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An almost nowhere Fréchet smooth norm on superreflexive spaces

by

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Abstract. Every separable infinite-dimensional superreflexive Banach space admits an equivalent norm which is Fréchet differentiable only on an Aronszajn null set.

Introduction. Every convex continuous function on a separable Banach space X is Gateaux differentiable on a dense G_δ-set by a theorem of Mazur. If the dual of X is separable it is even Fréchet differentiable on a dense G_δ-set. If we confine ourselves to the weaker notion of Gateaux differentiability, then locally Lipschitz functions, and in particular convex continuous functions, are also differentiable on a set which is large in the sense of measure.

The strongest present result in this direction is due to Mankiewicz [Man] and Aronszajn [A]. They defined in every separable Banach space a family $\mathcal{A}$ of sets which mimics the family of Lebesgue null sets in finite dimensions. The definitions of the family $\mathcal{A}$ (now usually called the Aronszajn null sets, see Section 2) used by Mankiewicz and by Aronszajn are formally different; it was recently shown by Csörnyei that they both coincide with the so called Gaussian null sets [C]. Mankiewicz and Aronszajn proved that every locally Lipschitz function is Gateaux differentiable almost everywhere, that is, except on a set belonging to $\mathcal{A}$. For Fréchet differentiability this fails except for finite dimensions, where the classical theorem of Rademacher is available. If X is a separable and infinite-dimensional Banach space then by a result of Preiss and Tišer [PT] there is a Lipschitz function f on X such that the set of points where f *is* Fréchet differentiable is Aronszajn null.

In [MM] it was shown that it is of no help to consider only convex continuous functions. There exists an equivalent norm p on the separable Hilbert space ℓ_2 such that the set of points where p *is* Fréchet differentiable

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is Aronszajn null. Here we prove the result of [MM] in any separable infinite-dimensional superreflexive Banach space. To construct a "bad" norm on a superreflexive Banach space we combine the simplified proof of [MM] given by Preiss with the existence of very "nice" renormings of superreflexive spaces.

In [PZ] Preiss and Zajíček ask if the set of points where a convex continuous function on a Banach space with a separable dual is not Fréchet differentiable can be covered by countably many closed convex sets with empty interior and countably many δ-convex surfaces. This was disproved by Konyagin [K]. Notice that our example also provides a counterexample to this question. By [M1] and [M2] in superreflexive Banach spaces every closed convex set with empty interior is Haar null (this is another replacement of Lebesgue null sets in infinite dimensions) and δ-convex surfaces are easily seen to be Haar null. A countable union of Haar null sets is Haar null, and the union of a Haar null set and of an Aronszajn null set cannot be the entire space.

If X is a Banach space, $x \in X$ and $r > 0$, we denote by $B(x,r)$ the closed ball with center x and radius r; B_X is the closed unit ball of X. The modulus of convexity of a Banach space X is denoted by $\boldsymbol{\delta}_X$. The Banach space $\mathbb{R}^n$ is considered with the ℓ_2-norm if not stated otherwise; we denote by λ_n the n-dimensional Lebesgue measure. If A is a subset of a Banach space X we define $\operatorname{cone} A = \bigcup_{t>0} tA$.

2. Sets small in measure. The following notion of a null set was introduced by Aronszajn [A]; for equivalent definitions see [C].

DEFINITION 2.1. Let X be a separable Banach space and let A be a Borel subset of X. The set A is called *Aronszajn null* if for every sequence $(x_i)_{i=1}^{\infty}$ in X whose closed linear span is X there exist Borel sets $A_i \subset X$ such that $A = \bigcup_{i=1}^{\infty} A_i$ and the intersection of A_i with any line in direction x_i has one-dimensional Lebesgue measure zero, for each $i \in \mathbb{N}$.

Suppose $n \in \mathbb{N}$ and A is a Borel subset of a Banach space X such that A intersected with any n-dimensional affine subspace of X is of n-dimensional Lebesgue measure zero. It is an easy consequence of Fubini's theorem that A is Aronszajn null. If A is a cone-like set, that is, $cA \subset A$ for any $c > 0$, then in the above situation it is enough to consider all n-dimensional subspaces rather than all n-dimensional *affine* subspaces.

LEMMA 2.2. *Let X be a separable Banach space and A a Borel subset of X so that $cA \subset A$ for any $c > 0$. Suppose $n \in \mathbb{N}$ is such that A intersected with the unit ball of any n-dimensional subspace of X is of n-dimensional Lebesgue measure zero. Then A is Aronszajn null.*

Proof. If $n = 1$ then $A \subset \{0\}$. Suppose that $n > 1$ and that there is an $(n-1)$-dimensional affine subspace $Y \subset X$ with $0 \notin Y$ and with $A \cap Y$ having positive $(n-1)$-dimensional Lebesgue measure. Since $\operatorname{cone}(A \cap Y) \subset A \cap \operatorname{span} Y$, the n-dimensional Lebesgue measure of $A \cap B_{\operatorname{span} Y}$ is positive, which is a contradiction. By Fubini's theorem the n-dimensional Lebesgue measure of $A \cap Z$ is zero for any n-dimensional affine $Z \subset X$, and A is Aronszajn null.

We will need the following simple lemma to construct a "bad" norm on a general separable superreflexive Banach space from a bad norm on its quotient with a basis.

LEMMA 2.3. *Let X, Y be separable infinite-dimensional Banach spaces, and $T : X \to Y$ a continuous linear surjective mapping. Let $A \subset Y$ be Aronszajn null. Then $T^{-1}(A)$ is Aronszajn null.*

Proof. Suppose $E \subset X$ is a countable set with $\overline{\operatorname{span}}\, E = X$. Let $(y_n)_{n=1}^{\infty}$ be an enumeration of $T(E) \setminus \{0\}$. Then $\overline{\operatorname{span}}\{y_n\}_{n=1}^{\infty} = Y$. Let $A_n \subset Y$ be Borel so that $A_n \cap (y + \operatorname{span} y_n)$ has one-dimensional measure zero for each $y \in Y$ and $\bigcup_{n\in\mathbb{N}} A_n = A$. Since T is continuous, each set $T^{-1}(A_n)$ is Borel. Clearly, $\bigcup_{n\in\mathbb{N}} T^{-1}(A_n) = T^{-1}(A)$. If $n \in \mathbb{N}$ is arbitrary, $z \in T^{-1}(y_n)$ and $x \in X$ then T restricted to $x + \operatorname{span} z$ is an affine homeomorphism. Since $T(T^{-1}(A_n) \cap (x + \operatorname{span} z)) = A_n \cap (T(x) + \operatorname{span} y_n)$, the set $T^{-1}(A_n) \cap (x + \operatorname{span} z)$ has one-dimensional measure zero.

Let X be a Banach space and $n \in \mathbb{N}$. For each n-dimensional subspace Z of X fix an isomorphism $T_Z : Z \to \mathbb{R}^n$ with $\|T_Z\| = 1$ and $\|T_Z^{-1}\| \le n$. In the sequel, "$\lambda_Z(A \cap Z) < \varepsilon$ for any n-dimensional subspace $Z \subset X$" means that the n-dimensional measures λ_Z come from these isomorphisms.

LEMMA 2.4. *Let X be a Banach space with a uniformly convex norm so that $\boldsymbol{\delta}_X(\varepsilon) \ge c\varepsilon^p$ for some $p \ge 2$ and $c > 0$. Suppose Z is an n-dimensional subspace of X and $v \in X^*$ is such that $\sup_{x\in B_Z} \langle v, x\rangle \le 1 + \varrho$ for some $\varrho > 0$. Then $\lambda_Z(C) \le \beta \varrho^{(n-1)/p}$ for $C = B_Z \cap \{x \in X : v(x) > \|x\|\}$, where β is an absolute constant.*

Proof. Denote by $\|v\|_Z$ the norm of v when restricted to Z and by u the point of B_Z where it is attained. Observe that $C = B_Z \cap \operatorname{cone}\{x \in B_Z : v(x) > 1\}$. Since $1/\|v\|_Z \ge 1/(1+\varrho) \ge 1 - \varrho$, we have

$$\{x \in B_Z : v(x) \ge 1\} \subset \{x \in B_Z : v(x)/\|v\|_Z \ge 1 - \varrho\}.$$

Since Z is uniformly rotund with $\boldsymbol{\delta}_Z(\varepsilon) \ge c\varepsilon^p$, the diameter of the latter set is at most $\alpha\varrho^{1/p}$, where $\alpha > 0$ is a constant (see e.g. [D], p. 58). Hence $C \subset \operatorname{conv}(B(u, \alpha\varrho^{1/p}) \cup B(0, \alpha\varrho^{1/p}))$ and

$$\lambda_Z(C) = \lambda_n(T_Z(C)) \le \lambda_n(\operatorname{conv}(B_{\mathbb{R}^n}(T_Z(u), \alpha\varrho^{1/p}) \cup B_{\mathbb{R}^n}(0, \alpha\varrho^{1/p})).$$

The latter is at most $\beta\varrho^{(n-1)/p}$ for some constant β since $\|T_Z(u)\| \le 1$.

PROPOSITION 2.5. *Let X be a uniformly convex Banach space with a uniformly convex dual so that $\delta_X(\varepsilon) \ge c\varepsilon^p$ for some $p \ge 2$ and $c > 0$. Suppose X has a basis. Then there exists $N \in \mathbb{N}$ such that for each $\varepsilon > 0$ there is $\delta > 0$ and a countable symmetric set $(v_n)_{n\in\mathbb{N}}$ in X^* with $1 + \delta \le \|v_n\| \le 2$ so that for $S = \bigcup_{n\in\mathbb{N}}\{x \in X : v_n(x) > \|x\|\}$,*

(A) $\overline{S} = X$, *and*

(B) $\lambda_Z(S \cap B_Z) \le \varepsilon$ *for each N-dimensional subspace $Z \subset X$.*

Proof. Let (e_n) be the basis of X and (f_n) the dual basis; we can suppose that $m \le \|e_n\| \le M$ and $m \le \|f_n\| \le M$ for all n and some $M, m > 0$. By the theorem of Gurarii and Gurarii (see e.g. [BL]) there are $q > 1$ and $\gamma > 0$ so that

$$\|x\| \ge \gamma\Big(\sum_{i=1}^{\infty} |\langle f_i, x\rangle|^q\Big)^{1/q}$$

for all $x \in X$. For $x \in X$ we define the support of x as $\operatorname{spt} x = \{i \in \mathbb{N} : \langle f_i, x\rangle \ne 0\}$. Let $(x_k)_{k=1}^{\infty}$ be a dense sequence in the unit sphere of X with each x_k finitely supported. For each k choose $x_k^* \in X^*$ with $\|x_k^*\| = 1 = \langle x_k^*, x_k\rangle$. Choose $n_1 < n_2 < \dots$ such that $\max \operatorname{spt}(x_k) < n_k$. Fix some $r > 0$ small (to be specified later) and define $v_k = x_k^* + rf_{n_k}$. Since

$$\langle v_k, x_k\rangle = \langle x_k^*, x_k\rangle + r\langle f_{n_k}, x_k\rangle = 1 + 0 = 1,$$

we get $\|v_k\| \ge 1$ and, similarly, $\big\|x_k^* + \frac{1}{2}rf_{n_k}\big\| \ge 1$. The uniform rotundity of X^* implies that

$$1 \le \big\|x_k^* + \tfrac{1}{2}rf_{n_k}\big\| = \tfrac{1}{2}\|x_k^* + v_k\| \le \|v_k\| - \delta_{X^*}(r\|f_{n_k}\|).$$

Since $m \le \|f_{n_k}\| \le M$, this means that $1 + \delta_{X^*}(rm) \le \|v_k\| \le 2$ for all $k \in \mathbb{N}$; the norms of v_k's are bounded away from 1.

To verify the condition (A), for $k \in \mathbb{N}$ choose $w_k \in X$ so that $\|w_k\| = 1 < \langle v_k, w_k\rangle$. Then for any $t > 0$,

$$\langle v_k, x_k + tw_k\rangle = 1 + t\langle v_k, w_k\rangle > 1 + t \ge \|x_k + tw_k\|,$$

hence $x_k + tw_k \in S$. Consequently, $\{x_n : n \in \mathbb{N}\} \subset \overline{S}$ and since $(x_n)_{n\in\mathbb{N}}$ is dense in the sphere of X and $\operatorname{cone} S = S$ we get $\overline{S} = X$.

To verify the condition (B), fix $N \in \mathbb{N}$ (again to be specified later). Let Z be an N-dimensional subspace of X. We will show that for $\varrho > 0$ not too many v_k's can attain the value $1 + \varrho$ on B_Z. From Lemma 2.4 we know that those v_k's that are at most $1 + 2\varrho$ on B_Z do not contribute too much measure to $\lambda_Z(S \cap B_Z)$; N will be chosen so that the products of these numbers add up.

There exist $u_1, \dots, u_N \in B_Z$ so that $B_Z \subset \{\sum_{i=1}^N a_iu_i : |a_i| \le N\}$. For $j = 1, 2, \dots$ define

$$I_j = \{k \in \mathbb{N} : 1 + 2^{-j} \le \max_{x\in B_Z}\langle v_k, x\rangle \le 1 + 2^{-j+1}\},$$

and $S_j = \bigcup_{n\in I_j}\{x \in X : v_n(x) > \|x\|\}$. If $v_n(y) > \|y\|$ for some $y \in B_Z$ and $n \in \mathbb{N}$, then $v_n(y/\|y\|) > 1$ and $\max_{x\in B_Z}\langle v_k, x\rangle > 1$. Therefore $B_Z \cap S = \bigcup_{j=1}^{\infty}(B_Z \cap S_j)$. By Lemma 2.4,

$$\lambda_Z(S \cap B_Z) \le \sum_{j=1}^{\infty} \beta 2^{(1-j)(N-1)/p}|I_j|. \tag{1}$$

To estimate $|I_j|$, suppose that $k \in I_j$ and $x \in B_Z$ are such that $\langle v_k, x\rangle \ge 1 + 2^{-j}$. Since $x = \sum_{i=1}^N a_iu_i$ for some suitable $|a_i| \le N$ we can estimate

$$1 + 2^{-j} \le \langle x_k^* + rf_{n_k}, x\rangle \le 1 + r\sum_{i=1}^{N} a_i\langle f_{n_k}, u_i\rangle$$
$$\le 1 + rN^2 \max_i |\langle f_{n_k}, u_i\rangle|.$$

This means that if $k \in I_j$ there is some $i = 1, \dots, N$ for which

$$|\langle f_{n_k}, u_i\rangle| \ge \frac{1}{2^jrN^2}.$$

This cannot happen for too many k's, since

$$1 \ge \|u_i\| \ge \gamma\Big(\sum_{n=1}^{\infty} |\langle f_n, u_i\rangle|^q\Big)^{1/q};$$

each u_i can have at most $(2^jrN^2/\gamma)^q$ coordinates that are not smaller than $1/(2^jrN^2)$. Therefore $|I_j| \le \alpha_1 2^{jq}r^qN^{2q+1}$ for a suitable constant $\alpha_1 > 0$. Finally, by substituting into (1), we arrive at

$$\lambda_Z(S \cap B_Z) \le \alpha_2 r^q N^{2q+1} \sum_{j=1}^{\infty} 2^{j(q-(N-1)/p)}.$$

If N is such that $q - (N-1)/p < 0$, then this is at most ε for r small enough. To obtain, for a given $\varepsilon > 0$, a symmetric set $C = (v_k)$ it is enough to put $C = C' \cup (-C')$, where C' is a set which works for $\varepsilon/2$.

3. Convex functions. Suppose that g_n are convex continuous functions, and the subdifferential of each of them has a large jump somewhere close to a given point x for which $g_1(x) = g_2(x) = \dots$ Then the pointwise supremum of (g_n) (if it exists) is not Fréchet differentiable at x.

LEMMA 3.1. *Let X be a Banach space, $x \in X$, $x^* \in X^*$. Let $g, g_1, g_2, \dots$ be convex continuous functions on X with $g(x) = g_1(x) = g_2(x) = \dots,$*

$g_n \le g$, and $x^* \in \partial g_n(x)$ for all $n \in \mathbb{N}$. Suppose there are $a > 0$, $y_n \in X$ and $y_n^* \in \partial g_n(y_n)$ with $\lim y_n = x$ and $\|x^* - y_n^*\| > a$. Then the convex continuous function $\widetilde{g} = \sup g_n$ is not Fréchet differentiable at x.

Proof. We can suppose that $x = 0$, $x^* = 0$, and $g(x) = 0$. Since the functions g_n are uniformly bounded on some neighborhood of x, they are all Lipschitz with the same constant $c > 0$ on some neighborhood of x. Suppose that all y_n are contained in this neighborhood. Then for $v \in X$ we can estimate

$$g_n(v) \ge g_n(y_n) + \langle y_n^*, v - y_n\rangle \ge \langle y_n^*, v\rangle - 2c\|y_n\|.$$

Choose $u_n \in X$ with $\|u_n\| = 1$ and $\langle y_n^*, u_n\rangle \ge \|y_n^*\| - 1/n$ and put $h_n = \sqrt{\|y_n\|}u_n$. Then $\lim h_n = 0$ and

$$\limsup_{n\to\infty} \frac{\widetilde{g}(h_n) - \widetilde{g}(0)}{\|h_n\|} \ge \limsup_{n\to\infty} \frac{1}{\sqrt{\|y_n\|}}(\langle y_n^*, \sqrt{\|y_n\|}u_n\rangle - 2c\|y_n\|)$$
$$= \limsup_{n\to\infty} \langle y_n^*, u_n\rangle - 2c\sqrt{\|y_n\|} \ge a,$$

and $\widetilde{g}$ is not Fréchet differentiable at x.

If f is a function on a Banach space X we denote by D_f the set of points in X where f is Fréchet differentiable.

THEOREM 3.2. *Let X be an infinite-dimensional separable superreflexive Banach space. Then there exists an equivalent norm p on X such that the set of points where p is Fréchet differentiable is Aronszajn null.*

Proof. We can suppose that X has a basis. Indeed, since X is separable, it admits a quotient space Y with a basis; denote the quotient mapping by T. The Banach space Y is also superreflexive; if we assume the statement of the theorem being true for spaces with basis, there is an equivalent norm $\widetilde{p}$ on Y so that $D_{\widetilde{p}}$ is Aronszajn null. For $x \in X$ define $p(x) = \|x\| + \widetilde{p}(T(x))$. Then p is an equivalent norm on X and $D_p \subset T^{-1}(D_{\widetilde{p}})$. The set $T^{-1}(D_{\widetilde{p}})$ is Aronszajn null by Lemma 2.3.

Now suppose X has a basis. Since X is superreflexive, it admits a norm which is both power type rotund and uniformly smooth (see e.g. [BL]). Then the dual norm is uniformly convex. Let $N \in \mathbb{N}$ be as in Proposition 2.5. Fix some $\varepsilon > 0$ and choose v_n, $n \in \mathbb{N}$, and $S_\varepsilon = S$ as in Proposition 2.5. Define $g_n(x) = \max\{v_n(x), \|x\|\} \le 2\|x\|$. Then $g_n = v_n$ on $C_n = \{x \in X : v_n(x) > \|x\|\}$, hence $v_n \in \partial g_n(x)$ for $x \in C_n$. For $x \in X \setminus C_n$ we have $g_n(x) = \|x\|$ and for any $x^* \in \partial\|x\| \subset \partial g_n(x)$, $\|x^*\| \le 1 < 1 + \delta \le \|v_n\|$. Put $f_\varepsilon = \sup g_n$. Then f_ε is an equivalent norm. Since $S_\varepsilon = \bigcup_{n\in\mathbb{N}} C_n$ is dense in X, by Lemma 3.1 the function f_ε can be Fréchet differentiable only at the points of S_ε. Put $p = \sum_{n=1}^\infty 2^{-n} f_{1/n}$. Then p is an equivalent norm on X which can be Fréchet differentiable only at the points of $D = \bigcap_{n=1}^\infty S_{1/n}$. The set D is Aronszajn null by Lemma 2.2 and (B) of Proposition 2.5.

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