STUDIA MATHEMATICA 133 (1) (1999)

Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains

by

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Abstract. The purpose of the paper is to extend results of the potential theory of the classical Schrödinger operator to the α-stable case. To obtain this we analyze a weak version of the Schrödinger operator based on the fractional Laplacian and we prove the Conditional Gauge Theorem.

1. Introduction. The purpose of the paper is to develop the potential theory of the Schrödinger operator S^α based on the fractional Laplacian $\Delta^{\alpha/2}$.

The operator S^α was investigated (along with the so-called relativistic Schrödinger operator) in connection with the problem of relativistic stability of matter (see e.g. [Li], [W], [Fe] and [CMS] and the references given there). At that time the potential theory for the operator $\Delta^{\alpha/2}$ on bounded domains was not fully developed; in particular, precise estimates for the Green function and the Boundary Harnack Principle were not available even for smooth domains. In this paper we treat advanced aspects of the potential theory of S^α taking into account recent progress in this field.

The paper is organized as follows. In Section 2 we collect basic facts concerning symmetric α-stable Lévy processes, α-harmonic functions, Green function, Kato class $\mathcal{J}^\alpha$ and conditional α-stable Lévy motion. These facts are either standard or their proofs are direct adaptations of well-known results on Brownian motion. A good source of classical results is [ChZ]; for results on symmetric α-stable processes, see [BG], [B1] and [K]; for information on the Riesz kernels and fractional Laplacian we refer to [La] and [S]. We assume that reader is familiar with the foundations of Markov processes in the setting e.g. of [BG], Ch. I. In Section 3 we define the weak

1991 *Mathematics Subject Classification*: Primary 31B25, 60J50.

Key words and phrases: α-stable Lévy processes, α-stable Feynman–Kac semigroup, weak fractional Laplacian, α-stable Schrödinger operator, potential theory, q-harmonic functions, conditional gauge theorem.

fractional Laplacian $\widetilde{\Delta}^{\alpha/2}$ and prove an analogue of Weyl's lemma. Then we introduce the weak Schrödinger operator S^α based on $\widetilde{\Delta}^{\alpha/2}$ and reveal its basic properties.

Section 4 contains the proof of the Conditional Gauge Theorem (CGT), which is the main result of the paper. Although in the proof of CGT we generally follow the approach of [CFZ], there are substantial difficulties to overcome, due to discontinuities of the paths of our process.

In Section 5 we define q-harmonic functions u on D and show that they coincide with solutions of the equation $S^\alpha u = 0$ in D. We point out that most of important results for the classical Schrödinger operator and q-harmonic functions [ChZ] carry over to the present context but we give alternative methods of proof.

Section 6 contains results concerning potential theory for the operator S^α such as the Harnack Inequality, Boundary Harnack Principle and Martin representation for nonnegative q-harmonic functions. The main technical tool exploited throughout the paper is the Boundary Harnack Principle for $\Delta^{\alpha/2}$, proved in [B1] and [BB], and its consequences such as the Martin representation [B2] and 3G Theorem (described below). We rely on these results; otherwise the paper is essentially self-contained.

After the first version of the manuscript had been submitted to the Editors, the authors learned about the recent paper [CS]. Some results of [CS] are related to ours; however, they are obtained by completely different methods and restricted to $C^{1,1}$ domains.

2. Preliminaries

2.1. *Notation and terminology.* We basically adopt the notation of [B1]. Let us only remark that all functions considered in this paper are defined on the whole of $\mathbb{R}^d$, as a consequence of nonlocality of the theory of α-harmonic functions. We always require Borel measurability on $\mathbb{R}^d$. Thus, for a subset $D \subseteq \mathbb{R}^d$, we denote by $L^\infty(D)$ the class of all Borel measurable functions on $\mathbb{R}^d$ that are bounded on D. The similar convention applies to the definition of $L^p(D)$, for $1 \le p < \infty$. Analogously $C(D)$ denotes the class of Borel functions on $\mathbb{R}^d$ that are continuous on an open subset $D \subseteq \mathbb{R}^d$. $C_0(D)$ is a subclass of $C(D)$ consisting of the functions that are continuous everywhere and vanish on D^c. $C_c(D)$ is the class of functions with compact support contained in D. We denote by $C^k(D)$ the class of functions that are k times continuously differentiable on D, and $C_c^k(D) = C^k(D) \cap C_c(D)$. For $\varphi \in C^k(\mathbb{R}^d)$ we put $\|\varphi\|_k = \sup_{|\gamma| \le k} \sup_{x \in \mathbb{R}^d} |D^\gamma \varphi(x)|$.

We will also consider less standard classes of functions; suitable definitions are postponed to subsequent sections.

Many results of the paper are restricted to bounded *Lipschitz* domains D; we refer the reader to [B1] for their definition and basic properties.

We adopt the convention that constants may change their value but not the dependence from one use to the next. The notation $C(a, b, \ldots)$ means that C is a constant depending *only* on $a, b, \ldots$ Unless specified otherwise all constants depend on D, α and q (a function in the Kato class $\mathcal{J}^\alpha$). Constants are always positive and finite.

2.2. *Symmetric α-stable processes and α-harmonic functions.* For the rest of the paper, let $\alpha \in (0, 2)$ and $d > \alpha$, e.g. $d \ge 2$. We denote by (X_t, P^x) the standard rotation invariant ("symmetric") α-stable Lévy process in $\mathbb{R}^d$ (i.e. homogeneous, with independent increments), with index of stability α, and with characteristic function of the form $E^0 e^{iuX_t} = e^{-t|u|^\alpha}$, $u \in \mathbb{R}^d$, $t \ge 0$. As usual, E^x denotes the expectation with respect to the distribution P^x of the process starting from $x \in \mathbb{R}^d$. We always assume that sample paths of X_t are right-continuous and have left-hand limits a.s. The process (X_t) is Markov with transition probabilities given by $P_t(x, A) = P^x(X_t \in A) = \mu_t(A - x)$, where μ_t is the one-dimensional distribution of X_t with respect to P^0. We have $P_t(x, A) = \int_A p(t; x, y)\, dy$, where $p(t; x, y) = p_t(x - y)$ are the *transition densities* of X_t. The function $p_t(x) = p_t(-x)$ is continuous in (t, x), $t > 0$, and has the following useful *scaling property*: $p_t(x) = t^{-d/\alpha} p_1(x/t^{1/\alpha})$. The process (X_t, P^x) is strong Markov with respect to the so-called "standard filtration" $\{\mathcal{F}_t : t \ge 0\}$, and quasi left-continuous on $[0, \infty]$. The *shift operator* is denoted by θ_t. The operator θ_t is also extended to Markov times τ and is then denoted by θ_τ. The process X_t has the potential kernel

$$K_\alpha(x) = \mathcal{A}(d, \alpha)|x|^{\alpha - d},$$

where $\mathcal{A}(d, \gamma) = \Gamma((d - \gamma)/2)/(2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|)$ (cf. [La]; see also [S]).

For $A \subset \mathbb{R}^d$, we put $\tau_A = \inf\{t \ge 0 : X_t \notin A\}$, the *first exit time* from A. $x \in \mathbb{R}^d$ is called *regular* for a (Borel) set A if $P^x\{\tau_{\overline{A}} = 0\} = 1$; A itself is called *regular* if all $x \in A^c$ are regular for A.

Let u be a Borel measurable function on $\mathbb{R}^d$. We say that u is *α-harmonic* in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = E^x u(X_{\tau_U}), \quad x \in U, \tag{1}$$

for every bounded open set U with closure $\overline{U}$ contained in D. It is called *regular α-harmonic* in D if (1) holds for $U = D$. If D is unbounded then by the usual convention, $E^x u(X_{\tau_D}) = E^x[\tau_D < \infty; u(X_{\tau_D})]$. Under (1) it is always assumed that the expectation in (1) is absolutely convergent.

By the strong Markov property of X_t, a regular α-harmonic function is necessarily α-harmonic. The converse is not generally true [B2].

When $B = B(0, r) \subset \mathbb{R}^d$, $r > 0$, and $|x| < r$, the P^x distribution of X_{τ_B} has the density function $P_r(x, \cdot)$ (the *Poisson kernel*), explicitly given by the formula

$$P_r(x,y) = C_\alpha^d\left[\frac{r^2-|x|^2}{|y|^2-r^2}\right]^{\alpha/2}|x-y|^{-d} \quad \text{for } |y|>r, \tag{2}$$

with $C_\alpha^d = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$, and equal to 0 elsewhere [BGR].

2.3. *Killed symmetric α-stable Lévy motion.* Let D be a bounded domain. We often assume that D is regular. We denote by (P_t^D) the semigroup generated by the process (X_t) killed on exiting D. The semigroup (P_t^D) is determined by transition densities $p_t^D(x,y)$ which are symmetric, that is, $p_t^D(x,y)=p_t^D(y,x)$, and continuous in (t,x,y) for $t>0$ and $x,y\in D$. Thus, for any nonnegative Borel f on $\mathbb{R}^d$ we have $P_t^D f(x) = E^x[t<\tau_D;\ f(X_t)] = \int_D f(y)p_t^D(x,y)\,dy$.

We call $L^p(D)$ $(1\le p<\infty)$ or, for regular D, $C_0(D)$, an *appropriate space* for the semigroup $(P_t^D)_{t>0}$. The semigroup acts on each appropriate space as a strongly continuous semigroup of contractions.

The *Green operator* for D is denoted by G_D. We set

$$G_D(x,y) = \int_0^\infty p_t^D(x,y)\,dt$$

and call $G_D(x,y)$ the *Green function* for D. We obtain

$$G_D f(x) = E^x\Big[\int_0^{\tau_D} f(X_t)\,dt\Big] = \int_D G_D(x,y)\,f(y)\,dy$$

for nonnegative Borel functions f on $\mathbb{R}^d$. When D is fixed we write $G(x,y)$ instead of $G_D(x,y)$. If D is regular then $G_D(x,y)$ has the following properties: $G_D(x,y)=G_D(y,x)$; $G_D(x,y)$ is positive for $x,y\in D$ and continuous at $x,y\in\mathbb{R}^d$, $x\ne y$; $G_D(x,y)=0$ if x or y belongs to D^c. For $x,y\in\mathbb{R}^d$ we have (unless $x=y\in D^c$)

$$G_D(x,y) = K_\alpha(x,y) - E^x K_\alpha(X_{\tau_D},y), \tag{3}$$

where $K_\alpha(x,y)=K_\alpha(x-y)$.

Let us recall that the domain $D\subseteq\mathbb{R}^d$ is called *Green-bounded* if $\sup_{x\in\mathbb{R}^d}E^x\tau_D<\infty$. Since $E^x\tau_D = G_D\mathbf{1}(x)$, the above property is equivalent to the condition $\|G_D\mathbf{1}\|_\infty<\infty$. But $0\le G_D(x,y)\le K_\alpha(x,y)$, so we obtain $\sup_{x\in D}G_D\mathbf{1}(x)\le\sup_{x\in D}\int_D K_\alpha(x,y)\,dy<\infty$ for a bounded Borel set D. Since for $x\in D^c$ we have $E^x\tau_D=0$, bounded Borel sets are Green-bounded. That sets D of finite Lebesgue measure are also Green-bounded follows from a direct modification of the proof of Theorem 1.17 from [ChZ]: we then have $\sup_{x\in\mathbb{R}^d}E^x\tau_D\le Cm(D)^{\alpha/d}$.

2.4. *Kato class $\mathcal{J}^\alpha$.* We say that a Borel function q belongs to the *Kato class* $\mathcal{J}^\alpha$ if q satisfies either of the two equivalent conditions (see [Z]):

$$\lim_{r\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le r}|q(y)|K_\alpha(x-y)\,dy=0, \tag{4}$$

$$\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}\int_0^t P_s|q|(x)\,ds=0. \tag{5}$$

We write $q\in\mathcal{J}^\alpha_{\mathrm{loc}}$ if $\mathbf{1}_Bq\in\mathcal{J}^\alpha$ for every bounded Borel subset $B\subseteq\mathbb{R}^d$. If $q\in\mathcal{J}^\alpha$ then $\sup_{x\in\mathbb{R}^d}\int_{|x-y|<1}|q(y)|\,dy<\infty$; in particular, $\mathcal{J}^\alpha_{\mathrm{loc}}\subseteq L^1_{\mathrm{loc}}$. If $f\in L^\infty(\mathbb{R}^d)$ then $f, fq\in\mathcal{J}^\alpha$.

Let D be a Green-bounded domain in $\mathbb{R}^d$ and $q\in\mathcal{J}^\alpha$. For any $b>0$ there exists $a>0$ depending only on q and b such that

$$G_D|q|\le aG_D\mathbf{1}+b.$$

Consequently, for a fixed $q\in\mathcal{J}^\alpha$, we have $\|G_Dq\|_\infty\to 0$ if $\|G_D\mathbf{1}\|_\infty\to 0$. Also $G_Dq\in L^\infty(\mathbb{R}^d)\cap C(D)$, and $\lim_{x\to z}G_Dq(x)=0$ if z is regular for D. Hence $G_Dq\in C_0(D)$ provided D is regular.

2.5. *Feynman–Kac semigroups.* For $q\in\mathcal{J}^\alpha$ we define an additive functional $A(t)=\int_0^t q(X_s)\,ds$, $t\ge 0$. The corresponding multiplicative functional $e_q(t)$ is defined by $e_q(t)=\exp(A(t))$, $t\ge 0$. For all $s,t\ge 0$ we have $e_q(s+t)=e_q(s)\{e_q(t)\circ\theta_s\}$. If now τ is a Markov time such that for every $t\ge 0$ we have $\tau\le t+\tau\circ\theta_t$ on $\{t<\tau\}$ then for $q\ge 0$ we obtain the following important fact, referred to in the sequel as *Khasminski's lemma:*

$$\text{If}\quad \sup_{x\in\mathbb{R}^d}E^xA(\tau)=\varepsilon<1\quad\text{then}\quad\sup_{x\in\mathbb{R}^d}E^xe_q(\tau)<(1-\varepsilon)^{-1}. \tag{6}$$

We also have $\lim_{t\downarrow 0}\sup_{x\in\mathbb{R}^d}E^xe_{|q|}(t)=1$ and $\sup_{x\in\mathbb{R}^d}E^xe_{|q|}(t)\le e^{C_0+C_1t}$ for some $C_0,C_1>0$ and all $t>0$.

Now, we denote by $(T_t)=(T_t^D)$ the Feynman–Kac semigroup killed on exiting D. Thus, for nonnegative Borel f we have

$$T_tf(x)=E^x[t<\tau_D;\ e_q(t)f(X_t)].$$

(T_t) is a strongly continuous semigroup of bounded operators on each appropriate space for the semigroup (P_t^D), and $\|T_t\|_p\le\|T_t\|_\infty\le e^{C_0+C_1t}$ for each $1\le p\le\infty$. Every T_t is also a bounded operator from L^p into L^∞, and is determined by a symmetric transition density function u_t which is in $C_0(D\times D)$ for regular D. For each $f\in L^p$ $(1\le p\le\infty)$ we thus have

$$T_tf(x)=\int_D u_t(x,y)f(y)\,dy.$$

Moreover, if D is regular, then T_t maps L^1 into $C_0(D)$ for $t>0$.

The potential operator V for (T_t) is introduced as follows:

$$Vf(x) = \int_0^\infty T_t f(x)\,dt = E^x\Big[\int_0^{\tau_D} e_q(t) f(X_t)\,dt\Big],$$

where f is nonnegative and Borel measurable on D. We call V the *q-Green operator*. If $\int_0^\infty \|T_t\|_\infty\,dt < \infty$, with the operator norm taken in $L^\infty(D)$, then V is bounded on L^p, $1 \le p \le \infty$. In particular, $V\mathbf{1} \in L^\infty(D)$ and for regular D the operator V has a symmetric kernel $V(x,y)$ called the *q-Green function* which is given by the formula

$$V(x,y) = \int_0^\infty u_t(x,y)\,dt. \tag{7}$$

Thus, $Vf(x) = \int_D V(x,y) f(y)\,dy$.

We let $\mathbf{F}(D,q)$ denote the space of all Borel functions f such that for some constants C_1, C_2 we have $|f(x)| \le C_1 + C_2|q(x)|$ for all $x \in D$. If $f \in \mathbf{F}(D,q)$ then $Vf \in C_0(D)$ whenever $V\mathbf{1} \in L^\infty(D)$ and D is regular.

The following identities exemplify relations between the operators V and G_D. We assume here that $q \in \mathcal{J}^\alpha$ and $V\mathbf{1} \in L^\infty(D)$. For $x \in D$ we obtain

$$Vf(x) = G_D f(x) + V(qG_D f)(x) \tag{8}$$

whenever $V(|q|G_D|f|)(x) < \infty$. If $G_D(|q|V|f|)(x) < \infty$ then

$$Vf(x) = G_D f(x) + G_D(qVf)(x). \tag{9}$$

2.6. *Stopped Feynman–Kac functional. Gauge Theorem.* Let D be a domain in $\mathbb{R}^d$ and let $q \in \mathcal{J}^\alpha$. We will usually assume that D is bounded or of finite Lebesgue measure. Then $\tau_D < \infty$ a.s. by 2.3. Since also $\int_0^t |q(X_s)|\,ds < \infty$ a.s. for each $t > 0$, the random variable $e_q(\tau_D)$ is well defined. The function

$$u(x) = E^x e_q(\tau_D)$$

is called the *gauge* (function) for (D,q); when it is bounded in D, hence in $\mathbb{R}^d$, we say that (D,q) is *gaugeable*. For a fixed $q \in \mathcal{J}^\alpha$ but a variable domain D we use the alternative notation u_D for the gauge for (D,q). If $G_D q$ is bounded below then by Jensen's inequality we obtain

$$\inf_{x\in\mathbb{R}^d} u_D(x) > 0. \tag{10}$$

This is the case when D is Green-bounded and $q \in \mathcal{J}^\alpha$.

If (10) holds and (D,q) is gaugeable, then (E,q) is gaugeable for any domain $E \subseteq D$. In fact $\|u_E\|_\infty \le \|u_D\|_\infty \|u_D^{-1}\|_\infty$.

The theorem below provides the fundamental property of the gauge and clarifies gaugeability conditions. For the proof, we refer to [ChZ] (§5.6 and Theorem 4.19).

GAUGE THEOREM. *Let D be a domain with $m(D) < \infty$ and let $q \in \mathcal{J}^\alpha$. If $u(x_0) < \infty$ for some $x_0 \in D$ then u is bounded in $\mathbb{R}^d$. Moreover, the following conditions are equivalent*:

(i) *(D,q) is gaugeable*;
(ii) *The semigroup T_t satisfies $\int_0^\infty \|T_t\|_\infty\,dt < \infty$*;
(iii) *$V\mathbf{1} \in L^\infty(\mathbb{R}^d)$*;
(iv) *$V|q| \in L^\infty(\mathbb{R}^d)$*.

In the sequel, for brevity, we often write $V\mathbf{1} \in L^\infty(\mathbb{R}^d)$ to indicate that (D,q) is gaugeable.

2.7. *Conditional α-stable Lévy motion.* We now introduce the notion of conditional α-stable Lévy motion. Although it is analogous to the conditional Brownian motion, some of its basic features are different, due to existence of jumps of the underlying process.

Let D be a domain in $\mathbb{R}^d$, $d > \alpha$, and let $h \ge 0$ be a function which is α-harmonic and positive on D. We define

$$p_h^D(t;x,y) = h(x)^{-1} p_t^D(x,y) h(y), \quad t > 0,\ x,y \in D,$$

where $p_t^D(x,y)$ is the transition density of P_t^D. It is not difficult to check that p_h^D is the density for a transition sub-probability. According to the general theory of Markov processes [BG], p_h^D defines a Markov process on the state space $D_\partial = D \cup \{\partial\}$, where ∂ is the extra point ("cemetery") needed to define the transition probabilities. This process is called *h-conditioned symmetric α-stable Lévy motion* or briefly: *α-stable h-Lévy motion*. Its *lifetime* is defined as $T_\partial = \tau_D$. The process remains at ∂ in $[\tau_D,\infty)$. The h-conditioned process is still denoted by X_t, but we use P_h^x and E_h^x for the corresponding probabilities and expectations. Thus, for any bounded Borel measurable function f on D we obtain

$$\begin{aligned} E_h^x f(X_t) &= h(x)^{-1} \int_D p^D(t;x,y) h(y) f(y)\,dy \\ &= h(x)^{-1} E^x[t < \tau_D;\ f(X_t)h(X_t)]. \end{aligned}$$

We list some elementary properties of α-stable h-Lévy motion. Let $t > 0$ and let $\Phi \ge 0$ be a function measurable with respect to $\mathcal{F}_t$. For $x \in D$ we have

$$E_h^x[t < \tau_D;\ \Phi] = h(x)^{-1} E^x[t < \tau_D;\ \Phi h(X_t)]. \tag{11}$$

We recall that for a given Markov time τ we denote by $\mathcal{F}_\tau$ the family of all sets $A \in \mathcal{F}$ with $A \cap \{\tau \le t\} \in \mathcal{F}_t$ for every $t > 0$. In our further considerations we will also use the σ-field $\mathcal{F}_{\tau-}$ which is generated by $\mathcal{F}_{0+}$ and the sets $A_t \cap \{\tau > t\}$, where $A_t \in \mathcal{F}_t$. We have $\{T < \tau\} \in \mathcal{F}_{\tau-}$ for any Markov time T.

If T is a Markov time and $\Phi \geq 0$ is a function measurable with respect to $\mathcal{F}_T$ we obtain

$$E_h^x[T < \tau_D; \Phi] = h(x)^{-1} E^x[T < \tau_D; \Phi h(X_T)]. \tag{12}$$

The next relation exhibits the strong Markov property of α-stable h-Lévy motion. Namely, for any Markov time T, $A \in \mathcal{F}_T$ and any function $\Phi \geq 0$ measurable with respect $\mathcal{F}_{\tau_D -}$ we have

$$E_h^x[A \cap \{T < \tau_D\}; \Phi \circ \theta_T] = E_h^x[A \cap \{T < \tau_D\}; E_h^{X_T} \Phi]. \tag{13}$$

Let D be a bounded Lipschitz domain in $\mathbb{R}^d$. For fixed $x_0 \in D$, we denote by K the corresponding *Martin kernel*

$$K(x,\xi) = \lim_{D \ni y \to \xi} \frac{G(x,y)}{G(x_0,y)}, \quad x \in \mathbb{R}^d,\ \xi \in \partial D,$$

where $G = G_D$. Then K is a unique function satisfying the following conditions:

(i) for any fixed $\xi \in \partial D$, $K(\cdot,\xi)$ is a strictly positive α-harmonic function in D, vanishing on D^c;

(ii) for any fixed $x \in D$, $K(x,\cdot)$ is a continuous function on ∂D;

(iii) for any $\xi \in \partial D$, $\eta \in \partial D$, $\eta \neq \xi$ we have $\lim_{x \to \eta} K(x,\xi) = 0$;

(iv) $K(x_0,\xi) = 1$ for all $\xi \in \partial D$.

The properties of K are nontrivial consequences of BHP (see [B2]). The corresponding results for more general domains are given in [MS].

For $\xi \in \partial D$ the α-stable h-Lévy motion determined by $h(\cdot) = K(\cdot,\xi)$ is called *α-stable ξ-Lévy motion*. The associated probability and expectation will be denoted by P_ξ^x and E_ξ^x. We have $E_\xi^x \tau_D < \infty$ and $P_\xi^x\{\lim_{t \uparrow\uparrow \tau_D} X_t = \xi\} = 1$ for every $\xi \in \partial D$ and $x \in D$.

In the classical case of Brownian motion the ξ-Brownian motion plays a dominant role, because of the fact that on exiting D the process hits the boundary. For our α-stable process, the situation is dramatically different: on leaving D the process jumps into the interior of D^c, with probability one, whenever D is a bounded domain with the *exterior cone property* (see Lemma 6 in [B1]). In consequence, another version of conditional process becomes even more important. This is the α-stable h-Lévy motion conditioned by $h(\cdot) = G(\cdot,y)$ with $y \in D$ and called *α-stable y-Lévy motion*. The y-process is defined on $D \setminus \{y\}$ and turns out to be the main tool in the potential theory of the α-stable Schrödinger operator. Let ζ denote $\tau_{D\setminus\{y\}}$, where D is (more generally) a bounded regular domain in $\mathbb{R}^d$. For every $y \in D$ and $x \in D \setminus \{y\}$ we have $E_y^x \zeta < \infty$ and $P_y^x\{\lim_{t \uparrow\uparrow \zeta} X_t = y\} = 1$.

By the above statements we can redefine α-stable ξ-Lévy and y-Lévy motions as follows. α-stable ξ-Lévy motion is defined on the state space $D \cup \{\xi\}$, and takes the value ξ at and after its lifetime τ_D. Similarly, α-stable y-Lévy motion is defined on the state space D and takes the value y at and after its lifetime ζ.

The following result is very important in the sequel (see [IW] and [B1], Lemmas 6 and 17 for justification). Let D be a bounded domain with the exterior cone property. Then the distribution of the pair $(X_{\tau_D -}, X_{\tau_D})$ with respect to P^x $(x \in D)$ is concentrated on $D \times D^c$ with the density function $g^x(v,y)$ given by the following explicit formula:

$$g^x(v,y) = \frac{\mathcal{A}(d,-\alpha)}{|v-y|^{d+\alpha}} G(x,v), \quad (v,y) \in D \times D^c. \tag{14}$$

Integrating (14) over D we obtain

$$g^x(y) = \int_D \frac{\mathcal{A}(d,-\alpha) G(x,v)}{|v-y|^{d+\alpha}}\, dv, \quad y \in D^c, \tag{15}$$

which gives the density function of the *α-harmonic measure* $\omega_D^x(dy) = P^x\{X_{\tau_D} \in dy\}$ of the set D.

By (14) and routine arguments, for $\Phi \geq 0$ measurable with respect to $\mathcal{F}_{\tau_D -}$, and any Borel $f \geq 0$, we obtain the following important formula:

$$E^x[f(X_{\tau_D})\Phi] = E^x[f(X_{\tau_D}) E^x_{X_{\tau_D -}}[\Phi]], \quad x \in D. \tag{16}$$

By (16) we easily obtain, for $x \in D$,

$$E^x[e_q(\tau_D) f(X_{\tau_D})] = \int_D \int_{D^c} E_v^x[e_q(\tau_{D\setminus\{v\}})] f(y) g^x(v,y)\, dy\, dv. \tag{17}$$

3. α-stable Schrödinger operator. The purpose of this section is to introduce the notion of the α-stable Schrödinger operator and establish its basic properties. To achieve this, we first analyze the operator $\Delta^{\alpha/2}$ (see also [S]). Our approach is based on distribution theory, and we begin with introducing an appropriate L^1 space.

DEFINITION 3.1. We denote by $\mathcal{L}^1 = \mathcal{L}^1(dx/(1+|x|)^{d+\alpha})$ the space of all Borel functions f on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^{d+\alpha}}\, dx < \infty.$$

DEFINITION 3.2. Let $f \in \mathcal{L}^1$. For $\varepsilon > 0$ and $x \in \mathbb{R}^d$ we define

$$\Delta_\varepsilon^{\alpha/2} f(x) = \mathcal{A}(d,-\alpha) \int_{|y-x|>\varepsilon} \frac{f(y)-f(x)}{|y-x|^{d+\alpha}}\, dy,$$

and put $\Delta^{\alpha/2} f(x) = \lim_{\varepsilon \downarrow 0} \Delta_\varepsilon^{\alpha/2} f(x)$ whenever the limit exists.

LEMMA 3.3. *Assume that*

$$(1)\qquad \int\limits_{|y-x|>\varepsilon} \frac{|f(x)g(y)|}{|y-x|^{d+\alpha}}\,dx\,dy<\infty \quad \textit{and} \quad \int\limits_{\mathbb{R}^d} |f(x)g(x)|\,dx<\infty.$$

Then

$$(2)\qquad \int\limits_{\mathbb{R}^d} f(x)\Delta_\varepsilon^{\alpha/2}g(x)\,dx=\int\limits_{\mathbb{R}^d} g(x)\Delta_\varepsilon^{\alpha/2}f(x)\,dx.$$

Proof. The first part of (1) justifies the following application of Fubini's theorem:

$$\mathcal{A}(d,-\alpha)\int\limits_{\mathbb{R}^d} f(x)\int\limits_{|y-x|>\varepsilon}\frac{g(y)}{|y-x|^{d+\alpha}}\,dy\,dx$$
$$=\mathcal{A}(d,-\alpha)\int\limits_{\mathbb{R}^d} g(y)\int\limits_{|y-x|>\varepsilon}\frac{f(x)}{|y-x|^{d+\alpha}}\,dx\,dy.$$

We obtain the conclusion by subtracting the expression

$$\mathcal{A}(d,-\alpha)\int\limits_{\mathbb{R}^d}\int\limits_{|z|>\varepsilon}\frac{f(x)g(x)}{|z|^{d+\alpha}}\,dz\,dx=\mathcal{A}(d,-\alpha)\int\limits_{\mathbb{R}^d}\int\limits_{|z|>\varepsilon}\frac{f(y)g(y)}{|z|^{d+\alpha}}\,dz\,dy.$$

REMARK 3.4. Suppose that $f\in\mathcal{L}^1$ and $g\in C_c(\mathbb{R}^d)$. Then f and g satisfy the condition (1) of Lemma 3.3 for all $\varepsilon>0$.

LEMMA 3.5. *Let D be an open subset of $\mathbb{R}^d$. Then $\Delta^{\alpha/2}\varphi(x)$ exists for every $\varphi\in C_c^2(D)$ and is a continuous function of $x\in\mathbb{R}^d$. If D is bounded then for all $x\in\mathbb{R}^d$ and $\varepsilon<1$,*

$$|\Delta_\varepsilon^{\alpha/2}\varphi(x)|\le C(D,\alpha)\frac{\|\varphi\|_2}{(1+|x|)^{d+\alpha}}.$$

The proof is based on Taylor's expansion of φ and the symmetry of the kernel $|y|^{-d-\alpha}$ and is omitted.

Lemma 3.5 allows us to introduce the following

DEFINITION 3.6. We define the distribution k^α by the formula

$$(3)\qquad (k^\alpha,\varphi)=\Delta^{\alpha/2}\varphi(0),\quad \varphi\in C_c^\infty(\mathbb{R}^d)\ (\text{or } \varphi\in C_c^2(\mathbb{R}^d)).$$

By this definition

$$(4)\qquad k^\alpha*\varphi(x)=\Delta^{\alpha/2}\varphi(x),\quad \varphi\in C_c^2(\mathbb{R}^d),$$

where as usual ([R])

$$k^\alpha*\varphi(x)=(k^\alpha,\tau_x\check{\varphi})\quad\text{with}\quad \tau_x\check{\varphi}(y)=\varphi(x-y).$$

We now introduce the *weak fractional Laplacian* $\widetilde{\Delta}^{\alpha/2}$.

DEFINITION 3.7. For $f\in\mathcal{L}^1$ we define the distribution $\widetilde{\Delta}^{\alpha/2}f$ by the formula

$$(\widetilde{\Delta}^{\alpha/2}f,\varphi)=(f,\Delta^{\alpha/2}\varphi),\quad \varphi\in C_c^\infty(\mathbb{R}^d).$$

By Lemma 3.5, $(1+|x|)^{d+\alpha}\Delta^{\alpha/2}\varphi_n(x)$ converges to 0 uniformly whenever $\varphi_n\in C_c^\infty(\mathbb{R}^d)$ converges to 0 in the usual topology of the test functions. By bounded convergence we get

$$(f,\Delta^{\alpha/2}\varphi_n)=\int\limits_{\mathbb{R}^d} f(x)\Delta^{\alpha/2}\varphi_n(x)\,dx\to 0,\qquad n\to\infty,$$

so that the functional $\widetilde{\Delta}^{\alpha/2}f$ is continuous, i.e. it is a distribution.

Assume that $f\in C^2(D)\cap\mathcal{L}^1$, where as before D is an open subset of $\mathbb{R}^d$. We claim that $\Delta^{\alpha/2}f(x)$ exists and is a continuous function of $x\in D$. Indeed, fix $\varepsilon>0$ and define $D_\varepsilon=\{x\in D:\operatorname{dist}(x,D^c)>\varepsilon\}$. Assume that D is bounded and write $f=f_1+f_2$, where $f_1\in C_c^2(D)$ and $f_2=0$ on D_ε. We observe that $\Delta^{\alpha/2}f_2(x)=\Delta_\varepsilon^{\alpha/2}f_2(x)$ for $x\in D_{2\varepsilon}$. Application of Lemma 3.5 for f_1 proves the claim.

LEMMA 3.8. *Let D be an open subset of $\mathbb{R}^d$ and let $f\in C^2(D)\cap\mathcal{L}^1$. Then $\widetilde{\Delta}^{\alpha/2}f=\Delta^{\alpha/2}f$ as distributions in D.*

Proof. By Lemma 3.3 we have $(f,\Delta_\varepsilon^{\alpha/2}\varphi)=(\Delta_\varepsilon^{\alpha/2}f,\varphi)$ for $\varphi\in C_c^\infty(D)$ and $\varepsilon>0$. By the above remarks on the existence of $\Delta^{\alpha/2}f(x)$ and by Lemma 3.5 we see that $\Delta_\varepsilon^{\alpha/2}f(x)\to\Delta^{\alpha/2}f(x)$ boundedly on compact subsets of D as $\varepsilon\downarrow 0$. By bounded convergence we get

$$(f,\Delta^{\alpha/2}\varphi)=(\Delta^{\alpha/2}f,\varphi),\quad \varphi\in C_c^\infty(D),$$

which finishes the proof.

The following theorem and Theorem 3.12 below, describing the α-harmonic functions as those annihilated by $\Delta^{\alpha/2}$, are the main results of this section. Although at least a version of Theorem 3.9 seems to be well known [BC], we were unable to point out an actual reference and we give the proof in detail.

THEOREM 3.9. *Let D be an open subset of $\mathbb{R}^d$. A function f on $\mathbb{R}^d$ is α-harmonic in D if and only if it is continuous in D and $\Delta^{\alpha/2}f=0$ in D.*

Proof. If f is α-harmonic in D then the Dynkin [D] characteristic operator $\mathcal{U}$ clearly vanishes, i.e. for $x\in D$,

$$\mathcal{U}f(x)=\lim_{r\downarrow 0}\frac{E^xf(X_{\tau_{B(x,r)}})-f(x)}{E^x\tau_{B(x,r)}}=0.$$

Of course, $f\in C^\infty(D)$. By the scaling property of the process we have

$E^x\tau_{B(x,r)} = r^\alpha E^0\tau_{B(0,1)}$ for all $x \in \mathbb{R}^d$ and $r > 0$. We easily obtain

$$0 = \mathcal{U}f(x) = \lim_{r\downarrow 0} \frac{C_\alpha^d}{E^0\tau_{B(0,1)}} \int\limits_{|y-x|>r} \frac{f(y)-f(x)}{|x-y|^d(|x-y|^2-r^2)^{\alpha/2}}\,dy$$
$$= \frac{C_\alpha^d}{E^0\tau_{B(0,1)}} \lim_{r\downarrow 0} \int\limits_{|x-y|>r} \frac{f(y)-f(x)}{|x-y|^{d+\alpha}}\,dy$$

for $x \in D$, so that $\Delta^{\alpha/2}f(x) = 0$.

To prove the other implication, we fix an open set $D_1 \neq \emptyset$ relatively compact in D. Without any loss of generality we assume that D_1 has the exterior cone property. We define $\widetilde{f}(x) = E^x f(X_{\tau_{D_1}})$, $x \in \mathbb{R}^d$. The function $\widetilde{f}$ is regular α-harmonic in D_1, since $E^x|f(X_{\tau_{D_1}})| < \infty$ for $x \in \mathbb{R}^d$. Indeed, let D_2 be another open set relatively compact in D such that $\overline{D}_1 \subset D_2$. There exist functions f_1, f_2 on D_1^c such that $f = f_1 + f_2$, f_1 is continuous and bounded on D_1^c and $f_2 \equiv 0$ in D_2. We have

$$\widetilde{f}(x) = E^x f_1(X_{\tau_{D_1}}) + E^x f_2(X_{\tau_{D_1}}) \quad \text{for } x \in \mathbb{R}^d.$$

The first expectation is clearly absolutely convergent, hence regular α-harmonic in D_1. It is also continuous in $\overline{D}_1$ or, in other words, D_1 is regular for the corresponding Dirichlet problem. The last assertion follows from the exterior cone property (see [B1], Lemma 10, or [PS], Theorem 4.2.2).

By [B1] (Lemmas 6 and 7, and Remark 4), the P^x distribution of $X_{\tau_{D_1}}$ has the density function $g^x(y)$ satisfying the inequality

$$g^x(y) \le c_1 \frac{s(x)}{\operatorname{dist}(y, D_1)^{d+\alpha}}, \quad x \in D_1,\ y \in D_2^c,$$

where s is a continuous function in D_1 such that $s(x) \to 0$ whenever $\operatorname{dist}(x, \partial D_1) \to 0$. Given that $\Delta^{\alpha/2}f(x)$ exists at two different points x, we have $\int_{\mathbb{R}^d} (|f(y)|/(1+|y|)^{d+\alpha})\,dy < \infty$. In consequence the second expectation above is absolutely convergent (so it is regular α-harmonic in D_1) and $|E^x f_2(X_{\tau_{D_1}})| \le c_2 s(x)$ for $x \in D_1$. We see that $\widetilde{f}$ is continuous in $\overline{D}_1$ and regular α-harmonic in D_1. Also, $\widetilde{f} = f$ on D_1^c.

Let $h = \widetilde{f} - f$. We now verify that $h \equiv 0$, so that $f \equiv \widetilde{f}$ is regular α-harmonic in D_1. We have $\Delta^{\alpha/2}h(x) = 0$ for $x \in D_1$. The function h is continuous and compactly supported. If it has a (strictly) positive maximum at $x_0 \in D_1$, then $\Delta^{\alpha/2}h(x_0) \le -\mathcal{A}(d,-\alpha)h(x_0)\int_{D_1^c}|y-x_0|^{\alpha-d}\,dy < 0$, which gives a contradiction. Similarly, h must be nonnegative.

COROLLARY 3.10. *If $f \in C^2(D) \cap \mathcal{L}^1$ and $\widetilde{\Delta}^{\alpha/2}f = 0$ in D then f is α-harmonic in D.*

Let us recall that $P_r(x,y)$ denotes the Poisson kernel (2.2). It has a singularity at $|y| = r$. To remove this inconvenience, we fix a nonnegative function $\phi \in C_c^\infty((1/2,1))$ such that $\int_{1/2}^1 \phi(r)\,dr = 1$ and we define

$$\psi(y) = \int\limits_{1/2}^{1} \phi(r)P_r(0,y)\,dr = C_\alpha^d |y|^{-d} \int\limits_{|y|\wedge 1/2}^{|y|\wedge 1} \frac{r^\alpha \phi(r)}{(|y|^2-r^2)^{\alpha/2}}\,dr$$

for $y \in \mathbb{R}^d$. It is not difficult to check the following

LEMMA 3.11. *The function ψ is in $C^\infty(\mathbb{R}^d)$. Moreover, for all $y \in \mathbb{R}^d$ and every multiindex γ we have*

$$|D^\gamma \psi(y)| \le \frac{C(d,\alpha,\gamma)}{(1+|y|)^{d+\alpha+|\gamma|}}.$$

We define $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(x/\varepsilon)$, where $x \in \mathbb{R}^d$ and $\varepsilon > 0$.

THEOREM 3.12. *Let D be an open subset of $\mathbb{R}^d$. Suppose that $f \in \mathcal{L}^1$ and $\widetilde{\Delta}^{\alpha/2}f$ vanishes in D. Then there exists a function u α-harmonic in D such that $f = u$ almost everywhere.*

Proof. The proof is a modification of classical arguments (see e.g. [Fo]).

STEP 1. Assume that additionally $f \in C^2(D)$. Applying Corollary 3.10 we obtain the conclusion. By an application of Fubini's theorem and by the scaling property $\varepsilon^{-d}P_r(0,x/\varepsilon) = P_{r\varepsilon}(0,x)$ of the Poisson kernel we obtain

$$f(x) = f * \psi_\varepsilon(x) \quad \text{for all } x \in D_{3\varepsilon}.$$

STEP 2. In the general case we fix a nonnegative function $g \in C_c^\infty(\mathbb{R}^d)$ which is symmetric (i.e. $g(-x) = g(x)$), supported in $B(0,1)$ and such that $\int_{\mathbb{R}^d} g(y)\,dy = 1$. We define $g_\delta(x) = \delta^{-d}g(x/\delta)$, $\delta > 0$, $x \in \mathbb{R}^d$. Obviously, $g_\delta * f \in C^\infty(\mathbb{R}^d)$. Also, $g_\delta * f \in \mathcal{L}^1$. For a test function φ by symmetry of g and Fubini's theorem we get

$$(g_\delta * f, \Delta^{\alpha/2}\varphi) = (f, g_\delta * \Delta^{\alpha/2}\varphi). \tag{5}$$

By (4) and the usual properties of convolution we also obtain

$$g_\delta * \Delta^{\alpha/2}\varphi = g_\delta * k^\alpha * \varphi = k^\alpha * (g_\delta * \varphi) = \Delta^{\alpha/2}(g_\delta * \varphi). \tag{6}$$

By (5) and (6) we thus have, for $\varphi \in C_c^\infty(D_\delta)$,

$$(\widetilde{\Delta}^{\alpha/2}(g_\delta * f), \varphi) = (f, g_\delta * \Delta^{\alpha/2}\varphi) = (\widetilde{\Delta}^{\alpha/2}f, g_\delta * \varphi) = 0.$$

By Step 1, $g_\delta * f$ is α-harmonic in D_δ and for $x \in D_{\delta+3\varepsilon}$,

$$g_\delta * f(x) = (g_\delta * f) * \psi_\varepsilon(x) = (g_\delta * \psi_\varepsilon) * f(x). \tag{7}$$

The left-hand side of (7) converges to f in the sense of distributions as $\delta \downarrow 0$. We also have, for $\delta < 1$,

$$g_\delta * \psi_\varepsilon(y) \le C(\varepsilon)(1+|y|)^{-d-\alpha}, \quad y \in \mathbb{R}^d.$$

The convergence of the right-hand side of (7) to $\psi_\varepsilon * f$ as $\delta \downarrow 0$ is therefore pointwise and locally bounded (hence distributional). Thus, $f = \psi_\varepsilon * f$ as distributions in $D_{3\varepsilon}$. From Lemma 3.11 we infer that $\psi_\varepsilon * f \in C^2(D_{3\varepsilon})$, thus $f \in C^2(D)$ as a distribution, which completes the proof of the theorem.

The following observation will be crucial in the sequel.

PROPOSITION 3.13. *Let D be a bounded open subset of $\mathbb{R}^d$ and let $\psi \in L^1(D)$. Then*

$$\widetilde{\Delta}^{\alpha/2} G_D \psi = -\psi \quad \text{in } D.$$

Proof. Recall (see (2.3)) that $G_D(x,y) = K_\alpha(x,y) - E^x K_\alpha(X_{\tau_D}, y)$ for $x, y \in D$. For $\psi \in \mathcal{J}^\alpha_{\mathrm{loc}}$ and $x \in D$ we thus have

$$\begin{aligned} G_D\psi(x) &= \int_D G_D(x,y)\psi(y)\,dy \\ &= \int_D K_\alpha(x,y)\psi(y)\,dy - E^x\Big[\int_D K_\alpha(X_{\tau_D}, y)\psi(y)\,dy\Big]. \end{aligned}$$

The application of Fubini's theorem is justified since all the integrals are absolutely convergent by the defining condition (2.4) on $\psi \in \mathcal{J}^\alpha_{\mathrm{loc}}$ and boundedness of D. The second term above is regular α-harmonic in D so $\widetilde{\Delta}^{\alpha/2}$ annihilates it in D. Thus, for $\varphi \in C_c^\infty(D)$ we obtain

$$\begin{aligned} (\widetilde{\Delta}^{\alpha/2} G_D\psi, \varphi) &= (\widetilde{\Delta}^{\alpha/2} K_\alpha\psi, \varphi) = (K_\alpha\psi, \Delta^{\alpha/2}\varphi) \\ &= \iint K_\alpha(x,y)\psi(y)\Delta^{\alpha/2}\varphi(x)\,dy\,dx = (\psi, K_\alpha\Delta^{\alpha/2}\varphi). \end{aligned}$$

By [La], Ch. I, we obtain $K_\alpha \Delta^{\alpha/2}\varphi = -\varphi$, which completes the proof for $\psi \in \mathcal{J}^\alpha_{\mathrm{loc}}$. For general $\psi \in L^1(D)$ we easily get $\|G_D|\psi|\|_1 \le \|G_D\mathbf{1}\|_\infty \|\psi\|_1$. By an approximation argument our result carries over to such general ψ.

We now define the *weak Schrödinger operator* S^α.

DEFINITION 3.14. Let D be an open set in $\mathbb{R}^d$. For $u \in \mathcal{L}^1$ such that $uq \in L^1_{\mathrm{loc}}(D)$ we define the distribution $S^\alpha u$ in D by the formula

$$S^\alpha u = -(\widetilde{\Delta}^{\alpha/2} + q)u$$

and call S^α the (weak) *Schrödinger operator* in D.

In the following theorem we prove an important uniqueness result for S^α (compare Theorem 3.21 in [ChZ]).

THEOREM 3.15. *Let D be a bounded and regular domain in $\mathbb{R}^d$ and let $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$. Assume that $V\mathbf{1} \in L^\infty(\mathbb{R}^d)$, $\phi \in C_0(D)$, and $S^\alpha\phi$ vanishes in D. Then $\phi = 0$ in $\mathbb{R}^d$.*

Proof. Put $f = \phi - G_D(q\phi)$. Since $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$, we have $q\phi \in \mathcal{J}^\alpha$. Therefore $G_D(q\phi) \in C_0(D)$. We obtain

$$\widetilde{\Delta}^{\alpha/2} f = \widetilde{\Delta}^{\alpha/2}\phi - \widetilde{\Delta}^{\alpha/2} G_D(q\phi) = (\widetilde{\Delta}^{\alpha/2} + q)\phi = -S^\alpha\phi = 0$$

in D. By Theorem 3.12 and continuity of f it follows that f is α-harmonic in D. By the maximum principle (see the proof of Theorem 3.9), $f = 0$ in $\mathbb{R}^d$. Thus $\phi = G_D(q\phi)$. We now apply (2.8) with f replaced by $q\phi$. First observe that $G_D(|q\phi|) \in L^\infty(\mathbb{R}^d)$ implies $|q|G_D(|q\phi|) \in \mathbf{F}(D,q)$, so $V(|q|G_D(|q\phi|)) < \infty$ in D. We then obtain $V(q\phi) = G_D(q\phi) + V(qG_D(q\phi)) = \phi + V(q\phi)$ in D. Since $\phi = 0$ in D^c, we have $\phi = 0$ in $\mathbb{R}^d$.

PROPOSITION 3.16. *Let D be a bounded regular domain, $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$ and let $V\mathbf{1} \in L^\infty(\mathbb{R}^d)$. Then*

$$S^\alpha V f = f \quad \text{in } D \tag{8}$$

for $f \in \mathbf{F}(D,q)$. If $S^\alpha f \in \mathbf{F}(D,q)$ and $f \in C_0(D)$ then

$$V S^\alpha f = f \quad \text{in } D. \tag{9}$$

Proof. If $f \in \mathbf{F}(D,q)$ then $V|f| \in C_0(D)$, which yields $G_D(|q|V|f|) < \infty$ in D. Hence, by (2.9) we obtain $Vf = G_D f + G_D(qVf)$. By Proposition 3.13, $\widetilde{\Delta}^{\alpha/2} Vf = -f - qVf$, and so $f = -\widetilde{\Delta}^{\alpha/2} Vf - qVf$ in D, which is (8).

If $S^\alpha f \in \mathbf{F}(D,q)$ then $VS^\alpha f \in C_0(D)$ by regularity of D. Given that $f \in C_0(D)$ we have $VS^\alpha f - f \in C_0(D)$. Applying (8) with f replaced by $S^\alpha f$, we get $S^\alpha V S^\alpha f = S^\alpha f$, hence $S^\alpha(VS^\alpha f - f) = 0$. By Theorem 3.15, we obtain (9).

For a given $f \in \mathbf{F}(D,q)$ we consider the inhomogeneous equation

$$S^\alpha \phi = f \quad \text{in } D.$$

It follows from the preceding proposition that, under the above conditions on D and q, its unique (weak) solution $\phi \in C_0(D)$ is given by $\phi = Vf$.

We now discuss the connection of the operator S^α with the infinitesimal generator of the Feynman–Kac semigroup $(T_t)_{t>0}$. We recall that (T_t) acts as a strongly continuous semigroup on each of the appropriate spaces $\mathcal{X} = \mathcal{X}(D) = L^p(D)$ for a bounded open subset D of $\mathbb{R}^d$ $(1 \le p < \infty)$, and $C_0(D)$ whenever D is regular. The infinitesimal generator A is defined as an operator on $\mathcal{D}(A) \subseteq \mathcal{X}$ by the formula

$$Af = \lim_{t\downarrow 0}(1/t)[T_t f - f] \quad \text{for } f \in \mathcal{D}(A),$$

where the limit is taken in the norm of $\mathcal{X}$ and $\mathcal{D}(A)$ is the set of all functions f for which the limit exists. If (D,q) is gaugeable then V is a bounded operator in each appropriate space. General semigroup theory then yields

$$\mathcal{D}(A) = V[\mathcal{X}], \quad A = -V^{-1}. \tag{10}$$

PROPOSITION 3.17. *Let D be a bounded regular domain. The domain $\mathcal{D}(A_0)$ of the infinitesimal generator A_0 for (T_t) acting on $C_0(D)$ consists of all $f \in C_0(D)$ such that $S^\alpha f \in C_0(D)$. If $f \in \mathcal{D}(A_0)$ then*

$$A_0 f = -S^\alpha f. \tag{11}$$

Proof. As a particular case of (10) we have $\mathcal{D}(A_0) = V[C_0(D)]$ provided (D,q) is gaugeable. If $f \in C_0(D)$ and $S^\alpha f \in C_0(D)$ then $S^\alpha f \in \mathbf{F}(D,q)$. Hence by (9), $f = VS^\alpha f \in V[C_0(D)]$. Conversely, if $f = Vg$ with $g \in C_0(D)$, then $f \in C_0(D)$ and by (8), $S^\alpha f = S^\alpha V g = g \in C_0(D)$. Since by (10), $g = V^{-1} f = -A_0 f$, we obtain (11). If (D,q) is not gaugeable, we get (11) by considering $(T_t^\lambda) = (e^{-\lambda t} T_t)$ for suitable $\lambda > 0$.

4. Conditional gauge and q-Green function. In this section we prove the Conditional Gauge Theorem (CGT) which is the main result of the paper. We assume in the sequel, unless stated otherwise, that D is a bounded Lipschitz domain in $\mathbb{R}^d$ and $d > \alpha$. We first state for our α-stable process an important result on the Green function of D called the "3G Theorem". The proof, which strongly relies on BHP (see [B1]), is omitted because it is a direct adaptation of arguments developed for Brownian motion by Cranston, Fabes and Zhao [CFZ].

3G THEOREM. *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, $d > \alpha$, and let G be the Green function of D. There exists a constant C such that for all $x, y, z \in D$ we have*

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le C(K_\alpha(x,y) + K_\alpha(y,z)),$$

and

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le C\left(\frac{|x-z|}{|x-y||y-z|}\right)^{d-\alpha},$$

unless $x = y = z$. In fact, the constant C above depends on D only through its Lipschitz character and diameter ([CFZ]).

As a limiting case, by the definition of the Martin kernel K (see 2.7) we obtain, with the same constant C,

COROLLARY 4.1. *For all $x, y \in D$ and $\xi \in \partial D$ we also have*

$$\frac{G(x,y)K(y,\xi)}{K(x,\xi)} \le C(K_\alpha(x,y) + K_\alpha(y,\xi)).$$

In the sequel we assume that $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$. The following lemma is a "conditional" version of Khasminski's lemma (see (2.6)). The proof relies on the 3G Theorem as in [CFZ] and is omitted.

LEMMA 4.2. *For every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, D, q)$ such that for every open set $U \subseteq D$ with $m(U) < \eta$ we have, for all $y \in D$,*

$$\sup_{u \in D,\, u \ne y} E_y^u\Big[\int_0^{\tau_{U\setminus\{y\}}} |q(X_t)|\,dt\Big] < \varepsilon,$$

and if $0 < \varepsilon < 1$ then

$$0 < \inf_{u \in D,\, u \ne y} E_y^u e_{|q|}(\tau_{U\setminus\{y\}}) \le \sup_{u \in D,\, u \ne y} E_y^u e_{|q|}(\tau_{U\setminus\{y\}}) < (1-\varepsilon)^{-1}.$$

We first prove some lemmas. The first is a more precise version of Lemma 4.4 of [CFZ].

LEMMA 4.3. *Let D and U be bounded regular domains, $U \subseteq D$, and let $y \in U$. Set $D_0 = D \setminus \overline{U}$ and $\zeta = \tau_{D\setminus\{y\}}$. Let $u \in D$, $u \ne y$, and $x \in D_0$. Then*

$$P_y^u\{\tau_{U\setminus\{y\}} = \zeta\} = \frac{G_U(u,y)}{G_D(u,y)}, \qquad P_y^x\{\tau_{D_0} = \zeta\} = 0. \tag{1}$$

Proof. For $y \in D$ we obtain $G_D(u,y)^{-1} E^u[\tau_U < \tau_D;\, G_D(X_{\tau_U}, y)] = P_y^u\{\tau_{U\setminus\{y\}} < \zeta\}$, because $\{y\}$ is a polar set. By definition of Green function we have $G_D(z,y) = K_\alpha(z,y) - E^z K_\alpha(X_{\tau_D}, y)$. This and the strong Markov property yield

$$\begin{aligned} E^u[\tau_U < \tau_D;\, G_D(X_{\tau_U}, y)] &= E^u[\tau_U < \tau_D;\, K_\alpha(X_{\tau_U}, y)] - E^u[\tau_U < \tau_D;\, E^{X_{\tau_U}}[K_\alpha(X_{\tau_D}, y)]] \\ &= E^u[\tau_U < \tau_D;\, K_\alpha(X_{\tau_U}, y) - K_\alpha(X_{\tau_D}, y)] \\ &= E^u[K_\alpha(X_{\tau_U}, y) - K_\alpha(X_{\tau_D}, y)] = G_D(u,y) - G_U(u,y). \end{aligned}$$

This shows the first part of (1).

For the second part, observe that if $x \in D_0$ then $E^x G_D(X_{\tau_{D_0}}, y) = G_D(x,y)$, since $G_D(\cdot, y)$ is regular α-harmonic in D_0. Thus

$$\begin{aligned} P_y^x\{\tau_{D_0} < \zeta\} &= G_D(x,y)^{-1} E^x[\tau_{D_0} < \tau_D;\, G_D(X_{\tau_{D_0}}, y)] \\ &= G_D(x,y)^{-1} E^x G_D(X_{\tau_{D_0}}, y) = G_D(x,y)^{-1} G_D(x,y) = 1. \end{aligned}$$

This completes the proof of (1).

As a corollary we obtain (cf. Lemma 4.4 in [CFZ])

COROLLARY 4.4. *Assume that $y \in D$ with $d(y, D^c) > 3\delta$. Put $U = B(y, 3\delta)$. Then*

$$\inf_{u \in \overline{B(y,\delta)}} P^u\{\tau_U = \tau_D\} > 0, \qquad \inf_{u \in \overline{B(y,\delta)}\setminus\{y\}} P_y^u\{\tau_{U\setminus\{y\}} = \zeta\} > 0. \tag{2}$$

Proof. We first prove the second part of (2). In view of Lemma 4.3 we have the identity

$$P_y^u\{\tau_{U\setminus\{y\}}=\zeta\}=\frac{G_U(u,y)}{G_D(u,y)}\ge\frac{G_U(u,y)}{K_\alpha(u,y)}$$
$$=1-\mathcal{A}(d,\alpha)^{-1}|u-y|^{d-\alpha}E^uK_\alpha(X_{\tau_U},y).$$

Observe that $|u-y|\le\delta$ for $u\in\overline{B(y,\delta)}$ and also $|X_{\tau_U}-y|>3\delta$, which yields $E^uK_\alpha(X_{\tau_U},y)\le\mathcal{A}(d,\alpha)(3\delta)^{\alpha-d}$. This completes the proof of the second part of (2).

We now prove the first part. Set $R=\operatorname{diam}(D)$. Then by (2.2),

$$\begin{aligned}P^u\{\tau_U=\tau_D\}&=P^u\{X_{\tau_U}\in D^c\}\\&=P^{u-y}\{X_{\tau_{B(0,3\delta)}}\in D^c-y\}\ge P^{u-y}\{X_{\tau_{B(0,3\delta)}}\in B(0,R)^c\}\\&=C_\alpha^d\int_{|z|>R}\left(\frac{(3\delta)^2-|u-y|^2}{|z|^2-(3\delta)^2}\right)^{\alpha/2}\frac{dz}{|u-y-z|^d}\\&\ge C_\alpha^d(8\delta)^{\alpha/2}\omega_d\int_R^\infty\frac{\varrho^{d-1}\,d\varrho}{\varrho^\alpha(\varrho+\delta)^d}\ge\frac{C_\alpha^d(8\delta)^{\alpha/2}\omega_d}{(7/6)^d\alpha R^\alpha},\end{aligned}$$

because $\varrho\ge R>6\delta$ in the integrand. We denote by ω_d the surface measure of the unit sphere in $\mathbb{R}^d$.

By the above corollary and Lemma 4.2 we easily obtain the following result (cf. Lemma 4.3 in [CFZ]).

LEMMA 4.5. *Under the notation of Corollary 4.4 there exist constants C_1 and C_2 such that for every $u,v\in\overline{B(y,\delta)}$, $v\ne y$, with $\delta>0$ small enough we have*

$$C_1\le E^u[\tau_U=\tau_D;\,e_q(\tau_D)]\le C_2,\qquad C_1\le E_y^v[\tau_{U\setminus\{y\}}=\zeta;\,e_q(\zeta)]\le C_2.$$

Before stating the next lemma, we introduce some notation. For $y\in\mathbb{R}^d$, $|y|>1$, let

$$I_1(y)=\int_{B(0,1)}\frac{\mathcal{A}(d,\alpha)\,du}{|u-y|^{d+\alpha}|u|^{d-\alpha}},\qquad I_2(y)=\int_{B(0,1)}\frac{\mathcal{A}(d,\alpha)\,du}{|u-y|^{d+\alpha}}.$$

To simplify formulas, we write e.g. $I_1(y)\approx I_2(y)$ for $y\in A$ if there exist constants C_1, C_2 not depending on y such that $C_1I_1(y)\le I_2(y)\le C_2I_1(y)$ for all $y\in A$.

The next lemma contains an essential argument to be used in the proof of CGT. The following short proof has been communicated to us by K. Samotij.

LEMMA 4.6. *For all $y\in\mathbb{R}^d$ such that $|y|>1$ we have*

$$I_1(y)\approx I_2(y).$$

Proof. Clearly, we have $I_1(y)\ge I_2(y)$. To show a reverse inequality, define $A(y)=B(y/|y|,1/2)\cap B(0,1)$, $B(y)=B(0,1)\setminus A(y)$ and $M(y)=\sup_{u\in B(y)}|y-u|^{-d-\alpha}$, $m(y)=\inf_{u\in B(y)}|y-u|^{-d-\alpha}$. It is not difficult to see that $M(y)\le Cm(y)$. Consequently,

$$\begin{aligned}\int_{B(y)}\frac{du}{|u-y|^{d+\alpha}|u|^{d-\alpha}}&\le M(y)\int_{|u|<1}\frac{du}{|u|^{d-\alpha}}\\&\le C\frac{\omega_d}{\alpha}m(y)\le C\frac{\omega_d}{\alpha|B(y)|}\int_{B(y)}\frac{du}{|u-y|^{d+\alpha}}.\end{aligned}$$

However, for $u\in A(y)$ we have $|u|>1/2$, so

$$\int_{A(y)}\frac{du}{|u-y|^{d+\alpha}|u|^{d-\alpha}}\le2^{d-\alpha}\int_{A(y)}\frac{du}{|u-y|^{d+\alpha}}.$$

This clearly ends the proof.

In the sequel we always assume that (D,q) is gaugeable. We recall the notion of the q-Green function V and the transition density u_t for the killed Feynman–Kac semigroup T_t. Both V and u_t are symmetric in $(x,y)\in D\times D$ and u_t is continuous. The following lemma yields that V is also continuous in $x,y\in D$ provided $x\ne y$. The proof is an adaptation of that of Lemma 6.1 in [ChZ] and is omitted.

LEMMA 4.7. *Let (D,q) be gaugeable. For every $0<\delta\le d+\alpha$ there exist constants t_0, ε and C depending on δ such that*

$$u_t(x,y)\le\begin{cases}Ct^{1-\delta/\alpha}|x-y|^{\delta-d-\alpha}&\text{for }0<t\le t_0,\\C\exp(-\varepsilon t)&\text{for }t>t_0.\end{cases}$$

Taking $\delta=\alpha$ in the above lemma we obtain the continuity of V.

In the Brownian case the corresponding estimates (see Lemma 6.1 in [ChZ]) are much stronger than those of the above lemma, due to the exponential character of the transition density u_t, and yield a very important inequality: $V(x,y)\le CK_2(x,y)$, where $K_2(x,y)$ is the classical (Newtonian) kernel. The lack of such an estimate at this stage is a source of substantial difficulty, which is resolved in Step 2 of the proof of Theorem 4.9 below.

The next lemma is a direct adaptation of Theorem 3.20 from [ChZ].

LEMMA 4.8. *If (D,q) is gaugeable then for each $x\in D$ and almost all $y\in D$,*

$$V(x,y)=G(x,y)+\int_D V(x,u)q(u)G(u,y)\,du. \tag{3}$$

We now prove the Conditional Gauge Theorem asserting the boundedness of the conditional gauge $u(x,y)$. A final form of CGT will be given in Theorem 4.10. As in the classical case, CGT plays a crucial role in the theory of the α-stable Schrödinger operator. We recall that the function u is defined as

$$u(x,y)=E_y^x e_q(\tau_{D\setminus\{y\}}),\qquad (x,y)\in D\times\overline{D},\ x\neq y.$$

THEOREM 4.9 (Conditional Gauge Theorem). *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, $d>\alpha$, and $q\in\mathcal{J}^{\alpha}_{\mathrm{loc}}$. If (D,q) is gaugeable then $u(x,y)$ is symmetric and bounded in $D\times D$ for $x\neq y$.*

Proof. We divide the proof into several steps. For $\delta>0$ we put $D_\delta=\{x\in D: d(x,D^c)>3\delta\}$. We choose and fix throughout the proof δ and a Lipschitz domain U^δ such that $D\setminus D_\delta\subseteq U^\delta\subseteq D$ and for all $y\in D$,

$$\sup_{u\in D,\,u\neq y} E_y^u\Big[\int_0^\tau |q(X_t)|\,dt\Big]<1/2,\qquad \sup_{u\in\mathbb{R}^d} E^u\Big[\int_0^\tau |q(X_t)|\,dt\Big]<1/2,$$

with $\tau=\tau_{U^\delta\setminus\{y\}}$ or $\tau=\tau_{B(y,3\delta)\setminus\{y\}}$. By (2.6) and Lemma 4.2 we have

$$\sup_{u\in D,\,u\neq y} E_y^u e_{|q|}(\tau)\le 2,\qquad \sup_{u\in\mathbb{R}^d} E^u e_{|q|}(\tau)\le 2. \tag{4}$$

STEP 1. In this step we show for $x,y\in D_\delta$, $x\neq y$, the following:

$$u(x,y)<C,\qquad \text{where } C=C(D,\alpha,q,\delta). \tag{5}$$

Fix $x,y\in D_\delta$ and define $D_0=D\setminus\overline{B(y,\delta)}$, $U=B(y,3\delta)\setminus\{y\}$ and

$$T_1=\tau_{D_0},\qquad T_n=S_{n-1}+\tau_{D_0}\circ\theta_{S_{n-1}},$$
$$S_0=0,\qquad S_n=T_n+\tau_U\circ\theta_{T_n},\qquad n=1,2,\ldots$$

Put $\zeta=\tau_{D\setminus\{y\}}$. We claim that

$$E_y^x e_q(\zeta)=\sum_{n=1}^{\infty}E_y^x[T_n<\zeta,\ S_n=\zeta;\ e_q(\zeta)]. \tag{6}$$

Indeed, since $\zeta<\infty$ a.s., we have either $T_n=\zeta$ or $S_n=\zeta$, for some n. If $T_n=\zeta$, then the corresponding term of the series representing $E_y^x e_q(\zeta)$ is of the form

$$\begin{aligned}E_y^x&[S_{n-1}<\zeta, T_n=\zeta;\ e_q(\zeta)]\\&=E_y^x[S_{n-1}<\zeta;\ e_q(S_{n-1})\{\tau_{D_0}=\zeta;\ e_q(\zeta)\}\circ\theta_{S_{n-1}}].\end{aligned}$$

By the strong Markov property (2.13) of the conditional process we deduce that the last term above is equal to

$$E_y^x[S_{n-1}<\zeta;\ e_q(S_{n-1})E_y^{X_{S_{n-1}}}[\tau_{D_0}=\zeta;\ e_q(\zeta)]].$$

However, $X_{S_{n-1}}\in D_0$ if $S_{n-1}<\zeta$, so by the second formula of (1) in Lemma 4.3 we conclude that the above expression is zero. This justifies our claim.

Taking into account one term of the series (6) we get as above

$$\begin{aligned}G(x,y)&E_y^x[T_n<\zeta,\ S_n=\zeta;\ e_q(\zeta)]\\&=G(x,y)E_y^x[T_n<\zeta;\ e_q(T_n)E_y^{X_{T_n}}[\tau_U=\zeta;\ e_q(\zeta)]]\\&=E^x[T_n<\tau_D;\ e_q(T_n)G(X_{T_n},y)E_y^{X_{T_n}}[\tau_U=\zeta;\ e_q(\zeta)]].\end{aligned}$$

Since $X_{T_n}\in\overline{B(y,\delta)}$ whenever $T_n<\tau_D$, by Lemma 4.5 we obtain

$$\begin{aligned}E^x&[T_n<\tau_D;\ e_q(T_n)G(X_{T_n},y)E_y^{X_{T_n}}[\tau_U=\zeta;\ e_q(\zeta)]]\\&\approx E^x[T_n<\tau_D;\ e_q(T_n)G(X_{T_n},y)]\\&=E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})E^{X_{S_{n-1}}}[\tau_{D_0}<\tau_D;\ e_q(\tau_{D_0})G(X_{\tau_{D_0}},y)]]\\&=E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})E^{X_{S_{n-1}}}[e_q(\tau_{D_0})G(X_{\tau_{D_0}},y)]].\end{aligned}$$

Using the formula (2.17) for D_0 and Lemma 4.6, we have, for $z\in D_0$,

$$\begin{aligned}E^z&[e_q(\tau_{D_0})G(X_{\tau_{D_0}},y)]\\&=\int_{D_0}\int_{D_\delta^c\cap D}\widetilde{u}(z,v)G(w,y)\frac{\mathcal{A}(d,-\alpha)}{|v-w|^{d+\alpha}}G_{D_0}(z,v)\,dw\,dv\\&\le\int_{D_0}\int_{B(y,\delta)}\widetilde{u}(z,v)K_\alpha(w,y)\frac{\mathcal{A}(d,-\alpha)}{|v-w|^{d+\alpha}}G_{D_0}(z,v)\,dw\,dv\\&=\int_{D_0}\widetilde{u}(z,v)\mathcal{A}(d,-\alpha)\delta^{-d}I_1\Big(\frac{y-v}{\delta}\Big)G_{D_0}(z,v)\,dv\\&\approx\int_{D_0}\widetilde{u}(z,v)\mathcal{A}(d,-\alpha)\delta^{\alpha-d}\delta^{-\alpha}I_2\Big(\frac{y-v}{\delta}\Big)G_{D_0}(z,v)\,dv\\&\approx\delta^{\alpha-d}\int_{D_0}\int_{B(y,\delta)}\widetilde{u}(z,v)\frac{\mathcal{A}(d,-\alpha)}{|v-w|^{d+\alpha}}G_{D_0}(z,v)\,dw\,dv\\&\le\delta^{\alpha-d}\int_{D_0}\int_{D_0^c}\widetilde{u}(z,v)\frac{\mathcal{A}(d,-\alpha)}{|v-w|^{d+\alpha}}G_{D_0}(z,v)\,dw\,dv\\&=\delta^{\alpha-d}E^z e_q(\tau_{D_0})\approx\delta^{\alpha-d}E^z e_q(\tau_D)\approx\delta^{\alpha-d}\end{aligned}$$

by gaugeability. Here $\widetilde{u}(z,v)=\widetilde{E}_v^z e_q(\tau_{D_0\setminus\{v\}})$ is the conditional gauge of the set D_0.

If $T_{n-1}<\tau_D$ for $n\ge 2$ then $X_{T_{n-1}}\in\overline{B(y,\delta)}$, hence by (4) and Lemma 4.5 we obtain

$$\begin{aligned}G(x,y)E_y^x[T_n<\zeta,\ S_n=\zeta;\ e_q(\zeta)] &\le C\delta^{\alpha-d}E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})]\\ &= C\delta^{\alpha-d}E^x[T_{n-1}<\tau_D;\ e_q(T_{n-1})E^{X_{T_{n-1}}}[\tau_U<\tau_D;\ e_q(\tau_U)]]\\ &\le 2C\delta^{\alpha-d}E^x[T_{n-1}<\tau_D;\ e_q(T_{n-1})]\\ &\approx 2C\delta^{\alpha-d}E^x[T_{n-1}<\tau_D;\ e_q(T_{n-1})E^{X_{T_{n-1}}}[\tau_U=\tau_D;\ e_q(\tau_U)]]\\ &= 2C\delta^{\alpha-d}E^x[T_{n-1}<\tau_D,\ S_{n-1}=\tau_D;\ e_q(\tau_D)].\end{aligned}$$

For $n=1$ we have $G(x,y)E_y^x[T_n<\zeta,\ S_n=\zeta;\ e_q(\zeta)]\le C\delta^{\alpha-d}$ by the first inequality above. Thus

$$\begin{aligned}G(x,y)E_y^x e_q(\zeta) &= G(x,y)\sum_{n=1}^{\infty}E_y^x[T_n<\zeta,\ S_n=\zeta;\ e_q(\zeta)]\\ &\le C\delta^{\alpha-d}\Big(1+\sum_{n=2}^{\infty}E^x[T_{n-1}<\tau_D,\ S_{n-1}=\tau_D;\ e_q(\tau_D)]\Big)\\ &\le C\delta^{\alpha-d}(1+E^x e_q(\tau_D)).\end{aligned}$$

Recall that x,y satisfy the conditions $d(x,D^c)>3\delta$, $d(y,D^c)>3\delta$ and $|x-y|\le \operatorname{diam}(D)<\infty$. We obtain (cf. [ChZ], Lemma 6.7)

$$G(x,y)\ge C'|x-y|^{\alpha-d}\ge C'(\operatorname{diam}(D))^{\alpha-d}$$

with $C'=C'(D,\alpha,q,\delta)$. This clearly ends the proof of Step 1.

STEP 2. In this step we show that for all $x,y\in D$, $x\ne y$,

$$VqG(\cdot,y)(x)=G(x,y)u(x,y)-G(x,y). \tag{7}$$

For this purpose we first verify that for x, y as above

$$V|q|G(\cdot,y)(x)<\infty. \tag{8}$$

For fixed $x,y\in D$, $x\ne y$, we choose $0<\delta_1\le\delta$ such that $x,y\in D_{\delta_1}$. We use the notation U, D_0 as in Step 1 with δ there replaced by δ_1. Then (4) holds for $\tau=\tau_U$. We have

$$\begin{aligned}V|q|G(\cdot,y)(x) &= \int_0^{\infty}T_s|q|G(\cdot,y)(x)\,ds\\ &= \int_0^{\infty}E^x[s<\tau_D;\ e_q(s)|q(X_s)|G(X_s,y)]\,ds\\ &= G(x,y)\int_0^{\infty}E_y^x[s<\zeta;\ e_q(s)|q(X_s)|]\,ds\\ &= G(x,y)E_y^x\Big[\int_0^{\zeta}e_q(s)|q(X_s)|\,ds\Big].\end{aligned}$$

Defining

$$B(t)=\int_0^t e_q(s)|q(X_s)|\,ds$$

we get $V|q|G(\cdot,y)(x)=G(x,y)E_y^xB(\zeta)$.

Next, as in Step 1, we obtain

$$E_y^xB(\zeta)=\sum_{n=1}^{\infty}E_y^x[T_n<\zeta,\ S_n=\zeta;\ B(\zeta)].$$

Since $B(T_n+\zeta\circ\theta_{T_n})=B(T_n)+e_q(T_n)\{B(\zeta)\circ\theta_{T_n}\}$, by the strong Markov property of the conditional process each term of the above series can be transformed as follows, for $n\ge1$:

$$\begin{aligned}E_y^x[T_n<\zeta,\ S_n=\zeta;\ B(\zeta)] &= E_y^x[T_n<\zeta;\ B(T_n)P_y^{T_n}\{\tau_U=\zeta\}]\\ &\quad+E_y^x[T_n<\zeta;\ e_q(T_n)E_y^{T_n}[\tau_U=\zeta;\ B(\zeta)]].\end{aligned} \tag{9}$$

By Lemma 4.3 we have

$$P_y^z\{\tau_U=\zeta\}=G(z,y)^{-1}G_U(z,y)\quad \text{ for } z\in D,\ z\ne y.$$

Hence the first expression on the right-hand side of (9) is of the form

$$G(x,y)^{-1}E^x[T_n<\tau_D;\ B(T_n)G_U(X_{T_n},y)].$$

Since $G_U(X_{\tau_{D_0}},y)=0$ whenever $\tau_{D_0}=\tau_D$, we obtain

$$\begin{aligned}&E^x[T_n<\tau_D;\ B(T_n)G_U(X_{T_n},y)]\\ &\quad= E^x[S_{n-1}<\tau_D;\ B(S_{n-1})E^{X_{S_{n-1}}}[G_U(X_{\tau_{D_0}},y)]]\\ &\quad\quad+E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})E^{X_{S_{n-1}}}[B(\tau_{D_0})G_U(X_{\tau_{D_0}},y)]].\end{aligned}$$

Now, for $u\in D_0$ we consider the expressions

$$E^u[G_U(X_{\tau_{D_0}},y)], \tag{10}$$

$$E^u[B(\tau_{D_0})G_U(X_{\tau_{D_0}},y)]. \tag{11}$$

Using (2.17) we see that the second one is of the form

$$\int_{D_0}\int_{D_0^c}\widetilde{E}_v^u[B(\tau_{D_0\setminus\{v\}})]G_U(w,y)\frac{\mathcal{A}(d,-\alpha)}{|v-w|^{d+\alpha}}\,G_{D_0}(u,v)\,dw\,dv$$

while the first is similar but without the factor $\widetilde{E}_v^uB(\tau_{D_0\setminus\{v\}})$ in the integrand. Again, $\widetilde{E}_v^u$ indicates that we condition with respect to the Green function of D_0, instead of D.

As in Step 1 and in view of Lemma 4.6 we estimate (11) as follows:

$$E^u[B(\tau_{D_0})G_U(X_{\tau_{D_0\setminus\{v\}}},y)]$$
$$\le \int_{D_0}\int_{B(y,\delta)} \widetilde{E}_v^u[B(\tau_{D_0\setminus\{v\}})]K_\alpha(w,y)\frac{\mathcal{A}(d,-\alpha)}{|v-w|^{d+\alpha}}G_{D_0}(u,v)\,dw\,dv$$
$$= \int_{D_0} \widetilde{E}_v^u[B(\tau_{D_0\setminus\{v\}})]\mathcal{A}(d,-\alpha)\delta_1^{-d}I_1\Big(\frac{y-v}{\delta_1}\Big)G_{D_0}(u,v)\,dv$$
$$\approx \int_{D_0} \widetilde{E}_v^u[B(\tau_{D_0\setminus\{v\}})]\mathcal{A}(d,-\alpha)\delta_1^{\alpha-d}\delta_1^{-\alpha}I_2\Big(\frac{y-v}{\delta_1}\Big)G_{D_0}(u,v)\,dv$$
$$\le C\delta_1^{\alpha-d}E^uB(\tau_{D_0}) \le C\delta_1^{\alpha-d}E^uB(\tau_D)\le \delta_1^{\alpha-d}C',$$

because $E^uB(\tau_D) = V|q|(u) \le \|V|q|\|_\infty < \infty$, by gaugeability of (D,q). Analogously, we estimate (10) by $C\delta_1^{\alpha-d}$. Thus, we have

$$E^x[T_n<\tau_D;\ B(T_n)G_U(X_{T_n},y)]$$
$$\le C\delta_1^{\alpha-d}E^x[S_{n-1}<\tau_D;\ B(S_{n-1})] + C\delta_1^{\alpha-d}E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})].$$

The second expression on the right-hand side of (9) is bounded from above by

$$E_y^x\Big[T_n<\zeta;\ e_q(T_n)E_y^{X_{T_n}}\Big[\int_0^{\tau_U} e_{|q|}(t)|q(X_t)|\,dt\Big]\Big]$$
$$= E_y^x[T_n<\zeta;\ e_q(T_n)E_y^{X_{T_n}}[e_{|q|}(\tau_U)-1]] \le E_y^x[T_n<\zeta;\ e_q(T_n)]$$
$$= G(x,y)^{-1}E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})E^{X_{S_{n-1}}}[e_q(\tau_{D_0})G(X_{\tau_{D_0}},y)]]$$
$$\le C\delta_1^{\alpha-d}G(x,y)^{-1}E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})]$$

(see Step 1). Observe that by Step 1 we have

$$\sum_{n=1}^\infty E^x[S_{n-1}<\tau_D;\ e_q(S_{n-1})] \le C(1+E^xe_q(\tau_D)) < C' < \infty.$$

Therefore, we obtain

$$\sup_{x,y\in D_{\delta_1},\,x\ne y} G(x,y)E_y^xB(\zeta)$$
$$\le C_1\delta_1^{\alpha-d}\Big(\sum_{n=1}^\infty E^x[S_{n-1}<\tau_D;\ B(S_{n-1})] + E^xe_q(\tau_D)+1\Big).$$

Hence, it remains to estimate $\sum_{n=1}^\infty E^x[S_{n-1}<\tau_D;\ B(S_{n-1})]$. Let $n\ge 2$. If $T_{n-1}<\tau_D$ then $X_{T_{n-1}}\in\overline{B(y,\delta_1)}$ and, by (4), $E^{X_{T_{n-1}}}B(\tau_U)\le 1$, as in

estimating the last term in (9). Thus, we have

$$E^x[S_{n-1}<\tau_D;\ B(S_{n-1})]$$
$$\le E^x[T_{n-1}<\tau_D;\ B(T_{n-1})] + E^x[T_{n-1}<\tau_D;\ e_q(T_{n-1})E^{X_{T_{n-1}}}[B(\tau_U)]]$$
$$\le E^x[T_{n-1}<\tau_D;\ B(T_{n-1})] + E^x[T_{n-1}<\tau_D;\ e_q(T_{n-1})].$$

Using Corollary 4.4 and Lemma 4.5 and adding one more nonnegative term we further estimate the above by the following expressions:

$$CE^x[T_{n-1}<\tau_D;\ B(T_{n-1})P^{X_{T_{n-1}}}\{\tau_U=\tau_D\}]$$
$$+CE^x[T_{n-1}<\tau_D;\ e_q(T_{n-1})E^{X_{T_{n-1}}}[\tau_U=\tau_D;\ B(\tau_U)]]$$
$$+CE^x[T_{n-1}<\tau_D;\ e_q(T_{n-1})E^{X_{T_{n-1}}}[\tau_U=\tau_D;\ e_q(\tau_U)]]$$
$$= CE^x[T_{n-1}<\tau_D,\ S_{n-1}=\tau_D;\ B(S_{n-1})]$$
$$+CE^x[T_{n-1}<\tau_D,\ S_{n-1}=\tau_D;\ e_q(S_{n-1})].$$

Observe finally that the sum over $n\ge 2$ of the first terms on the right-hand side is estimated from above by $CE^xB(\tau_D)$, which is bounded, as already explained, while the sum of the second terms is estimated by $CE^xe_q(\tau_D)$, which is obviously also bounded. This completes the proof of (8).

It is not difficult to verify that $T_s|q|G(\cdot,y)(x)<\infty$, because $q\in\mathcal{J}^\alpha_{\mathrm{loc}}$. This justifies the identity

$$T_sqG(\cdot,y)(x) = G(x,y)E_y^x[s<\zeta;\ e_q(s)q(X_s)].$$

This and (8) yield, by Fubini's theorem,

$$VqG(\cdot,y)(x) = \int_0^\infty T_sqG(\cdot,y)(x)\,ds$$
$$= G(x,y)E_y^x\Big[\int_0^\zeta e_q(s)q(X_s)\,ds\Big] = G(x,y)E_y^x[e_q(\zeta)-1]$$
$$= G(x,y)u(x,y) - G(x,y),$$

which gives (7) and completes this step.

Step 3. In this step we remove the condition $d(x,D^c)>3\delta$ imposed on $x\in D$ in (5).

To do this, assume that $y\in D_\delta$ but $d(x,D^c)\le 3\delta$. Let U^δ be as in Step 1. Define $U=U^\delta\setminus\{y\}$. Then

$$u(x,y) = E_y^x[\tau_U=\zeta;\ e_q(\tau_U)] + E_y^x[\tau_U<\zeta;\ e_q(\tau_U)u(X_{\tau_U},y)]$$
$$\le E_y^xe_{|q|}(\tau_U)(1+\sup_{w\in D_\delta,\,w\ne y}u(w,y)).$$

By Step 1 and properties of U^δ we obtain the conclusion.

STEP 4. In this step we prove that $u(x,y)$ is symmetric for $x,y\in D$, $x\neq y$.

It is enough to show that $VqG(\cdot,y)(x)$ is symmetric in $x,y\in D$. Indeed, as $G(x,y)$ is symmetric, the identity (7) yields the same property for u. By (8) and the definition of V we only have to verify that for each n the following function is symmetric in $x,y\in D$:

$$E^x\Big[\int_0^{\tau_D} A(s)^n q(X_s)G(X_s,y)\,ds\Big],$$

where $A(s)=\int_0^s q(X_t)\,dt$. We remark at this point that although the proof is similar to that of the symmetry of the Feynman–Kac semigroup, the above property is much more delicate and requires an independent justification.

We first state an auxiliary result: for each $n\geq 0$ and each collection of positive numbers $t,r_1,\ldots,r_n,s$ with $r_1<\ldots<r_n<s$,

$$\begin{aligned}(12)\quad & E^x[s<\tau_D;\ q(X_{r_1})\ldots q(X_{r_n})q(X_s)p_t^D(X_s,y)]\\ &= E^y[s+t-r_1<\tau_D;\ q(X_t)q(X_{s+t-r_n})\ldots q(X_{s+t-r_1})p_{r_1}^D(X_{s+t-r_1},x)].\end{aligned}$$

Here $p_t^D(w,v)=p_t^D(v,w)$, $w,v\in D$, denotes the transition density of the process X_t killed on exiting D. The equality (12) follows easily by the successive application of the Markov property and the symmetry of p_r^D. We omit the details.

We next observe that

$$\begin{aligned}(13)\quad & E^x[s<\tau_D;\ A(s)^n q(X_s)p_t^D(X_s,y)]\\ &= E^x\Big[s<\tau_D;\ \int_0^s q(X_{r_1})\,dr_1\ldots\int_0^s q(X_{r_n})\,dr_n q(X_s)p_t^D(X_s,y)\Big]\\ &= n!E^x\Big[s<\tau_D;\ \int_0^s q(X_{r_1})\int_{r_1}^s q(X_{r_2})\ldots\\ &\qquad\ldots\int_{r_{n-1}}^s q(X_{r_n})q(X_s)p_t^D(X_s,y)\,dr_n\ldots dr_1\Big]\\ &= n!\int_0^s\int_{r_1}^s\ldots\int_{r_{n-1}}^s E^x[s<\tau_D;\ q(X_{r_1})\ldots\\ &\qquad\ldots q(X_{r_n})q(X_s)p_t^D(X_s,y)]\,dr_n\ldots dr_1.\end{aligned}$$

We integrate both sides of (13) with respect to s from 0 to ∞. Taking into account (12) and introducing new variables: $r_1'=r_1$, $r_2'=r_2-r_1,\ldots,r_n'=r_n-r_1$, $s'=s+t-r_1$ we obtain

$$\begin{aligned}(14)\quad & E^x\Big[\int_0^{\tau_D} A(s)^n q(X_s)p_t^D(X_s,y)\,ds\Big]\\ &= n!\int_0^\infty\int_0^\infty\int_{r_2'}^\infty\ldots\int_{r_{n-1}'}^\infty\int_{r_n'+t}^\infty E^y[s'<\tau_D;\ q(X_t)q(X_{s'-r_n'})\ldots\\ &\qquad\ldots q(X_{s'-r_2'})q(X_{s'})p_{r_1'}^D(X_{s'},x)]\,ds'\,dr_n'\ldots dr_2'\,dr_1'\\ &= n!\int_0^\infty\int_{r_2'}^\infty\ldots\int_{r_{n-1}'}^\infty\int_{r_n'+t}^\infty E^y[s'<\tau_D;\ q(X_t)q(X_{s'-r_n'})\ldots\\ &\qquad\ldots q(X_{s'-r_2'})q(X_{s'})G(X_{s'},x)]\,ds'\,dr_n'\ldots dr_2'.\end{aligned}$$

Integrating (14) with respect to t from 0 to ∞ we finally obtain

$$\begin{aligned}& E^x\Big[\int_0^{\tau_D} A(s)^n q(X_s)G(X_s,y)\,ds\Big]\\ &= n!\int_0^\infty\int_0^{s'}\int_{r_2'}^{s'}\ldots\int_{r_{n-1}'}^{s'} E^y\Big[s'<\tau_D;\ \int_0^{s'-r_n'} q(X_t)\,dt\,q(X_{s'-r_n'})\ldots\\ &\qquad\ldots q(X_{s'-r_2'})q(X_{s'})G(X_{s'},x)\Big]\,dr_n'\ldots dr_2'\,ds'\\ &= n!\int_0^\infty\int_0^{s'}\int_{r_2'}^{s'}\ldots\int_{r_{n-1}'}^{s'} E^y[s'<\tau_D;\ A(s'-r_n')q(X_{s'-r_n'})\ldots\\ &\qquad\ldots q(X_{s'-r_2'})q(X_{s'})G(X_{s'},x)]\,dr_n'\ldots dr_2'\,ds'\\ &= \frac{1}{2}n!\int_0^\infty\int_0^{s'}\int_{r_2'}^{s'}\ldots\int_{r_{n-2}'}^{s'} E^y[s'<\tau_D;\ A(s'-r_{n-1}')^2 q(X_{s'-r_{n-1}'})\ldots\\ &\qquad\ldots q(X_{s'-r_2'})q(X_{s'})G(X_{s'},x)]\,dr_{n-1}'\ldots dr_2'\,ds'\\ &= \ldots = \int_0^\infty E^y[s'<\tau_D;\ A(s')^n q(X_{s'})G(X_{s'},x)]\,ds'\\ &= E^y\Big[\int_0^{\tau_D} A(s')^n q(X_{s'})G(X_{s'},x)\,ds'\Big].\end{aligned}$$

This ends the proof of this step.

STEP 5. In this step we apply the symmetry of $u(x,y)$ in $x,y\in D$ to finish the proof of the boundedness of u.

Observe that the symmetry of u along with Step 3 settles the case when $x\in D_\delta$ and $y\in D_\delta^c$. It remains to consider the case when $x\neq y$, $x,y\in D_\delta^c$.

To resolve this case, we proceed exactly as in Step 3 to obtain

$$u(x,y)=E_y^x[\tau_U=\zeta;\ e_q(\tau_U)]+E_y^x[\tau_U<\zeta;\ e_q(\tau_U)u(X_{\tau_U},y)].$$

If $\tau_U<\zeta$ then $d(X_{\tau_U},D^c)>3\delta$, which reduces the proof to the case $x\in D_\delta$, $y\in D_\delta^c$; see also (4). By Step 3 and symmetry of u we obtain the conclusion. This completes the proof of the theorem.

So far, the conditional gauge $u(x,y)=E_y^x e_q(\tau_{D\setminus\{y\}})$ has only been defined for $(x,y)\in D\times\overline{D}$, $x\neq y$, and proved to be bounded for $(x,y)\in D\times D$, $x\neq y$. The next theorem contains a refinement of this result.

THEOREM 4.10. *Under the assumptions of Theorem 4.9 the function u has a jointly continuous symmetric extension to $\overline{D}\times\overline{D}$ denoted also by u. We have $0<C_1\le u(x,y)\le C_2<\infty$ and $u(x,x)=1$ for $x,y\in\overline{D}$. Furthermore,*

$$V(x,y)=G(x,y)u(x,y)\quad \textit{for } x,y\in\overline{D},\ x\neq y. \tag{15}$$

For $x\in D$, $\eta\in\partial D$, we have

$$u(x,\eta)=1+\int_D\frac{G(x,w)K(w,\eta)}{K(x,\eta)}u(x,w)q(w)\,dw. \tag{16}$$

For $\xi\in\partial D$, $\eta\in\partial D$, $\xi\neq\eta$, we obtain

$$u(\xi,\eta)=1+\int_D\frac{K(w,\xi)K(w,\eta)}{H(\xi,\eta)}u(\xi,w)q(w)\,dw, \tag{17}$$

where

$$H(\xi,\eta)=\lim_{D\ni x\to\xi}\frac{K(x,\eta)}{G(x,x_0)}.$$

Proof. Using the formulas (3) and (7) we obtain, for each $x\in D$,

$$V(x,y)=G(x,y)u(x,y)\quad\text{for almost all } y\in D. \tag{18}$$

To verify (16) we first check that $V|q|K(\cdot,\eta)(x)<\infty$ for $x\in D$ and $\eta\in\partial D$. Applying (18), Theorem 4.9 and Corollary 4.1 we obtain

$$\begin{aligned}K(x,\eta)^{-1}V|q|K(\cdot,\eta)(x)&=K(x,\eta)^{-1}\int_D V(x,y)|q(y)|K(y,\eta)\,dy\\&=\int_D u(x,y)|q(y)|\frac{G(x,y)K(y,\eta)}{K(x,\eta)}\,dy\\&\le\sup_{x,y\in D,\,x\neq y}u(x,y)C\sup_{x,z\in\overline{D}}\int_D|q(y)|[K_\alpha(x,y)+K_\alpha(y,z)]\,dy<\infty.\end{aligned}$$

This enables us to apply Fubini's theorem in calculating $VqK(\cdot,\eta)(x)$:

$$\begin{aligned}VqK(\cdot,\eta)(x)&=\int_0^\infty T_sqK(\cdot,\eta)(x)\,ds=K(x,\eta)E_\eta^x\Big[\int_0^{\tau_D}e_q(s)q(X_s)\,ds\Big]\\&=K(x,\eta)E_\eta^x[e_q(\tau_D)-1]=K(x,\eta)u(x,\eta)-K(x,\eta).\end{aligned}$$

This proves (16). Now, applying (18) we write the identity (7) as

$$u(x,y)=1+\int_D\frac{G(x,w)G(w,y)}{G(x,y)}u(x,w)q(w)\,dw, \tag{19}$$

where $x,y\in D$, $x\neq y$. Then, using symmetry of u and iterating (19) we obtain

$$\begin{aligned}u(x,y)&=1+\int_D\frac{G(x,w)G(w,y)}{G(x,y)}q(w)\,dw\\&\quad+\int_D\int_D\frac{G(x,w)G(w,y)}{G(x,y)}\cdot\frac{G(w,v)G(v,x)}{G(w,x)}u(w,v)q(w)q(v)\,dw\,dv.\end{aligned}$$

In view of boundedness of u and the 3G Theorem, Fubini's theorem shows that the integrands are integrable uniformly in (x,y) over D and $D\times D$, respectively. Clearly, they are continuous in (x,y) for almost all w, (w,v), respectively. This yields the continuity of $u(x,y)$ and the equality in (15) for $(x,y)\in D\times D$, $x\neq y$.

In what follows we assume that $D\times D\ni(x,y)\to(\xi,\eta)\in\overline{D}\times\overline{D}$ so that $x\neq y$. By a simple topological argument, to prove the existence of a continuous extension of u it is enough to verify that $\lim u(x,y)$ exists. To this end we again invoke the uniform integrability and convergence of the integrand in (19). The proof proceeds by inspecting various cases.

CASE 1: $\xi,\eta\in D$, $\xi\neq\eta$. Clearly this case was settled above.

CASE 2: $\xi=\eta\in\overline{D}$. By the 3G Theorem we obtain

$$\frac{G(x,w)G(w,y)}{G(x,y)}\le C\frac{|x-y|^{d-\alpha}}{|x-w|^{d-\alpha}|w-y|^{d-\alpha}},$$

hence the integrand in (19) tends to 0 almost everywhere, so $u(x,y)\to1$.

CASE 3: $\xi\in D$, $\eta\in\partial D$. By the definition of the Martin kernel, Case 1, (19) and (16) we obtain

$$u(x,y)\to1+\int_D\frac{G(\xi,w)K(w,\eta)}{K(\xi,\eta)}u(\xi,w)q(w)\,dw=u(\xi,\eta).$$

CASE 4: $\xi\in\partial D$, $\eta\in D$. Since $u(x,y)=u(y,x)$, this reduces to the previous case.

CASE 5: $\xi, \eta \in \partial D$, $\xi \neq \eta$. We first note that $H(\xi,\eta)$ exists by BHP. Let $r > 0$ be sufficiently small (see [B1]). By BHP for $y \in D \cap B(\eta, r)$ and BHP for $x \in D \cap B(\xi, r)$ there are constants C and ν such that

$$\left|\frac{G(x,y)}{G(x,x_0)G(x_0,y)} - \frac{K(x,\eta)}{G(x,x_0)}\right| \le C\left|\frac{y-\eta}{r}\right|^\nu \frac{K(x,\eta)}{G(x,x_0)} \le C^2\left|\frac{y-\eta}{r}\right|^\nu H(\xi,\eta) \to 0,$$

so that

$$\lim \frac{G(x,y)}{G(x,x_0)G(x_0,y)} = \lim \frac{K(x,\eta)}{G(x,x_0)} = H(\xi,\eta).$$

As usual, x_0 is the reference point in the definition of the Martin kernel. For arbitrary $w \in D$ we obtain

$$\frac{G(x,w)G(w,y)}{G(x,y)} = \frac{G(x,w)}{G(x,x_0)} \cdot \frac{G(w,y)}{G(x_0,y)} \cdot \frac{G(x,x_0)G(x_0,y)}{G(x,y)} \to K(w,\xi)K(w,\eta)H(\xi,\eta).$$

By (19) and Case 4 we conclude that $u(x,y) \to u(\xi,\eta)$, where $u(\xi,\eta)$ is defined by (17).

The existence of a continuous extension of u is justified. The lower boundedness of u follows from Jensen's inequality. Namely, for $x, y \in D$, $x \neq y$, by the 3G Theorem we have $u(x,y) \ge \exp(-C)$, where

$$C = \sup_{x,y\in D,\, x\neq y} \int_D |q(w)|(G(x,w)G(w,y)/G(x,y))\,dw < \infty.$$

The symmetry of (the extended) u is obvious. By continuity, (15) follows from (18).

5. q-harmonic functions and their representation

DEFINITION 5.1. Let u be a Borel measurable function on $\mathbb{R}^d$ and let $q \in \mathcal{J}^\alpha_{\rm loc}$. We say that u is *q-harmonic* in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = E^x[e_q(\tau_U)u(X_{\tau_U})], \quad x \in U, \tag{1}$$

for every bounded open set U with $\overline{U} \subseteq D$. It is called *regular q-harmonic* in D if (1) holds for $U = D$, and *singular q-harmonic* in D if it is q-harmonic in D and $u(x) = 0$ for $x \in D^c$.

We understand that the expectation in (1) is absolutely convergent. If D is unbounded then by the usual convention we have $E^x[e_q(\tau_D)u(X_{\tau_D})] = E^x[\tau_D < \infty;\ e_q(\tau_D)u(X_{\tau_D})]$. The equality in (1) trivially holds for $x \in U^c$ and so it holds in the whole of $\mathbb{R}^d$ provided it does in U. For $q \equiv 0$ we obtain the previous definition of α-harmonicity. By the strong Markov property of X_t a regular q-harmonic function u is necessarily q-harmonic. The converse is not generally true (see Section 6 below).

The main objective of this section is to identify q-harmonic functions as solutions of the equation $S^\alpha u = 0$. A corresponding analysis of the classical Schrödinger operator is given in [ChZ]; however, our methods are substantially different and some of the results (e.g. Theorem 5.3) are more complete.

Let D be a bounded Lipschitz domain and let $q \in \mathcal{J}^\alpha_{\rm loc}$ be such that (D,q) is gaugeable. Let f be a Borel function on D^c. For $x \in \mathbb{R}^d$ we write

$$F(x) = E^x f(X_{\tau_D}) \quad \text{and} \quad F_q(x) = E^x[e_q(\tau_D)f(X_{\tau_D})].$$

LEMMA 5.2. *F_q is well defined and finite for some and therefore all $x \in D$ if and only if so is F. In this case F_q is regular q-harmonic in D and locally bounded on D. If also $\lim f(y)$ exists as $D^c \ni y \to \xi \in \partial D$ then F_q is continuous at ξ. If f is nonnegative then*

$$C_1F(x) \le F_q(x) \le C_2F(x), \quad x \in \mathbb{R}^d, \tag{2}$$

where C_1, C_2 are the constants from Theorem 4.10. If also $0 \not\equiv F \not\equiv \infty$ and $f = 0$ on $D^c \cap B(\xi,r)$ for some $\xi \in \partial D$, $r > 0$, then $\lim F_q(x)/F(x)$ exists as $D \ni x \to \xi$. Moreover, $F_q(x)/F(x)$ extends continuously to $\overline{D} \cap B(\xi,r)$.

Proof. Observe that for $f \ge 0$ the inequalities in (2) for $x \in D$ are immediate consequences of (2.17) and CGT. Outside D we have $F_q = F$. Since $C_1 \le 1 \le C_2$, the inequalities in (2) hold in the whole of $\mathbb{R}^d$. The Harnack inequality for nonnegative α-harmonic functions [La] and (2) justify the first two statements of the lemma for nonnegative f. For arbitrary f, we consider appropriate functions F_q, F defined by $|f|$.

We now prove the last two statements of the lemma. Fix $x_0 \in D$. If $D \ni x \neq x_0$ then by (2.17),

$$\frac{F_q(x)}{F(x)} = \frac{\int_{D^c}\int_D u(x,v)\frac{G(x,v)}{G(x,x_0)}\left(\frac{1+|y|}{|v-y|}\right)^{d+\alpha} dv\,d\mu(y)}{\int_{D^c}\int_D \frac{G(x,v)}{G(x,x_0)}\left(\frac{1+|y|}{|v-y|}\right)^{d+\alpha} dv\,d\mu(y)},$$

where $d\mu(y) = (f(y)/(1+|y|)^{d+\alpha})\,dy$ is a finite measure on D^c. The integrands in the numerator and denominator are integrable with respect to $dv\,d\mu(y)$ uniformly in x (cf. [B2], the proof of Lemma 6). Letting $D \ni x \to \xi \in \partial D$ we get

$$\frac{F_q(x)}{F(x)} \to \frac{\int_{D^c}\int_D u(\xi,v)f(y)K(v,\xi)|v-y|^{-d-\alpha}\,dv\,dy}{\int_{D^c}\int_D f(y)K(v,\xi)|v-y|^{-d-\alpha}\,dv\,dy}.$$

By a simple topological argument it follows that the continuous extension of the quotient $F_q(x)/F(x)$ to $\overline{D}\cap B(\xi,r)$ exists. In particular $F_q(x)\to 0$ as $x\to\xi$ because so does F (see [B1], Lemma 3).

Let now f be arbitrary such that F_q is well defined and finite on D. Let $\xi\in\partial D$ and assume that $\lim f(y)=g$ exists as $D^c\ni y\to\xi$. We claim that F_q is continuous at ξ. If $f=0$ on $D^c\cap B(\xi,r)$ for some $r>0$, then $F_q(x)\to 0$ as $x\to\xi$, by the previous part of the proof. Therefore to prove our claim we may and do assume that $f(x)=0$ for $x\in D^c\setminus B(\xi,r)$, and $g-\varepsilon\le f(x)\le g+\varepsilon$ if $x\in D^c\cap B(\xi,r)$ with small fixed $\varepsilon>0$ and $r>0$. By (2.17) and (2.14) for $x\in D$ we have

$$F_q(x)=\int_D\int_{D^c} f(y)u(x,v)\mathcal{A}(d,-\alpha)G(x,v)|v-y|^{-d-\alpha}\,dy\,dv.$$

Let $D\cap B(\xi,r)\ni x\to\xi$. As $G(x,v)\to 0$, we obtain

$$\int_{D\setminus B(\xi,2r)}\int_{D^c} f(y)u(x,v)\mathcal{A}(d,-\alpha)G(x,v)|v-y|^{-d-\alpha}\,dy\,dv\to 0.$$

If $g>0$ we have

$$\begin{aligned}\limsup F_q(x)&=\limsup\int_{B(\xi,2r)}\int_{D^c} f(y)u(x,v)\frac{\mathcal{A}(d,-\alpha)G(x,v)}{|v-y|^{d+\alpha}}\,dy\,dv\\&\le(g+\varepsilon)\sup\{u(x,v):x,v\in D\cap B(\xi,r)\}\limsup\omega_D^x(B(\xi,r)).\end{aligned}$$

Since $\omega_D^x(B(\xi,r))\to 1$ (cf. [B1], Lemma 10), and ε and r may be arbitrarily small, by Theorem 4.10 we get $\limsup F_q(x)\le g$. Similarly $\liminf F_q(x)\ge g$. The cases $g=0$ and $g<0$ are left to the reader. This ends the proof.

By Lemma 5.2 every q-harmonic function u is bounded by an α-harmonic function. In particular $u\in\mathcal{L}^1=\mathcal{L}^1(dx/(1+|x|)^{d+\alpha})$.

The next theorem provides an important representation formula for q-harmonic functions.

THEOREM 5.3. *Let $D\subseteq\mathbb{R}^d$ be a bounded Lipschitz domain and let $q\in\mathcal{J}^\alpha_{\mathrm{loc}}$. Assume that (D,q) is gaugeable and let f be a regular q-harmonic function in D. Then f is continuous in D and for every open set $U\subseteq D$ and all $x\in\mathbb{R}^d$,*

$$f(x)=E^xf(X_{\tau_U})+G_U(qf)(x).\tag{3}$$

Proof. Let $\widetilde{f}(x)=E^x[e_q(\tau_D)|f(X_{\tau_D})|]$, $x\in\mathbb{R}^d$. By Lemma 5.2, $\widetilde{f}$ is regular q-harmonic in D and locally bounded on D. Clearly $|f|\le\widetilde{f}$.

We first verify (3) for arbitrary open Lipschitz U precompact in D. We put

$$\Phi(t)=\mathbf{1}_{\{t<\tau_U\}}q(X_t)f(X_{\tau_U})\exp\int_t^{\tau_U}q(X_s)\,ds,$$

$$\Psi(t)=\mathbf{1}_{\{t<\tau_U\}}|q(X_t)|\widetilde{f}(X_{\tau_U})\exp\int_t^{\tau_U}q(X_s)\,ds.$$

Observe that the Markov property yields

$$\begin{aligned}\int_0^\infty E^x[\Phi(t)]\,dt&=E^x\Big[\int_0^{\tau_U}q(X_t)E^{X_t}[e_q(\tau_U)f(X_{\tau_U})]\,dt\Big]\\&=E^x\Big[\int_0^{\tau_U}q(X_t)f(X_t)\,dt\Big]=G_U(qf)(x),\end{aligned}$$

provided we are able to justify the application of Fubini's theorem. This, however, follows by calculations as above, applied to Ψ and $\widetilde{f}$ instead of Φ and f:

$$\int_0^\infty E^x[\Psi(t)]\,dt\le CE^x\Big[\int_0^{\tau_U}|q(X_t)|\,dt\Big]=CG_U(|q|)(x)<\infty.$$

At the same time we obtain

$$\begin{aligned}\int_0^\infty E^x[\Phi(t)]\,dt&=E^x[\{e_q(\tau_U)-1\}f(X_{\tau_U})]\\&=E^x[e_q(\tau_U)f(X_{\tau_U})]-E^xf(X_{\tau_U})=f(x)-E^xf(X_{\tau_U}).\end{aligned}$$

This shows (3) for U as above. In particular, f is continuous in D and, by Proposition 3.13, $\widetilde{\Delta}^{\alpha/2}f=-qf$, that is, $S^\alpha f=0$ in D.

Let $\widetilde{F}(x)=E^x|f(X_{\tau_D})|$. By Lemma 5.2, $|f(x)|\le\widetilde{f}(x)\le C\widetilde{F}(x)$ for $x\in\mathbb{R}^d$, and $\widetilde{F}$ is regular α-harmonic in D.

We now prove (3) for $U=D$. Let $U_n=\{x\in D:\mathrm{dist}(x,D^c)>1/n\}$. For n sufficiently large, U_n is a Lipschitz domain and (3) holds with $U=U_n$. Fix $x\in D$. It is well known that $(\widetilde{F}(X_{\tau_{U_n}}),\mathcal{F}_{\tau_{U_n}})$ is a P^x-martingale closed by $\widetilde{F}(X_{\tau_D})$. In particular $(f(X_{\tau_{U_n}}))$ are uniformly integrable. Since $P^x(X_{\tau_{U_n}}=X_{\tau_D})\to 1$ as $n\to\infty$ ([B1], Lemma 17), we find that $E^xf(X_{\tau_{U_n}})\to E^xf(X_{\tau_D})$ as $n\to\infty$. It is also well known that $G_{U_n}(x,y)\uparrow G_D(x,y)$ as $n\to\infty$ for all $x,y\in D$. By bounded convergence, $G_{U_n}(qf)(x)\to G_D(qf)(x)$. Indeed, it is enough to verify that $G_D(|q|\widetilde{F})(x)<\infty$. But by (2.15) and the 3G Theorem we obtain

$$\begin{aligned}&\int_D G(x,z)|q(z)|\widetilde{F}(z)\,dz\\&\quad=\int_D dz\int_{D^c}dy\int_D G(x,z)G(z,v)|q(z)|\frac{\mathcal{A}(d,-\alpha)}{|v-y|^{d+\alpha}}|f(y)|\,dv\end{aligned}$$

$$= \int_D dv \int_{D^c} dy \frac{\mathcal{A}(d,-\alpha)}{|v-y|^{d+\alpha}} G(x,v)|f(y)| \int_D \frac{G(x,z)G(z,v)}{G(x,v)} |q(z)|\,dz$$
$$\le C \int_D dv \int_{D^c} \frac{\mathcal{A}(d,-\alpha)}{|v-y|^{d+\alpha}} G(x,v)|f(y)|\,dy = C\widetilde{F}(x) < \infty.$$

Thus we get (3) for $U = D$.

For arbitrary open $U \subset D$ a similar argument works except that $P^x(X_{\tau_{U_n}} = X_{\tau_U}) \to 1$ as $n \to \infty$ may fail for open Lipschitz U_n which are precompact in D and approximate U. However, by left quasi-continuity, $X_{\tau_{U_n}} \to X_{\tau_U}$ almost surely as $n \to \infty$. If $X_{\tau_{U_n}} \ne X_{\tau_U}$ for all n, then $X_{\tau_U} \in D$ almost surely. By continuity of f on D we have $f(X_{\tau_{U_n}}) \to f(X_{\tau_U})$ as $n \to \infty$, which yields $E^x f(X_{\tau_{U_n}}) \to E^x f(X_{\tau_U})$ as before. The arguments for the Green function do not change as $G_U(|q|\widetilde{F})(x) \le G_D(|q|\widetilde{F})(x) < \infty$.

The next lemma gives a characterization of regular α-harmonic functions.

Lemma 5.4. *Let D be a bounded domain in $\mathbb{R}^d$ with the exterior cone property and let $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$. Let f be q-harmonic in D. Then f is regular q-harmonic in D if and only if $|f|$ is bounded by a function which is regular q-harmonic in D. If $f \ge 0$ then it is sufficient that $|f|$ be bounded over D. If D is additionally Lipschitz and (D,q) is gaugeable then the majorants above may be required to be α-harmonic, instead of q-harmonic in D.*

Proof. If f is regular q-harmonic in D then for $x \in \mathbb{R}^d$ we obtain $|f(x)| \le E^x[e_q(\tau_D)|f(X_{\tau_D})|]$, the right-hand side being a nonnegative regular q-harmonic function in D. Let D_n be a nondecreasing sequence of open subsets of D with $\bigcup D_n = D$ and $\overline{D}_n \subseteq D$. Let $x \in \mathbb{R}^d$. Then $P^x\{X_{\tau_D} = X_{\tau_{D_n}}\} \to 1$ as $n \to \infty$ ([B1], Lemma 17). We have

$$\begin{aligned} f(x) &= E^x[e_q(\tau_{D_n}) f(X_{\tau_{D_n}});\ X_{\tau_{D_n}} \in D^c] \\ &\quad + E^x[e_q(\tau_{D_n}) f(X_{\tau_{D_n}});\ X_{\tau_{D_n}} \in D \setminus D_n] \\ &= E^x[e_q(\tau_D) f(X_{\tau_D});\ X_{\tau_{D_n}} \in D^c] \\ &\quad + E^x[e_q(\tau_{D_n}) f(X_{\tau_{D_n}});\ X_{\tau_{D_n}} \in D \setminus D_n]. \end{aligned}$$

As $n \to \infty$, the last term converges to 0 by the boundedness assumption and uniform integrability. By bounded convergence,

$$f(x) = \lim_{n\to\infty} E^x[e_q(\tau_D) f(X_{\tau_D});\ X_{\tau_{D_n}} \in D^c] = E^x[e_q(\tau_D) f(X_{\tau_D})].$$

If $f \ge 0$ then we need only use the monotone convergence theorem.

By (2) we can equivalently assume that f is bounded by a (nonnegative) regular α-harmonic function if D is a bounded gaugeable Lipschitz domain.

Theorem 5.5. *Let U be an open set in $\mathbb{R}^d$ and let $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$. If f is a q-harmonic function in U then $S^\alpha f = 0$ in U. Conversely, assume that U is bounded and (U,q) is gaugeable. If a function f satisfies $S^\alpha f = 0$ in U then after a modification on a set of Lebesgue measure zero, f is q-harmonic in U.*

Proof. By the first part of the proof of Theorem 5.3, S^α annihilates q-harmonic functions. To prove the converse, let $S^\alpha f = 0$ in U. Let (U,q) be gaugeable and let D be open, Lipschitz and relatively compact in U. Clearly, (D,q) is gaugeable. Let $h^D = f - G_D(qf)$. By the conditions in Definition 3.14, h^D is well defined and Proposition 3.13 yields $\widetilde{\Delta}^{\alpha/2} h^D = -qf + qf = 0$ in D. By Theorem 3.12 we may and do modify h^D on a set of Lebesgue measure 0 to be α-harmonic in D.

Suppose that f is continuous in U. It follows that $G_D(qf) \in C_0(D)$. Consequently, h^D is bounded in a neighborhood of $\overline{D}$ and $h^D \in \mathcal{L}^1$. Lemma 5.2 with $q \equiv 0$ yields that h^D is regular α-harmonic in D (see also the proof of Theorem 3.9). Since $G_D(qf)(X_{\tau_D}) = 0$, we obtain

$$f(x) = E^x f(X_{\tau_D}) + G_D(qf)(x), \qquad x \in \mathbb{R}^d.$$

Let $v(x) = E^x[e_q(\tau_D) f(X_{\tau_D})]$, $x \in \mathbb{R}^d$. By Lemma 5.2, v is regular q-harmonic in D. We observe that $w = f - v \in C_0(D)$ by Lemma 5.2 and continuity of f in a neighborhood of $\overline{D}$. We also have $S^\alpha w = 0$ in D. Applying Theorem 3.15 we obtain $w \equiv 0$.

We now remove the a priori assumption that f is continuous in U (cf. [ChZ], Theorem 5.21). Let B be a (nonempty open) ball relatively compact in D. By the first part of the proof we also have $f = h^B + G_B(qf)$ a.e. with h^B α-harmonic in B. By the strong Markov property we have $G_D(qf)(x) = G_B(qf)(x) + E^x G_D(qf)(X_{\tau_B})$. The equality holds in $L^1(\mathbb{R}^d)$ (in particular a.e.) and the integrals are absolutely convergent for almost every x (see also [ChZ]). We thus have a.e.

$$\begin{aligned} h^B(x) - h^D(x) &= G_D(qf)(x) - G_B(qf)(x) = E^x(f - h^D)(X_{\tau_B}) \\ &= E^x f(X_{\tau_B}) - h^D(x). \end{aligned}$$

But this implies that $E^x|f(X_{\tau_B})| < \infty$ for some and therefore all $x \in B$ and, by continuity, $h^B(x) = E^x f(X_{\tau_B})$ for $x \in B$. Thus h^B is regular α-harmonic in B and $f(x) = E^x f(X_{\tau_B}) + G_B(qf)(x)$. Let $u(x) = E^x[e_q(\tau_B) f(X_{\tau_B})]$. By Lemma 5.2, u is regular q-harmonic in B. Clearly $f = u$ on B^c. Theorem 5.3 applied to u yields $f - u = G_B(q(f-u))$. Let B be so small that $\|G_B|q|\|_\infty < 1$. Then $\|q(f-u)\|_1 \le \|G_B|q|\|_\infty \|q(f-u)\|_1$, which gives $q(f-u) = 0$ a.e. It follows that $f = u$ a.e. We conclude that f is locally essentially bounded (Lemma 5.2) in U. Hence it is essentially continuous in U and the proof is complete.

Theorem 5.6. *Let D be a bounded open subset of $\mathbb{R}^d$ with the exterior cone property and $q \in \mathcal{J}^\alpha_{\mathrm{loc}}$. Then there exists a nonnegative function u which* *is q-harmonic in D and satisfies the condition*

$$0 < \inf_{x \in \mathbb{R}^d} u(x) \le \sup_{x \in \mathbb{R}^d} u(x) < \infty$$

if and only if (D,q) is gaugeable.

Proof. If (D,q) is gaugeable, then the gauge function has the required properties. Conversely, assume that such a function u exists. As in the proof of Lemma 5.4, let D_n be a nondecreasing sequence of open subsets of D satisfying $D = \bigcup D_n$ and $\overline{D}_n \subseteq D$. By Fatou's lemma, for $x \in \mathbb{R}^d$ we have

$$\begin{aligned} u(x) &= E^x[e_q(\tau_D)u(X_{\tau_D}); X_{\tau_{D_n}} \in D^c] \\ &\quad + E^x[e_q(\tau_{D_n})u(X_{\tau_{D_n}}); X_{\tau_{D_n}} \in D \setminus D_n] \\ &\ge E^x[e_q(\tau_D)u(X_{\tau_D})]. \end{aligned}$$

It follows that the gauge $E^x e_q(\tau_D)$ is bounded from above in D. As D is Green-bounded, the gauge function is also bounded from below.

6. Potential theory for S^α. The purpose of this section is to prove some basic results on the potential theory of the Schrödinger operator S^α using CGT. We assume in the remainder that D is a bounded Lipschitz domain in $\mathbb{R}^d$, $d > \alpha$, $q \in \mathcal{J}^\alpha$, and (D,q) is gaugeable.

THEOREM 6.1 (Harnack Principle). *There are constants C_3 and C_4 such that for every ball $B(x,2r)$ with $r < C_4$ and every function $u \ge 0$ which is q-harmonic in $B(x,2r)$, we have*

$$u(y) \le C_3 u(x), \quad y \in B(x,r). \tag{1}$$

Proof. If $r > 0$ is sufficiently small, then by Khasminski's lemma we see that Lemma 5.2 holds for $(B(x,2r),q)$, $x \in \mathbb{R}^d$, with absolute constants e.g. $C_1 = 1/2$ and $C_2 = 2$. In the context of Lemma 5.2 we put $F_q = u$ and $F(y) = E^y u(X_{\tau_{B(x,3r/2)}})$, $x \in \mathbb{R}^d$. By the Harnack inequality for nonnegative functions α-harmonic in $B(x,3r/2)$ we obtain, for $y \in B(x,r)$,

$$u(y) = F_q(y) \le 2F(y) \le 2C(d)F(x) \le 4C(d)F_q(x) = 4C(d)u(x).$$

In a similar way we derive from [B1] the boundary Harnack principle for nonnegative q-harmonic functions and Lipschitz domains.

THEOREM 6.2 (Boundary Harnack Principle). *There exist constants C_5 and R such that for all $\xi \in \partial D$, $r \in (0, R/2)$ and nontrivial functions $u, v \ge 0$ which are q-harmonic in $D \cap B(\xi, 2r)$ and vanish continuously in $D^c \cap B(\xi, 2r)$ we have*

$$u(x)/v(x) \le C_5 u(y)/v(y), \quad x, y \in D \cap B(\xi, r), \tag{2}$$

and $\lim u(x)/v(x)$ *exists as $D \ni x \to \xi$.*

Proof. For $r > 0$ small enough every Lipschitz domain D_r such that $D \cap B(\xi, r/R) \subseteq D_r \subseteq D \cap B(\xi, r)$ with $R = R(D)$ (see [B1]) is gaugeable with the corresponding constants C_1/C_2 and C_2/C_1. By BHP and the Harnack chain inequality for α-harmonic functions [B1], the present result follows from Lemma 5.2.

Let us remark that BHP gives only relative estimates for q-harmonic functions in gaugeable Lipschitz domains. Individual estimates for growth properties of q-harmonic functions (e.g. the Carleson estimate) are simple consequences of Lemma 5.2 and the corresponding results for α-harmonic functions [B1].

We now give a representation theorem for nonnegative functions which are q-harmonic in the Lipschitz domain D. Let $x_0 \in D$ be fixed.

LEMMA 6.3. *For every $\xi \in \partial D$,*

$$K_q(x,\xi) = \lim_{D \ni y \to \xi} \frac{V(x,y)}{V(x_0,y)} \tag{3}$$

exists for all $x \in \mathbb{R}^d$. The mapping $(x,\xi) \mapsto K_q(x,\xi)$ is continuous in $D \times \partial D$. For every $\xi \in \partial D$ the function $K_q(\cdot,\xi)$ is singular q-harmonic in D with $K_q(x_0,\xi) = 1$. If $\xi, \eta \in \partial D$ and $\xi \ne \eta$ then $K_q(x,\xi) \to 0$ as $x \to \eta$.

Proof. The above lemma is proved in [B2] for $q \equiv 0$ and V replaced by G. By this result and Theorem 4.10 we obtain

$$\frac{V(x,y)}{V(x_0,y)} = \frac{u(x,y)}{u(x_0,y)} \cdot \frac{G(x,y)}{G(x_0,y)} \to \frac{u(x,\xi)}{u(x_0,\xi)} K(x,\xi),$$

where K is the Martin kernel (for α-harmonic functions) on D. Thus, we have obtained

$$K_q(x,\xi) = \frac{u(x,\xi)}{u(x_0,\xi)} K(x,\xi). \tag{4}$$

As the factor $u(x,\xi)/u(x_0,\xi)$ is bounded and continuous, we need only check that $K_q(x,\xi)$ is q-harmonic in D. If U is a Lipschitz domain relatively compact in D then $\tau_U < \zeta$ holds P^x_ξ a.s. By the strong Markov property (2.13), for $x \in U$ we obtain

$$\begin{aligned} E^x[e_q(\tau_U)K_q(X_{\tau_U},\xi)] &= u(x_0,\xi)^{-1}E^x[e_q(\tau_U)u(X_{\tau_U},\xi)K(X_{\tau_U},\xi)] \\ &= u(x_0,\xi)^{-1}K(x,\xi)E^x_\xi[e_q(\tau_U)e_q(\zeta)\circ\theta_{\tau_U}] = K_q(x,\xi). \end{aligned}$$

THEOREM 6.4 (Martin Representation). *For each finite Borel measure μ on ∂D the function*

$$f(x) = \int_{\partial D} K_q(x,\xi)\,\mu(d\xi) \tag{5}$$

is singular q-harmonic in D with $\mu(\mathbb{R}^d) = f(x_0)$. *Conversely, for each nonnegative function f which is singular q-harmonic in D there is a unique finite Borel measure* μ *on* ∂D *such that* (5) *holds.*

Proof. The first statement is an easy consequence of Lemma 6.3 and Fubini–Tonelli.

Let $f \geq 0$ be a singular q-harmonic function in D. We define $D_n = \{x \in D : \operatorname{dist}(x, D^c) > 1/n\}$. For n large enough each D_n is a Lipschitz domain with Lipschitz character essentially the same as D. Let V_n (resp. G_n) denote the q-Green (resp. Green) function for D_n, and $u_n(x,y)$ the conditional gauge for D_n. Since $V_n(x,y) \uparrow V(x,y)$ and $G_n(x,y) \uparrow G(x,y)$ we obtain $u_n(x,y) \to u(x,y)$ as $n \to \infty$, by (4.15) of Theorem 4.10.

We claim that the functions $\{u_n\}$ are equicontinuous. Indeed, the gauges u_{D_n} are uniformly bounded and by the proof of Theorem 4.9 the same is true for the conditional gauges u_n. Since the constants in the 3G Theorem depend on the domain only through its diameter and Lipschitz character, by the proof of Theorem 4.10 the functions u_n are equicontinuous. Clearly now $u_n(x,y) \to u(x,y)$ uniformly on their respective domains of definition.

We are ready for the selection of the representing measure μ for f. For $x \in D$ we have

$$\begin{aligned} f(x) &= E^x[e_q(\tau_{D_n}) f(X_{\tau_{D_n}})] \\ &= \int_{D\setminus D_n} \int_{D_n} \mathcal{A}(d,-\alpha) u_n(x,v) \frac{G_{D_n}(x,v)}{|v-y|^{d+\alpha}} f(y)\, dv\, dy \\ &= \int_{D_n} \frac{G_{D_n}(x,v)}{G_{D_n}(x_0,v)} u_n(x,v)\, \mu_n(dv), \end{aligned}$$

where

$$\mu_n(dv) = \Big(\int_{D\setminus D_n} \mathcal{A}(d,-\alpha) f(y) \frac{G_{D_n}(x_0,v)}{|v-y|^{d+\alpha}}\, dy \Big)\, dv.$$

By Lemma 5.2 and [B2], $\mu_n \Rightarrow \widetilde{\mu}$ (weakly), where $\widetilde{\mu}$ is a finite Borel measure supported by ∂D with $\widetilde{\mu}(\partial D) \leq \mathrm{const} \cdot f(x_0)$ and such that for $x \in D$,

$$\int_{D_n} \frac{G_{D_n}(x,v)}{G_{D_n}(x_0,v)}\, \mu_n(dv) \to \int_{\partial D} K(x,\xi)\, \widetilde{\mu}(d\xi).$$

By the uniform convergence of u_n it follows that

$$\int_{D_n} \frac{G_{D_n}(x,v)}{G_{D_n}(x_0,v)} u_n(x,v)\, \mu_n(dv) \to \int_{\partial D} K(x,\xi) u(x,\xi)\, \widetilde{\mu}(d\xi).$$

Taking $\mu(d\xi) = u(x_0,\xi)\widetilde{\mu}(d\xi)$, we obtain $u(x_0,v)\mu_n(dv) \Rightarrow \mu(dv)$ as $n \to \infty$, which together with (4) gives the representation (5). The uniqueness of (5) follows from the weak convergence.

The description of nonnegative q-harmonic functions is concluded in the following lemma.

LEMMA 6.5. *Every nonnegative function f which is q-harmonic in D has a unique representation*

$$f(x) = f_1(x) + f_2(x), \qquad x \in \mathbb{R}^d,$$

where the functions f_1, f_2 *are nonnegative regular and singular q-harmonic in D, respectively.*

The proof of Lemma 6.5 is analogous to the proof of the corresponding result (for $q \equiv 0$) in [B2] and is omitted.

Acknowledgments. The results of this paper were presented during the conference "Geometric Stochastic Analysis and Fine Properties of Stochastic Processes", March 23–27, 1998, MSRI, Berkeley. The authors are grateful to the organizers for this opportunity.

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Received March 23, 1998
Revised version June 29, 1998 (4072)

STUDIA MATHEMATICA 133 (1) (1999)

An almost nowhere Fréchet smooth norm on superreflexive spaces

by

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Abstract. Every separable infinite-dimensional superreflexive Banach space admits an equivalent norm which is Fréchet differentiable only on an Aronszajn null set.

Introduction. Every convex continuous function on a separable Banach space X is Gateaux differentiable on a dense G_δ-set by a theorem of Mazur. If the dual of X is separable it is even Fréchet differentiable on a dense G_δ-set. If we confine ourselves to the weaker notion of Gateaux differentiability, then locally Lipschitz functions, and in particular convex continuous functions, are also differentiable on a set which is large in the sense of measure.

The strongest present result in this direction is due to Mankiewicz [Man] and Aronszajn [A]. They defined in every separable Banach space a family $\mathcal{A}$ of sets which mimics the family of Lebesgue null sets in finite dimensions. The definitions of the family $\mathcal{A}$ (now usually called the Aronszajn null sets, see Section 2) used by Mankiewicz and by Aronszajn are formally different; it was recently shown by Csörnyei that they both coincide with the so called Gaussian null sets [C]. Mankiewicz and Aronszajn proved that every locally Lipschitz function is Gateaux differentiable almost everywhere, that is, except on a set belonging to $\mathcal{A}$. For Fréchet differentiability this fails except for finite dimensions, where the classical theorem of Rademacher is available. If X is a separable and infinite-dimensional Banach space then by a result of Preiss and Tišer [PT] there is a Lipschitz function f on X such that the set of points where f *is* Fréchet differentiable is Aronszajn null.

In [MM] it was shown that it is of no help to consider only convex continuous functions. There exists an equivalent norm p on the separable Hilbert space ℓ_2 such that the set of points where p *is* Fréchet differentiable

1991 *Mathematics Subject Classification*: Primary: 46B20, Secondary: 28C20.
Key words and phrases: Aronszajn null, convex, differentiable, Banach space.
The author was supported by J. Kepler Universität, and by the Czech Academy of Sciences grant GAAV-A1019705.