An almost nowhere Fréchet smooth norm on superreflexive spaces

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Abstract. Every separable infinite-dimensional superreflexive Banach space admits an equivalent norm which is Fréchet differentiable only on an Aronszajn null set.

Introduction. Every convex continuous function on a separable Banach space $X$ is Gateaux differentiable on a dense $G_δ$-set by a theorem of Mazur. If the dual of $X$ is separable it is even Fréchet differentiable on a dense $G_δ$-set.

If we confine ourselves to the weaker notion of Gateaux differentiability, then locally Lipschitz functions, and in particular convex continuous functions, are also differentiable on a set which is large in the sense of measure.

The strongest present result in this direction is due to Mankiewicz [Man] and Aronszajn [A]. They defined in every separable Banach space a family $\mathcal{A}$ of sets which mimics the family of Lebesgue null sets in finite dimensions. The definitions of the family $\mathcal{A}$ (now usually called the Aronszajn null sets, see Section 2) used by Mankiewicz and by Aronszajn are formally different; it was recently shown by Csoóry that they both coincide with the so-called Gaussian null sets [C]. Mankiewicz and Aronszajn proved that every locally Lipschitz function is Gateaux differentiable almost everywhere, that is, except on a set belonging to $\mathcal{A}$. For Fréchet differentiability this fails except for finite dimensions, where the classical theorem of Rademacher is available. If $X$ is a separable and infinite-dimensional Banach space then by a result of Preiss and Tiser [PT] there is a Lipschitz function $f$ on $X$ such that the set of points where $f$ is Fréchet differentiable is Aronszajn null.

In [MM] it was shown that it is of no help to consider only convex continuous functions. There exists an equivalent norm $p$ on the separable Hilbert space $\ell_2$ such that the set of points where $p$ is Fréchet differentiable

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is Aronszajn null. Here we prove the result of [MM] in any separable infinite-dimensional superreflexive Banach space. To construct a "bad" norm on a superreflexive Banach space we combine the simplified proof of [MM] given by Preiss with the existence of very "nice" renormings of superreflexive spaces.

In [PZ] Preiss and Zajíček ask if the set of points where a convex continuous function on a Banach space with a separable dual is not Fréchet differentiable can be covered by countably many closed convex sets with empty interior and countably many \( \delta \)-convex surfaces. This was disproved by Konyagin [K]. Notice that our example also provides a counterexample to this question. By [M1] and [M2] in superreflexive Banach spaces every closed convex set with empty interior is Haar null (this is another replacement of Lebesgue null sets in infinite dimensions) and \( \delta \)-convex surfaces are easily seen to be Haar null. A countable union of Haar null sets is Haar null, and the union of a Haar null set and of an Aronszajn null set cannot be the entire space.

If \( X \) is a Banach space, \( x \in X \) and \( r > 0 \), we denote by \( B(x, r) \) the closed ball with center \( x \) and radius \( r \); \( B_X \) is the closed unit ball of \( X \). The modulus of convexity of a Banach space \( X \) is denoted by \( \delta_X \). The Banach space \( \mathbb{K}^\mathbb{N} \) is considered with the \( \ell_2 \)-norm if not stated otherwise; we denote by \( \lambda_n \) the \( n \)-dimensional Lebesgue measure. If \( A \) is a subset of a Banach space \( X \), we define \( \text{cone} A = \bigcup_{t > 0} tA \).

2. Sets small in measure. The following notion of a null set was introduced by Aronszajn [A]; for equivalent definitions see [C].

**Definition 2.1.** Let \( X \) be a separable Banach space and let \( A \) be a Borel subset of \( X \). The set \( A \) is called **Aronszajn null** if for every sequence \( (x_i)_{i=1}^\infty \) in \( X \) whose closed linear span is \( X \) there exist Borel sets \( A_i \subset X \) such that \( A = \bigcup_{i=1}^\infty A_i \) and the intersection of \( A_i \) with any line in direction \( x_i \) has one-dimensional Lebesgue measure zero, for each \( i \in \mathbb{N} \).

Suppose \( n \in \mathbb{N} \) and \( A \) is a Borel subset of a Banach space \( X \) such that \( A \) is disconnected with any \( n \)-dimensional affine subspace of \( X \) is of \( n \)-dimensional Lebesgue measure zero. It is an easy consequence of Rubini's theorem that \( A \) is Aronszajn null. If \( A \) is a cone-like set, that is, \( cA \subset A \) for any \( c > 0 \), then in the above situation it is enough to consider all \( n \)-dimensional subspaces rather than all \( n \)-dimensional affine subspaces.

**Lemma 2.2.** Let \( X \) be a separable Banach space and \( A \) a Borel subset of \( X \) so that \( cA \subset A \) for any \( c > 0 \). Suppose \( n \in \mathbb{N} \) is such that \( A \) intersected with the unit ball of any \( n \)-dimensional subspace of \( X \) is of \( n \)-dimensional Lebesgue measure zero. Then \( A \) is Aronszajn null.

**Proof.** If \( n = 1 \) then \( A \subset \{0\} \). Suppose that \( n > 1 \) and that there is an \( (n-1) \)-dimensional affine subspace \( Y \subset X \) with \( 0 \not\in Y \) and with \( A \cap Y \) having positive \( (n-1) \)-dimensional Lebesgue measure. Since cone \( (A \cap Y) \subset A \cap \text{span}Y \), the \( n \)-dimensional Lebesgue measure of \( A \cap \text{span}Y \) is positive, which is a contradiction. By Fubini's theorem the \( n \)-dimensional Lebesgue measure of \( A \cap Z \) is zero for any \( n \)-dimensional affine \( Z \subset X \), and \( A \) is Aronszajn null.

We will need the following simple lemma to construct a "bad" norm on a general separable superreflexive Banach space from a bad norm on its quotient with a basis.

**Lemma 2.3.** Let \( X, Y \) be separable infinite-dimensional Banach spaces, and \( T : X \to Y \) a continuous linear surjective mapping. Let \( A \subset Y \) be Aronszajn null. Then \( T^{-1}(A) \) is Aronszajn null.

**Proof.** Suppose \( E \subset X \) is a countable set with \( \text{span} E = \{0\} \). Let \( (y_n)_{n=1}^\infty \) be an enumeration of \( T(E) \setminus \{0\} \). Then \( \text{span} (y_n)_{n=1}^\infty = Y \). Let \( A_n \subset Y \) be Borel so that \( A_n \cap (y + \text{span} y_n) \) has one-dimensional measure zero for each \( y \in Y \) and \( \bigcup_{n} A_n = A \). Since \( T \) is continuous, each set \( T^{-1}(A_n) \) is Borel. Clearly, \( \bigcup_n T^{-1}(A_n) = T^{-1}(A) \). If \( n \in \mathbb{N} \) is arbitrary, \( x \in T^{-1}(y_n) \) and \( x \in X \) then \( T^{-1}(A_n) \) restricted to \( x + \text{span} x \) is an affine homeomorphism. Since \( T(T^{-1}(A_n) \cap (x + \text{span} x)) = A_n \cap (T(x) + \text{span} y_n) \), the set \( T^{-1}(A_n) \cap (x + \text{span} x) \) has one-dimensional measure zero.

Let \( X \) be a Banach space and \( n \in \mathbb{N} \). For each \( n \)-dimensional subspace \( Z \) of \( X \) fix an isomorphism \( T_Z : Z \to \mathbb{K}^n \) with \( \|T_Z\| = 1 \) and \( \|T_Z^{-1}\| \leq n \). In the sequel, \( \lambda_Z(A \cap Z) < \varepsilon \) for any \( n \)-dimensional subspace \( Z \subset X \) means that the \( n \)-dimensional measures \( \lambda_Z \) come from these isomorphisms.

**Lemma 2.4.** Let \( X \) be a Banach space with a uniformly convex norm so that \( \delta_X(\varepsilon) > c\varepsilon^p \) for some \( p \geq 2 \) and \( c > 0 \). Suppose \( Z \) is an \( n \)-dimensional subspace of \( X \) and \( v \in X^* \) is such that \( \sup_{x \in B_Z} \langle v, x \rangle \leq 1 + \varepsilon \) for some \( \varepsilon > 0 \). Then \( \lambda_Z(C) \leq \beta g((n-1)/\alpha) \) for \( C = B_Z \cap \{x \in X \mid \langle v, x \rangle > \|x\|\} \), where \( \beta \) is an absolute constant.

**Proof.** Denote by \( \|v\|_Z \) the norm of \( v \) when restricted to \( Z \) and by \( u \) the point of \( B_Z \) where it is attained. Observe that \( C = B_Z \cap \text{cone}(x \in B_Z \mid \langle v, x \rangle > 1) \). Since \( 1/\|v\|_Z \geq 1/(1 + \varepsilon) \geq 1 - \varepsilon \), we have

\[
\{x \in B_Z : \langle v, x \rangle \geq 1\} \subset \{x \in B_Z : \langle v, x \rangle > 1 - \varepsilon\}.
\]

Since \( Z \) is uniformly rotund with \( \delta_Z(\varepsilon) > c\varepsilon^p \), the diameter of the latter set is at most \( \alpha g(1/p) \), where \( \alpha > 0 \) is a constant (see e.g. [D], p. 58). Hence \( C \subset \text{conv}(B(u, \alpha g(1/p)) \cup B(0, \alpha g(1/p))) \) and
There exist \( u_1, \ldots, u_N \in B_Z \) so that \( B_Z \subset \{ \sum_{i=1}^N a_i u_i : |a_i| \leq N \} \). For \( j = 1, 2, \ldots \) define
\[
I_j = \{ k \in \mathbb{N} : 1 + 2^{-j} \leq \max_{x \in B_Z} \langle u_k, x \rangle \leq 1 + 2^{-j+1} \},
\]
and \( S_j = \bigcup_{n \in I_j} \{ x \in X : u_n(x) > \| x \| \} \). If \( u_n(y) > \| y \| \) for some \( y \in B_Z \) and \( n \in \mathbb{N} \), then \( u_n(y/\| y \|) > 1 \) and \( \max_{x \in B_Z} \langle u_k, x \rangle > 1 \). Therefore \( B_Z \cap S = \bigcup_{j=1}^\infty (B_Z \cap S_j) \). By Lemma 2.4,
\[
\lambda_2(S \cap B_Z) \leq \sum_{j=1}^\infty 2(2^{1-j}(N-1)/r)^{1/q} |I_j|.
\]

To estimate \( |I_j| \), suppose that \( k \in I_j \) and \( x \in B_Z \) are such that \( \langle u_k, x \rangle \geq 1 + 2^{-j} \). Since \( x = \sum_{i=1}^N a_i u_i \) for some suitable \( |a_i| \leq N \) we can estimate
\[
1 + 2^{-j} \leq \langle x, u_k \rangle = 1 + r \sum_{i=1}^N a_i \langle f_{n_i}, u_i \rangle \\
\leq 1 + r N^2 \max_i \| f_{n_i} \|.
\]
This means that if \( k \in I_j \) there is some \( i = 1, \ldots, N \) for which
\[
\langle f_{n_i}, u_i \rangle \geq \frac{1}{2r N^2}.
\]
This cannot happen for too many \( k \)'s, since
\[
1 \geq \| u_k \| \geq \gamma \left( \sum_{n=1}^\infty \| f_{n_i} \| \right)^{1/q};
\]
each \( u_i \) can have at most \( (2 r N^2)^q \) coordinates that are not smaller than \( 1/(2 r N^2) \). Therefore \( |I_j| \leq \alpha_1 2r N^2 2^{j-1} \) for a suitable constant \( \alpha_1 > 0 \). Finally, by substituting into (1), we arrive at
\[
\lambda_2(S \cap B_Z) \leq \alpha_2 N^2 2^{j-1} \sum_{j=1}^\infty 2^{(q - (N-1)/r)}.
\]

If \( N \) is such that \( q - (N-1)/r < 0 \), then this is at most \( \varepsilon \) for \( r \) small enough. To obtain, for a given \( \varepsilon > 0 \), a symmetric set \( C = (u_k) \) it is enough to put \( C = C' \cup (-C') \), where \( C' \) is a set which works for \( \varepsilon /2 \).

3. Convex functions. Suppose that \( g_n \) are convex continuous functions, and the subdifferential of each of them has a large jump somewhere close to a given point \( x \) for which \( g_1(x) = g_2(x) = \ldots \). Then the pointwise supremum of \( (g_n) \) (if it exists) is not Fréchet differentiable at \( x \).

**Lemma 3.1.** Let \( X \) be a Banach space, \( x \in X \), \( x^* \in X^* \). Let \( g, g_1, g_2, \ldots \) be convex continuous functions on \( X \) with \( (g(x) = g_1(x) = g_2(x) = \ldots \),
can be Fréchet differentiable only at the points of \( D = \bigcap_{n=1}^{\infty} S_{1/n} \). The set
\( D \) is Aronszajn null by Lemma 2.2 and (B) of Proposition 2.5.

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References


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98 E. Matoušková

\( g_n \leq g \), and \( x^* \in \partial g_n(x) \) for all \( n \in \mathbb{N} \). Suppose there are \( a > 0 \), \( y_n \in X \) and \( u_n = \partial g_n(y_n) \), with \( \lim y_n = x \) and \( \| x^* - y_n^* \| > a \). Then the convex continuous function \( \tilde{g} = \sup g_n \) is not Fréchet differentiable at \( x \).

Proof. We can suppose that \( x = 0 \), \( x^* = 0 \), and \( g(x) = 0 \). Since the functions \( g_n \) are uniformly bounded on some neighborhood of \( x \), they are all Lipschitz with the same constant \( c > 0 \) on some neighborhood of \( x \). Suppose that all \( y_n \) are contained in this neighborhood. Then for \( v \in X \) we can estimate

\[
g_n(v) \geq g_n(y_n) + (y_n^*, v - y_n) \geq (y_n^*, v) - 2c\| y_n \|.
\]

Choose \( u_n \in X \) with \( \| u_n \| = 1 \) and \( (y_n^*, u_n) \geq \| y_n^* \| - 1/n \) and put \( h_n = \sqrt{\| y_n \|} \). Then \( \lim h_n = 0 \) and

\[
\limsup_{n \to \infty} \frac{\tilde{g}(h_n) - \tilde{g}(0)}{h_n} \leq \limsup_{n \to \infty} \frac{1}{\sqrt{\| y_n \|}} ((y_n^*, \sqrt{\| y_n \|} u_n) - 2c\| y_n \|) = \limsup_{n \to \infty} (y_n^*, u_n) - 2c\| y_n \| \geq a,
\]

and \( \tilde{g} \) is not Fréchet differentiable at \( x \).

If \( f \) is a function on a Banach space \( X \) we denote by \( D_f \) the set of points in \( X \) where \( f \) is Fréchet differentiable.

Theorem 3.2. Let \( X \) be an infinite-dimensional separable superreflexive Banach space. Then there exists an equivalent norm \( p \) on \( X \) such that the set of points where \( p \) is Fréchet differentiable is Aronszajn null.

Proof. We can suppose that \( X \) is separable, that is, it admits a quotient space \( Y \) with a basis, denote the quotient mapping by \( T \). The Banach space \( Y \) is also superreflexive; if we assume the statement of the theorem being true for spaces with basis, there is an equivalent norm \( \tilde{p} \) on \( Y \) so that \( D_{\tilde{p}} \) is Aronszajn null. For \( x \in X \) define \( p(x) = \| x \| + \tilde{p}(T(x)) \). Then \( p \) is an equivalent norm on \( X \) and \( D_p \subset T^{-1}(D_{\tilde{p}}) \). The set \( T^{-1}(D_{\tilde{p}}) \) is Aronszajn null by Lemma 2.3.

Now suppose \( X \) has a basis. Since \( X \) is superreflexive, it admits a norm which is both power type rotund and uniformly smooth (see e.g. [BL]). Then the dual norm is uniformly convex. Let \( N \in \mathbb{N} \) be as in Proposition 2.5. Fix some \( \varepsilon > 0 \) and choose \( v_n, n \in \mathbb{N} \), and \( S_\varepsilon = S \) as in Proposition 2.5. Define \( g_n(x) = \max\{v_n(x), \| x \|\} \leq 2\| x \| \). Then \( g_n = v_n \) on \( C_n = \{ x \in X : v_n(x) > \| x \| \} \), hence \( v_n \in \partial g_n(x) \) for \( x \in C_n \). For \( x \in X \setminus C_n \) we have \( g_n(x) = \| x \| \) and for any \( x^* \in \partial \| x \| \subset \partial g_n(x) \), \( \| x^* \| \leq 1 < 1 + \delta \leq \| v_n \| \). Put \( f_\varepsilon = \sup g_n \). Then \( f_\varepsilon \) is an equivalent norm. Since \( S_\varepsilon = \bigcup_{n \in \mathbb{N}} C_n \) is dense in \( X \), by Lemma 3.1 the function \( f_\varepsilon \) can be Fréchet differentiable only at the points of \( S_\varepsilon \). Put \( p = \sum_{n=1}^{\infty} 2^{-n} f_1/n \). Then \( p \) is an equivalent norm on \( X \) which