with respect to normalized Lebesgue measure, $dt/(2\pi)$.

Proof. It suffices to consider the case $k = 0$ for an arbitrary forest $\{\mathcal{L}^i\}$. Let $I_1, I_2 \in \mathcal{L}_A^j$ and $J_1, J_2 \in \mathcal{L}_B^j$. If

$$\int (f_{I_1} \bar{f}_{I_2}) \overline{(f_{J_1} \bar{f}_{J_2})} \ne 0$$

then $(I_1 - I_2) \cap (J_1 - J_2) \ne \emptyset$, and we must have $I_2 = I_1 + t$ and $J_2 = J_1 + t$ for some $t \in \mathbb{R}$ (by Proposition 2.6.1), i.e. $I_1 - I_2 = J_1 - J_2$, and thus $s(I_1, I_2, J_1, J_2) = s$ is a square in S^j. For each such square s, define

$$\pi(f, s) = |f_{I_1}| \cdot |f_{I_2}| \cdot |f_{J_1}| \cdot |f_{J_2}|,$$

and note the following two possible ways of estimating $\pi(f, s)$:

$$\text{(1)} \qquad \pi(f, s) \le \frac{1}{2} |f_{I_1}|^2 |f_{J_1}|^2 + \frac{1}{2} |f_{I_2}|^2 |f_{J_2}|^2 = \frac{1}{2} \sum_{(I,J)} \chi_{s,\lambda}(I, J) |f_I|^2 |f_J|^2,$$

$$\text{(2)} \qquad \pi(f, s) \le \frac{1}{2} |f_{I_1}|^2 |f_{J_2}|^2 + \frac{1}{2} |f_{I_2}|^2 |f_{J_1}|^2 = \frac{1}{2} \sum_{(I,J)} \chi_{s,\mu}(I, J) |f_I|^2 |f_J|^2,$$

where $\sum_{(I,J)}$ denotes the sum over *all* $(I, J) \in \mathcal{L}_A^j \times \mathcal{L}_B^j$.

In the following string of inequalities, the first factor of 2 occurs because each square appears at most twice, as $s = s(I_1, I_2, J_1, J_2) = s(I_2, I_1, J_2, J_1)$. We fix a decomposition $S^j = S_\lambda^j \cup S_\mu^j$ as in Corollary 4.1. Then

$$\int |f_A|^2 |f_B|^2 = \int f_A \bar{f}_A \overline{(f_B \bar{f}_B)}$$
$$= \int \Big(\sum_{I_1 \in \mathcal{L}_A^j} f_{I_1} \Big) \Big(\sum_{I_2 \in \mathcal{L}_A^j} \bar{f}_{I_2} \Big) \overline{\Big(\sum_{J_1 \in \mathcal{L}_B^j} f_{J_1} \Big) \Big(\sum_{J_2 \in \mathcal{L}_B^j} \bar{f}_{J_2} \Big)}$$

$$= \sum_{(I_1, I_2, J_1, J_2) \in \mathcal{L}_A^j \times \mathcal{L}_A^j \times \mathcal{L}_B^j \times \mathcal{L}_B^j} \int (f_{I_1} \bar{f}_{I_2}) \overline{(f_{J_1} \bar{f}_{J_2})}$$
$$= \sum_{(I_1, I_2, J_1, J_2),\, I_1 - I_2 = J_1 - J_2} \int (f_{I_1} \bar{f}_{I_2}) \overline{(f_{J_1} \bar{f}_{J_2})}$$
$$\le \int 2 \sum_{s \in S^j} \pi(f, s) = \int 2 \Big(\sum_{s \in S_\lambda^j} \pi(f, s) + \sum_{s \in S_\mu^j} \pi(f, s) \Big)$$
$$\le \int 2 \Big(\sum_{s \in S_\lambda^j} \frac{1}{2} \sum_{(I,J)} \chi_{s,\lambda}(I, J) |f_I|^2 |f_J|^2 + \sum_{s \in S_\mu^j} \frac{1}{2} \sum_{(I,J)} \chi_{s,\mu}(I, J) |f_I|^2 |f_J|^2 \Big)$$
$$= \int \sum_{(I,J)} \Big(\sum_{s \in S_\lambda^j} \chi_{s,\lambda}(I, J) + \sum_{s \in S_\mu^j} \chi_{s,\mu}(I, J) \Big) |f_I|^2 |f_J|^2$$
$$\le \int \sum_{(I,J)} 2 |f_I|^2 |f_J|^2 = 2 \int \Big(\sum_{I \in \mathcal{L}_A^j} |f_I|^2 \Big) \Big(\sum_{J \in \mathcal{L}_B^j} |f_J|^2 \Big).$$

The last factor of 2 is from Proposition 5.2.

6.2. THEOREM. *Let $\{\mathcal{L}^i\}_{i=0}^\infty$ be a forest such that $\mathcal{L}^0$ consists of exactly the interval I_0, i.e. $\mathcal{L}^0 = \{I_0\}$. Fix $j \ge 0$ and suppose that the trigonometric polynomial f satisfies*

$$\operatorname{Spec}(f) \subset \bigcup_{I \in \mathcal{L}^j} I, \quad \text{i.e.} \quad f = \sum_{I \in \mathcal{L}^j} f_I.$$

Then

$$\int_{\mathbb{T}} |f|^4 \le 4 \int_{\mathbb{T}} \Big(\sum_{I \in \mathcal{L}^j} |f_I|^2 \Big)^2,$$

where the integrals are with respect to normalized Lebesgue measure on $\mathbb{T}$.

Proof. The theorem follows from Lemma 6.1 by the method of [H1] or [H2]. We repeat it here for completeness. The idea is to use the simple tree structure of the subintervals of I_0. (This tree structure has nothing to do with the trees of Section 4.). Moreover, only the "trivial" part of [H1] or [H2] is needed here, since by hypothesis f has no spectrum in the complement of $\bigcup \mathcal{L}^j$.

We first note the following identity: If $I_1, I_2 \in \mathcal{L}^i$ are disjoint then

$$\int |f_{I_1} + f_{I_2}|^4 = \int (|f_{I_1}|^4 + |f_{I_2}|^4 + 4 |f_{I_1}|^2 \, |f_{I_2}|^2).$$

The proof is an exercise in checking which of the terms $\int f_{I_k} \bar{f}_{I_l} \overline{(f_{I_m} \bar{f}_{I_n})}$, $k, l, m, n = 1, 2$, are equal to 0, using the facts that $I_1 - I_1 = [-l_i, l_i] = I_2 - I_2$, and that the gap between I_1 and I_2 is $\ge 3 l_i$ by definition of $\mathcal{L}^i$. If I is an interval, put $\mathcal{L}_I^k = \{J \in \mathcal{L}^k : J \subset I\}$. Let $\mathcal{L}_2^k = \{I \in \mathcal{L}^k : I \text{ is of Type 2}\}$. If $I \in \mathcal{L}_2^k$ let $A(I), B(I)$ denote the two elements of $\mathcal{L}_I^{k+1}$ in some order (say $A(I)$ is to the left of $B(I)$). Now iterate the above identity to obtain

$$\int |f|^4 = \int \Big(\sum_{I\in\mathcal{L}^j} |f_I|^4 + \sum_{k=0}^{j-1} \sum_{K\in\mathcal{L}_2^k} 4|f_{A(K)}|^2 |f_{B(K)}|^2 \Big). \tag{6.3}$$

That is, if I_0 is of Type 2, apply the identity once to get

$$\int |f|^4 = \int |f_{I_0}|^4 = \int |f_{A(I_0)}|^4 + |f_{B(I_0)}|^4 + 4|f_{A(I_0)}|^2\, |f_{B(I_0)}|^2.$$

If I_0 is of Type 1, we can just write $f_{I_0} = f_{I_1}$, where $I_1 \subset I_0$, $I_1 \in \mathcal{L}^1$. Then the first application of the identity occurs in $\mathcal{L}^1$ or on some later level. Next, apply the identity to each of the 4th power terms whose index interval is of Type 2, and leave the mixed terms alone. Stop when all 4th power terms have an index interval I on level j.

Given the identity (6.3), we apply Lemma 6.1 to each of the mixed terms $\int |f_{A(K)}|^2 |f_{B(K)}|^2$. To do this, fix K and take $A = A(K)$, $B = B(K)$ in the lemma. (In the present notation these A, B are on level $k+1$). Thus (6.3) gives

$$\int |f|^4 \le \int \Big[\sum_{I\in\mathcal{L}^j} |f_I|^4 + 8 \sum_{k=0}^{j-1} \sum_{K\in\mathcal{L}_2^k} \Big(\sum_{I\in\mathcal{L}^j_{A(K)}} |f_I|^2 \Big) \Big(\sum_{J\in\mathcal{L}^j_{B(K)}} |f_J|^2 \Big) \Big]. \tag{6.4}$$

It remains to check that in (6.4) each term of type $|f_I|^2|f_J|^2$, $(I,J) \in \mathcal{L}^j \times \mathcal{L}^j$, occurs with a coefficient (multiplicity) of at most 4. First of all, since $A(K)$ is always to the left of $B(K)$, the $|f_I|^4$ terms occur only once, and $|f_I|^2|f_J|^2$, $I \ne J$, occurs only if $I < J$ (I is to the left of J). Next, if we omit the 8, each (I, J), $I < J$, occurs at most once, by the following.

6.5. Proposition. *Let $0 \le k_1, k_2 \le j-1$, $K_1 \in \mathcal{L}_2^{k_1}$ and $K_2 \in \mathcal{L}_2^{k_2}$. If $K_1 \ne K_2$ then*

$$(\mathcal{L}^j_{A(K_1)} \times \mathcal{L}^j_{B(K_1)}) \cap (\mathcal{L}^j_{A(K_2)} \times \mathcal{L}^j_{B(K_2)}) = \emptyset.$$

To prove Proposition 6.5 note that either $K_1 \cap K_2 = \emptyset$ or $K_1 \supset K_2$ or $K_1 \subset K_2$. If $K_1 \cap K_2 = \emptyset$ the result follows. So assume without loss of generality that $K_1 \supset K_2$. Then $K_2 \subset A(K_1)$ or $K_2 \subset B(K_1)$. Assume w.l.o.g. that $K_2 \subset A(K_1)$. Then $A(K_2), B(K_2) \subset A(K_1)$. Since $A(K_1) \cap B(K_1) = \emptyset$, we get $B(K_2) \cap B(K_1) = \emptyset$ and the result follows.

Returning to (6.4), the proof is finished as follows:

$$\int |f|^4 \le \int \Big[\sum_{I\in\mathcal{L}^j} |f_I|^4 + 8 \sum_{(I,J)\in\mathcal{L}^j\times\mathcal{L}^j,\, I<J} |f_I|^2 |f_J|^2 \Big]$$
$$\le \int 4 \sum_{(I,J)\in\mathcal{L}^j\times\mathcal{L}^j} |f_I|^2 |f_J|^2 = 4 \int \Big(\sum_{I\in\mathcal{L}^j} |f_I|^2 \Big)^2.$$

Appendix on finite graphs. In this paper a finite *graph* is denoted by $G = (V_G, E_G)$ where V_G and E_G are finite sets, called the set of *vertices* and the set of *edges* of G, respectively. This is assumed to be accompanied by an assignment of 2 vertices $V_1, V_2 \in V_G$ to each edge $e \in E_G$; we say e *joins* V_1 and V_2 (without regard to order) and we allow $V_1 = V_2$. Thus, formally we should write $G = (V_G, E_G, \pi_G)$ where $\pi_G : E_G \to \{\{V_1, V_2\} : V_1, V_2 \in V_G\}$ and the convention is that e joins V_1 and V_2 if $\{V_1, V_2\} = \pi_G(e)$. A *subgraph* of a graph G is a graph $g = (V_g, E_g, \pi_g)$ where $V_g \subset V_G$, $E_g \subset E_G$ and $\pi_g = \pi_G$ restricted to E_g. Thus V_g must contain $\pi_G(e)$ for all $e \in E_g$.

In fact, the graphs in the text have the additional feature that $V_G = \lambda \cup \mu$ where $\lambda \cap \mu = \emptyset$ and for each edge $e \in E_g$, $\pi_G(e) = \{\lambda_1, \mu_1\}$ for some $\lambda_1 \in \lambda$ and $\mu_1 \in \mu$, and we write $\lambda_1 = \lambda(e)$, $\mu_1 = \mu(e)$. In particular, the case $V_1 = V_2$ never actually occurs. But, in this appendix we ignore these additional features. The proofs of all results about graphs are left as exercises.

A1. A loop (see Definition 3.1) is called a *simple loop* if its vertices (and hence also edges) are distinct, i.e. if $V_1, \ldots, V_{n-1}$ are distinct.

A2. Given a loop, there exists a simple loop contained in it, i.e. a simple loop which is a subgraph of the given loop, when both are regarded as graphs in the obvious sense.

A3. A graph is said to be *connected* if for any 2 vertices $V \ne W$ there exist vertices $V_1, \ldots, V_n$ and edges $e_1, \ldots, e_{n-1}$ such that $V_1 = V$, $V_n = W$ and e_i joins V_i and V_{i+1} for $i = 1, \ldots, n-1$. We call this a *path* from V to W.

A4. If an edge e joins vertices V, W, we say V and W are the *endpoints* of e. A vertex V of a graph G is called an *endpoint* of G if there exists an edge e in G such that V is one endpoint of e, the other endpoint of e is not equal to V, and V is not an endpoint of any other edge in G. ("Isolated" points are thus not called endpoints by our definition.)

A5. Let G be a connected graph with no loops and with at least 2 distinct vertices. Then G has at least 2 distinct endpoints.

A6. Let $W_1 \ne W_2$ be two endpoints of the connected graph $G = (V_G, E_G)$. If there is an edge e joining W_1 and W_2 in G, then G is a graph with one edge: $V_G = \{W_1, W_2\}$, $E_G = \{e\}$.

A7. Let $G = (V_G, E_G)$, $H = (V_H, E_H)$ be graphs. A map $\varphi : V_G \cup E_G \to V_H \cup E_H$ is called a *(graph) homomorphism* if $\varphi(V_G) \subset V_H$, $\varphi(E_G) \subset E_H$, and $\varphi(e)$ joins $\varphi(v)$ and $\varphi(w)$ whenever e joins v and w, $e \in E_G$, $v, w \in V_G$. Write $\varphi : G \to H$.

A8. If $\varphi : G \to H$ is a graph homomorphism and $G = (V_G, E_G)$ is connected then the image graph $(\varphi(V_G), \varphi(E_G))$ is connected.

A9. Let the graph $g = (V_g, E_g)$ be a simple loop (i.e. the graph formed by taking the vertices and edges of a simple loop). Let $\varphi : g \to h$ be a graph homomorphism, with $h = (\varphi(V_g), \varphi(E_g))$. If h has an endpoint $W = \varphi(V)$, then there exist 2 edges $e_1 \neq e_2 \in E_g$ such that V is an endpoint of both e_1 and e_2, and $\varphi(e_1) = \varphi(e_2) =$ the unique edge in h with W as one endpoint.

A10. A *simple path* from a vertex V to a vertex W is a path (see A3) from V to W such that all of its edges $e_1, \ldots, e_{n-1}$ are distinct. We include the one-point path V_1 (with no edge), and any simple loop ($V_1 = V_n$). A graph G is called a *tree* if for any vertices V, W of G there is a unique simple path from V to W. We have the following results:

(1) G is a tree if and only if G is connected and has no loops.

(2) If G is a tree, r is a vertex of G and e is an edge of G, then there is a unique simple path $((V_1, \ldots, V_n), (e_1, \ldots, e_{n-1}))$ with $V_1 = r$ and $e_{n-1} = e$.

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Commutants of certain multiplication operators on Hilbert spaces of analytic functions

by

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Abstract. This paper characterizes the commutant of certain multiplication operators on Hilbert spaces of analytic functions. Let $A = M_z$ be the operator of multiplication by z on the underlying Hilbert space. We give sufficient conditions for an operator essentially commuting with A and commuting with A^n for some $n > 1$ to be the operator of multiplication by an analytic symbol. This extends a result of Shields and Wallen.

1. Introduction. Let H be a Hilbert space of complex-valued analytic functions on the open unit disc $\mathbb{D}$ such that point evaluations are bounded linear functionals on H. Then for every $w \in \mathbb{D}$ there exists a function k_w in H such that $f(w) = \langle f, k_w \rangle$ for all $f \in H$. Now if we define $K : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ by $K(z, w) = k_w(z)$, then K is a positive definite function with the reproducing property $f(w) = \langle f(\cdot), K(\cdot, w) \rangle$ for every $w \in \mathbb{D}$ and $f \in H$. The function K is called the *reproducing kernel* for H.

Recall that a function $K : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ is *positive definite* (denoted $K \gg 0$) provided

$$\sum_{j,k=1}^{n} a_j \overline{a}_k K(w_j, w_k) \geq 0$$

for any finite set of complex numbers $a_1, \ldots, a_n$ and any finite subset $w_1, \ldots, w_n$ of $\mathbb{D}$. Conversely, if $K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is positive definite then

$$\Big\{ \sum_{j=1}^{n} a_j K(\cdot, w_j) : a_1, \ldots, a_n \in \mathbb{C} \text{ and } w_1, \ldots, w_n \in \mathbb{D} \Big\}$$

has dense linear span in a Hilbert space $H(K)$ of functions with

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