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On the directional entropy for $\mathbb{Z}^2$-actions on a Lebesgue space

by

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Abstract. We define the concept of directional entropy for arbitrary $\mathbb{Z}^2$-actions on a Lebesgue space, we examine its basic properties and consider its behaviour in the class of product actions and rigid actions.

1. Introduction. The concept of directional entropy was defined by Milnor [10] and applied to investigate the dynamics of cellular automata. Examples of the computation of this entropy are given in [11].

The directional entropy (topological and metric) was used by Boyle and Lind in [2] to study expansive $\mathbb{Z}^2$-actions.

We extend this concept to the class of arbitrary $\mathbb{Z}^2$-actions on a Lebesgue space. We are mainly interested in the properties of the directional entropy $h_{\vec{v}}(\Phi)$ as a function of $\vec{v}$ and Φ, where $\vec{v} \in \mathbb{R}^2$ and Φ is a $\mathbb{Z}^2$-action.

The second author has shown in [13] that the function $\vec{v} \to h_{\vec{v}}(\Phi)$ is upper semicontinuous for $\mathbb{Z}^2$-actions Φ generated by two automorphisms, one of which has a finite entropy and the second has a finite expected code length. In this paper we show that for product $\mathbb{Z}^2$-actions this function is Lipschitz and, on the other hand, it is not continuous for a certain rigid Gaussian $\mathbb{Z}^2$-action. In the class of product actions this function is also convex.

In the next section we show that the function $\Phi \to h_{\vec{v}}(\Phi)$ shares some properties of $\Phi \to h(\Phi)$ where $h(\Phi)$ denotes the usual entropy of Φ (cf. [3]). Among other things we show the analogue of the Kolmogorov–Sinai theorem. It appears, however, that the well known continuity property of the mean entropy $h(\Phi, P)$, where P is a countable measurable partition of X with finite entropy, does not hold for the mean directional entropy function $h_{\vec{v}}(\Phi, P)$.

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We also show, using Walters's idea, that for actions with discrete spectra the directional entropy $h_{\vec{v}}(\Phi)$, similarly to entropy, equals zero for every $\vec{v} \in \mathbb{R}^2$. On the other hand, in contrast to entropy, the directional entropy can be nonzero for rigid Gaussian $\mathbb{Z}^2$-actions.

We are also interested in applying the directional entropy to the computation of the relative entropy of automorphisms with respect to their factors determined by $\mathbb{Z}^2$-actions. More precisely, an automorphism T is said to be a *factor* of an automorphism $\widehat{T}$ acting on a space $(X, \mathcal{B}, \mu)$ determined by a $\mathbb{Z}^2$-action Φ if the fiber automorphisms of $\widehat{T}$ with respect to T have the form $\Phi^{\varphi(x)}$, $x \in X$, where φ is a measurable function from X to $\mathbb{Z}^2$.

It is shown in [14] that for Φ generated by cellular automata we have $h(\widehat{T}) = h(T) + h_{\vec{v}}(\Phi)$ where $\vec{v} = (|\int_X \varphi_1 \, d\mu|, |\int_X \varphi_2 \, d\mu|)$, $\varphi = (\varphi_1, \varphi_2)$. In this paper we show that this equality also holds for product actions and in this case it is an immediate consequence of the Newton formula [12] (see also [1]).

The second author has shown in [15] that the equality fails to be true in general.

2. Directional entropy of a $\mathbb{Z}^2$-action. Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space and let $\mathcal{Z}$ be the set of all countable measurable partitions of X with finite entropy equipped with the Rokhlin metric

$$\varrho(P, Q) = H(P|Q) + H(Q|P).$$

Let Φ be a $\mathbb{Z}^2$-action on $(X, \mathcal{B}, \mu)$. For a set $A \subset \mathbb{R}^2$ and $P \in \mathcal{Z}$ we put

$$P(A) = \bigvee_{g \in A \cap \mathbb{Z}^2} \Phi^g P.$$

Let $\vec{v} = (x, y)$ be a fixed vector of $\mathbb{R}^2$ and let Γ denote the family of all bounded subsets of $\mathbb{R}^2$. Let (T, S) be an ordered pair of commuting automorphisms of X which generate Φ, i.e.

$$\Phi^g = T^m \circ S^n, \quad g = (m, n) \in \mathbb{Z}^2.$$

For a partition $P \in \mathcal{Z}$ we put

$$h_{\vec{v}}((T, S), P) = \sup_{B \in \Gamma} \overline{\lim_{t \to \infty}} \frac{1}{t} H(P(B + [0, t)\vec{v})).$$

It is known (cf. [16]) that in fact

$$h_{\vec{v}}((T, S), P) = \sup_{B \in \Gamma} \lim_{t \to \infty} \frac{1}{t} H(P(B + [0, t)\vec{v})).$$

If we pass in this definition from the pair (T, S) to another pair (T_1, S_1) of commuting generators of Φ such that

$$T = T_1^a S_1^b, \quad S = T_1^c S_1^d$$

then it is easy to see that

$$h_{\vec{v}}((T, S), P) = h_{A(\vec{v})}((T_1, S_1), P)$$

where $P \in \mathcal{Z}$ and $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$.

In the sequel we use the notation

$$h_{\vec{v}}(\Phi, P) = h_{\vec{v}}((T_0, S_0), P)$$

where $T_0 = \Phi^{(1,0)}$ and $S_0 = \Phi^{(0,1)}$. The quantity $h_{\vec{v}}(\Phi, P)$ is said to be the *directional mean entropy* of Φ with respect to P in direction $\vec{v}$.

It is not hard to show that

$$h_{\vec{v}}(\Phi, P) = \lim_{m \to \infty} \lim_{t \to \infty} \frac{1}{t} H(P(R(\vec{v}, m, t)))$$

where

$$R(\vec{v}, m, t) = \begin{cases} \{(i, j) \in \mathbb{Z}^2 : 0 \le j \le [ty],\ -m + jx/y < i \le m + jx/y\} & \text{if } y \ne 0, \\ \{(i, j) \in \mathbb{Z}^2 : -m < j \le m,\ 0 \le i \le [tx]\} & \text{if } y = 0. \end{cases}$$

The quantity

$$h_{\vec{v}}(\Phi) = \sup_{P \in \mathcal{Z}} h_{\vec{v}}(\Phi, P)$$

is said to be the *directional entropy* of Φ in direction $\vec{v}$. We assume in the sequel that $\vec{v} \ne (0, 0)$.

Basic properties of directional entropy. Using classical arguments, one easily shows that

(i) If Φ and Ψ are $\mathbb{Z}^2$-actions and Ψ is a factor of Φ then $h_{\vec{v}}(\Phi) \ge h_{\vec{v}}(\Psi)$.

This yields at once

(ii) The directional entropy is an isomorphism invariant of $\mathbb{Z}^2$-actions.

As a direct consequence of the definition we have

(iii) For every $\alpha \in \mathbb{R}^2$,

$$h_{\alpha\vec{v}}(\Phi) = |\alpha| h_{\vec{v}}(\Phi).$$

Now we prove the analogue of the Kolmogorov–Sinai theorem.

THEOREM 2.1. *If $P \in \mathcal{Z}$ is a generator of Φ then $h_{\vec{v}}(\Phi) = h_{\vec{v}}(\Phi, P)$.*

Proof. We consider only the case $\vec{v} \ne (x, 0)$, $x \in \mathbb{R}$. The proof for $\vec{v} = (x, 0)$, $x \in \mathbb{R}$, is similar. We may assume that $\vec{v} = (x, 1)$. Let $Q \in \mathcal{Z}$. We need to show that

$$h_{\vec{v}}(\Phi, Q) \le h_{\vec{v}}(\Phi, P). \tag{1}$$

Let $\varepsilon > 0$ and let m be a positive integer. Put $Q^{(m)} = \bigvee_{i=-m+1}^{m} \Phi^{(i,0)} Q = \bigvee_{i=-m+1}^{m} T^i Q$.

Since P is a generator, there exist finite partitions $\widetilde{P}=\widetilde{P}_{m,\varepsilon}$, $\widetilde{Q}=\widetilde{Q}_{m,\varepsilon}$ and a positive integer $l=l_{m,\varepsilon}$ with

$$Q^{(m)}\le \widetilde{P}\vee\widetilde{Q},\quad \widetilde{P}\le P^{l\times l},\quad H(\widetilde{Q})<\varepsilon$$

where $P^{l\times l}=P([-l,l]\times[-l,l])$. We have

$$\begin{aligned}H(Q(R(\vec{v},m,t)))&=H\Big(\bigvee_{j=0}^{[t]}\bigvee_{i=-m+[jx]+1}^{m+[jx]}T^iS^jQ\Big)\\&=H\Big(\bigvee_{j=0}^{[t]}T^{[jx]}S^jQ^{(m)}\Big)\le H\Big(\bigvee_{j=0}^{[t]}T^{[jx]}S^j(\widetilde{P}\vee\widetilde{Q})\Big)\\&\le H\Big(\bigvee_{j=0}^{[t]}T^{[jx]}S^j\widetilde{P}\Big)+H\Big(\bigvee_{j=0}^{[t]}T^{[jx]}S^j\widetilde{Q}\Big)\\&<H\Big(\bigvee_{j=0}^{[t]}\bigvee_{-l\le p,q\le l}T^{[jx]+p}S^{j+q}P\Big)+([t]+1)\varepsilon.\end{aligned}$$

Dividing by t and letting $t\to\infty$ and then $m\to\infty$, we get

$$h_{\vec{v}}(\Phi,Q)\le h_{\vec{v}}(\Phi,P)+\varepsilon.$$

Since $\varepsilon>0$ is arbitrary we obtain (1).

From Theorem 2.1 it follows that, for $\mathbb{Z}^2$-actions determined by cellular automata, our definition reduces to that of Milnor.

The following result says that directional entropy is an interesting invariant only for $\mathbb{Z}^2$-actions with zero entropy.

PROPOSITION 2.1. *If Φ is a $\mathbb{Z}^2$-action with $h(\Phi)>0$ then $h_{\vec{v}}(\Phi)=\infty$.*

Proof. Although it is not hard to prove this directly, for clarity we use the generalized Sinai theorem (cf. [8]) which says that there exists a partition $P\in\mathcal{Z}$ such that the partitions Φ^gP, $g\in\mathbb{Z}^2$, are independent.

We have

$$\begin{aligned}h_{\vec{v}}(\Phi)\ge h_{\vec{v}}(\Phi,P)&=\lim_{m\to\infty}\lim_{t\to\infty}\frac{1}{t}H(P(R(\vec{v},m,t)))\\&=\lim_{m\to\infty}\lim_{t\to\infty}(2m+1)\frac{[ty]+1}{t}H(P)=\infty.\end{aligned}$$

Proposition 2.1 yields at once

COROLLARY 2.1. *For every $\vec{v}\in\mathbb{Z}^2$ we have $h(\Phi)\le h_{\vec{v}}(\Phi)$.*

It is well known ([3]) that for every $\mathbb{Z}^2$-action Φ the function $P\to h(\Phi,P)$, $P\in\mathcal{Z}$, is continuous. The following result is an easy consequence of Proposition 2.1.

COROLLARY 2.2. *If $h(\Phi)>0$ then the function $P\to h_{\vec{v}}(\Phi,P)$ is not continuous.*

Indeed, take an arbitrary sequence $(P_n)\subset\mathcal{Z}$ convergent to the trivial partition ν of X and such that $P_n\ne\nu$ for $n\ge1$. In view of Proposition 2.1 we have $h_{\vec{v}}(\Phi,P_n)=\infty$ for $n\ge1$ and $h_{\vec{v}}(\Phi,\nu)=0$.

We show in the next section that there are also $\mathbb{Z}^2$-actions Φ with $h(\Phi)=0$ for which the function considered in Corollary 2.2 is not continuous.

PROPOSITION 2.2. *If Φ is ergodic and $(P_n)\subset\mathcal{Z}$ is such that $P_n\nearrow\mathcal{B}$ then*

$$\lim_{n\to\infty}h_{\vec{v}}(\Phi,P_n)=h_{\vec{v}}(\Phi).$$

Proof. If $h(\Phi)>0$ then Proposition 2.1 implies

$$\lim_{n\to\infty}h_{\vec{v}}(\Phi,P_n)=\infty=h_{\vec{v}}(\Phi).$$

Now consider the case $h(\Phi)=0$. Suppose $\vec{v}=(x,1)$. Let P be a generator for Φ and let $\varepsilon>0$. It follows from the definition of $h_{\vec{v}}(\Phi,P)$ that there exists a positive integer m with

$$\begin{aligned}h_{\vec{v}}(\Phi)=h_{\vec{v}}(\Phi,P)&\le\lim_{t\to\infty}\frac{1}{t}H(P(R(\vec{v},m,t)))+\varepsilon\\&=\lim_{t\to\infty}\frac{1}{t}H\Big(\bigvee_{j=0}^{[t]}T^{[jx]}S^jP^{(m)}\Big)+\varepsilon\end{aligned}$$

where

$$P^{(m)}=\bigvee_{i=-m+1}^{m}T^iP.$$

Let $\mathcal{B}_n=\bigvee_{g\in\mathbb{Z}^2}\Phi^gP_n$, $n\ge1$. Our assumption implies $\mathcal{B}_n\nearrow\mathcal{B}$. Arguing as in the proof of Theorem 2.1, we see that for sufficiently large n there exists a partition $Q_n\in\mathcal{Z}$ measurable with respect to $\mathcal{B}_n$ with

$$\lim_{t\to\infty}\frac{1}{t}H\Big(\bigvee_{j=0}^{[t]}T^{[jx]}S^jP^{(m)}\Big)\le\lim_{t\to\infty}\frac{1}{t}H\Big(\bigvee_{j=0}^{[t]}T^{[jx]}S^jQ_n\Big)+\varepsilon.$$

Therefore, for the factor action $\Phi/\widehat{\mathcal{B}}_n$ we have

$$h_{\vec{v}}(\Phi)\le h_{\vec{v}}(\Phi/\widehat{\mathcal{B}}_n)+2\varepsilon,$$

where $\widehat{\mathcal{B}}_n$ is the measurable partition of X determined by $\mathcal{B}_n$. Since P_n is a generator for $\Phi/\widehat{\mathcal{B}}_n$, from Theorem 2.1 we obtain

$$h_{\vec{v}}(\Phi)\le h_{\vec{v}}(\Phi,P_n)+2\varepsilon.$$

Letting $n\to\infty$, we obtain the desired result.

The following result is an easy consequence of Proposition 2.2.

COROLLARY 2.3. *If* $\vec{v} \in \mathbb{Z}^2$ *then* $h_{\vec{v}}(\Phi) = h(\Phi^{\vec{v}})$.

COROLLARY 2.4. *For any ergodic* $\mathbb{Z}^2$*-actions* Φ *and* Ψ *we have*

$$h_{\vec{v}}(\Phi \times \Psi) = h_{\vec{v}}(\Phi) + h_{\vec{v}}(\Psi).$$

Proof. It is enough to use classical arguments, Proposition 2.2 and the equality

$$H((P \times Q)(R(\vec{v}, m, t))) = H(P(R(\vec{v}, m, t))) + H(Q(R(\vec{v}, m, t)))$$

where P and Q are arbitrary partitions with finite entropy of the Lebesgue spaces on which Φ and Ψ act, respectively, $m \in \mathbb{Z}$ and $t > 0$.

In [9] Krug defined and investigated the sequence entropy for $\mathbb{Z}^d$-actions, $d \geq 2$. We recall the definition of this notion in the case $d = 2$ for convenience of the reader.

Let now $A = (A(n))$ be a sequence of elements of $\mathbb{Z}^2$. For a given partition $P \in \mathcal{Z}$ we put

$$h_A(\Phi, P) = \overline{\lim_{n\to\infty}} \frac{1}{n} H\Big(\bigvee_{k=0}^{n-1} \Phi^{A(k)} P\Big).$$

The quantity

$$h_A(\Phi) = \sup\{h_A(\Phi, P) : P \in \mathcal{Z}\}$$

is said to be the *sequence entropy* of Φ along A.

The following result is given in [9].

PROPOSITION 2.3. *If* (P_n) *is a sequence of partitions from* $\mathcal{Z}$ *such that* $P_n \nearrow \mathcal{B}$ *then*

$$h_A(\Phi) = \lim_{n\to\infty} h_A(\Phi, P_n).$$

Our present goal is to give a relation between sequence entropy and directional entropy.

If $\vec{v} = (x, 0)$, $x \in \mathbb{R}$, then taking $A(n) = (n, 0)$ for $n \geq 1$ one obviously obtains

$$h_{\vec{v}}(\Phi) = |x| h_A(\Phi).$$

Let us now consider the case $\vec{v} = (x, y) \in \mathbb{R}^2$ where $y \neq 0$.

PROPOSITION 2.4. *If* $A(n) = ([nx/y], n)$ *then*

$$h_{\vec{v}}(\Phi) = |y| h_A(\Phi).$$

Proof. We may assume that $y = 1$. Fix $P \in \mathcal{Z}$ and $t \geq 0$. We have

$$H\Big(\bigvee_{j=0}^{[t]} T^{[jx]} S^j P\Big) \leq H\Big(\bigvee_{j=0}^{[t]} T^{[jx]} S^j P^{(m)}\Big) = H(P(R(\vec{v}, m, t))), \quad m \geq 1.$$

Dividing by t and letting first $t \to \infty$ and then $m \to \infty$ we get $h_A(\Phi, P) \leq h_{\vec{v}}(\Phi, P)$. Hence $h_A(\Phi) \leq h_{\vec{v}}(\Phi)$.

Let now (P_k) be a sequence of partitions from $\mathcal{Z}$ such that $P_k \nearrow \mathcal{B}$. We have

$$H(P_k(R(\vec{v}, m, t))) = H\Big(\bigvee_{j=0}^{[t]} T^{[jx]} S^j P_k^{(m)}\Big) \leq H\Big(\bigvee_{j=0}^{[t]} T^{[jx]} S^j P_k^{m\times m}\Big), \quad m \geq 1.$$

Since the σ-algebra $\sigma(P_k^{m\times m}, m \geq 1)$ coincides with the factor σ-algebra $(P_k)_\Phi$ generated by P_k for $k \geq 1$, the above inequality and Proposition 2.3 imply

$$h_{\vec{v}}(\Phi, P_k) \leq h_A(\Phi/(P_k)_\Phi) \leq h_A(\Phi).$$

Taking the limit as $k \to \infty$ and applying Proposition 2.2 we get $h_{\vec{v}}(\Phi) \leq h_A(\Phi)$, which gives the desired equality.

If x/y is irrational then one obtains the same formula as in Proposition 2.4 if one takes

$$A(n) = ([[nx/y]], n), \quad n \geq 1,$$

where $[[t]]$ denotes the nearest integer to $t \in \mathbb{R} \setminus \mathbb{Q}$.

3. Directional entropy for product $\mathbb{Z}^2$-actions. The following class of $\mathbb{Z}^2$-actions was introduced in [5] to give, among other things, examples of actions with a given rank, covering number and simple spectrum.

A $\mathbb{Z}^2$-action Φ on a Lebesgue space $(Y, \mathcal{C}, \nu)$ is said to be a *product action* if there exist automorphisms S_1, S_2 acting on Lebesgue spaces $(Y_1, \mathcal{C}_1, \nu_1)$ and $(Y_2, \mathcal{C}_2, \nu_2)$ such that

$$(Y, \mathcal{C}, \nu) = (Y_1, \mathcal{C}_1, \nu_1) \times (Y_2, \mathcal{C}_2, \nu_2)$$

and

$$\Phi^{(m,n)}(y_1, y_2) = (S_1^m y_1, S_2^n y_2), \quad (m, n) \in \mathbb{Z}^2.$$

It was shown in [5] that $h(\Phi) = 0$ and Φ is ergodic (resp. weakly mixing) iff S_i, $i = 1, 2$, is ergodic (resp. weakly mixing).

PROPOSITION 3.1. *If* Φ *is ergodic then for every* $\vec{v} = (x, y) \in \mathbb{R}^2$ *we have*

$$h_{\vec{v}}(\Phi) = |x| h(S_1) + |y| h(S_2).$$

Proof. Let P and Q be finite measurable partitions of Y_1 and Y_2, respectively. We may assume that $x > 0$ and $y > 0$. For fixed $m, t > 0$ we have

$$H((P\times Q)(R(\vec v,m,t)))=H\Big(\bigvee_{(i,j)\in R(\vec v,m,t)} S_1^iP\times S_2^jQ\Big)$$
$$=H\Big(\bigvee_{j=0}^{[ty]} S_2^jQ\Big)+H\Big(\bigvee_{i=-m+[xt]}^{m+[xt]} S_1^iP\Big).$$

Hence, for every $m>0$ we have

$$\lim_{t\to\infty}\frac{1}{t}H((P\times Q)(R(\vec v,m,t)))=yh(S_2,Q)+xh(S_1,P).$$

Letting $m\to\infty$ we get

$$h_{\vec v}(P\times Q,\Phi)=xh(S_1,P)+yh(S_2,Q). \tag{2}$$

Let (P_n) and (Q_n) be sequences of finite partitions of Y_1 and Y_2, respectively, such that $P_n\nearrow \mathcal{C}_1$ and $\bigvee_{n=1}^{\infty}Q_n\nearrow\mathcal{C}_2$. We have $P_n\times Q_n\nearrow\mathcal{C}$.

Substituting, in (2), P_n (resp. Q_n) for P (resp. Q) and then letting $n\to\infty$ we obtain, by Proposition 2.2, the desired equality.

COROLLARY 3.1. *If $h(S_i)<\infty$, $i=1,2$, then the function $\vec v\to h_{\vec v}(\Phi)$ is convex and satisfies the Lipschitz condition.*

Let now T be an automorphism of a Lebesgue space $(X,\mathcal{B},\mu)$ and Φ be a $\mathbb{Z}^2$-action on a Lebesgue space $(Y,\mathcal{C},\nu)$. Let

$$\widehat{T}(x,y)=(Tx,\Phi^{\varphi(x)}y),\qquad (x,y)\in X\times Y,$$

where $\varphi=(\varphi_1,\varphi_2):X\to\mathbb{Z}^2$ is a measurable function and $\varphi_i\in L^1(X,\mu)$, $i=1,2$.

COROLLARY 3.2. *If Φ is a product $\mathbb{Z}^2$-action then*

$$h(\widehat{T})=h(T)+h_{\vec v}(\Phi)$$

where $\vec v=(|\int_X\varphi_1\,d\mu|,|\int_X\varphi_2\,d\mu|)$.

Proof. By assumption, $\widehat{T}$ acts on $X\times Y_1\times Y_2$ and

$$\widehat{T}(x,y_1,y_2)=(Tx,S_1^{\varphi_1(x)}y_1,S_2^{\varphi_2(x)}y_2).$$

Consider the automorphism U of $X\times Y_1$ defined by

$$U(x,y_1)=(Tx,S_1^{\varphi_1(x)}y_1),\qquad (x,y_1)\in X\times Y_1.$$

Hence,

$$\widehat{T}(x,y_1,y_2)=(U(x,y_1),S_2^{\varphi_2(x)}y_2).$$

Applying the Newton formula [12] (see also [1]), we obtain

$$h(\widehat{T})=h(U)+\Big|\int_X\varphi_2\,d\mu\Big|\cdot h(S_2).$$

Applying it now to the automorphism U, we get

$$h(U)=h(T)+\Big|\int_X\varphi_1\,d\mu\Big|\cdot h(S_1),$$

and so, by Proposition 3.1,

$$h(\widehat{T})=h(T)+\Big|\int_X\varphi_1\,d\mu\Big|\cdot h(S_1)+\Big|\int_X\varphi_2\,d\mu\Big|\cdot h(S_2)=h(T)+h_{\vec v}(\Phi)$$

where $\vec v=(|\int_X\varphi_1\,d\mu|,|\int_X\varphi_2\,d\mu|)$.

4. Directional entropy for rigid actions. A $\mathbb{Z}^2$-action Φ is said to be *rigid* if there exists a sequence $(m_k,n_k)\subset\mathbb{Z}^2$ such that for every $A\in\mathcal{B}$,

$$\lim_{k\to\infty}\mu(\Phi^{(m_k,n_k)}A\cap A)=\mu(A).$$

EXAMPLE 4.1. A $\mathbb{Z}^2$-action Φ on a Lebesgue space $(X,\mathcal{B},\mu)$ is said to have *discrete spectrum* if there exists an orthonormal basis in $L^2(X,\mu)$ consisting of eigenfunctions of Φ. It is shown in [6] that any such action is isomorphic to a rotation action defined as follows.

Let X be a compact abelian group equipped with the normalized Haar measure and let $a,b\in X$ be independent over $\mathbb{Z}$. The $\mathbb{Z}^2$-action $\Phi_{a,b}$ defined by

$$\Phi_{a,b}^{(m,n)}x=a^mb^nx,\qquad (m,n)\in\mathbb{Z}^2,\ x\in X,$$

is called the *rotation $\mathbb{Z}^2$-action*.

Applying classical arguments (cf. [18]), one shows that Φ is ergodic iff the set $\{a^mb^n:(m,n)\in\mathbb{Z}^2\}$ is dense in X.

Hence, in particular, there exists a sequence (m_k,n_k) such that $a^{m_k}b^{n_k}\to 1$. This property forces the rigidity of $\Phi_{a,b}$ and so the rigidity of an arbitrary action with discrete spectrum.

EXAMPLE 4.2. Let X be the set of all real-valued functions defined on $\mathbb{Z}^2$, and let $\mathcal{B}$ denote the product σ-algebra of subsets of X. Let $\xi_g:X\to\mathbb{R}$ be the projection onto the gth coordinate for $g\in\mathbb{Z}^2$. For a given finite symmetric measure ϱ on the two-dimensional torus $\mathbb{T}^2$ we denote by μ the (unique) probability measure on $\mathcal{B}$ such that the family $(\xi_g,g\in\mathbb{Z}^2)$ forms a stationary Gaussian random field with covariance function $R:\mathbb{Z}^2\to\mathbb{C}$ given by

$$R(g)=\int_{\mathbb{T}^2}z^mw^n\,\varrho(dzdw),\qquad g=(m,n)\in\mathbb{Z}^2.$$

The $\mathbb{Z}^2$-action Φ on $(X,\mathcal{B},\mu)$ defined by

$$(\Phi^gx)(h)=x(g+h),\qquad g,h\in\mathbb{Z}^2,$$

is called the *Gaussian $\mathbb{Z}^2$-action with spectral measure ϱ*.

Proceeding in a similar way to the one-dimensional case (cf. [17]), one shows that Φ is rigid iff there exists a sequence $(m_k, n_k) \subset \mathbb{Z}^2$ such that

$$\lim_{k\to\infty} \int_{\mathbb{T}^2} |z^{m_k} w^{n_k} - 1|^2 \, \varrho(dzdw) = 0.$$

It is easy to show that every rigid $\mathbb{Z}^2$-action has zero entropy.

We show in the sequel that this result fails for directional entropy.

THEOREM 4.1. *If Φ is an ergodic $\mathbb{Z}^2$-action with discrete spectrum then $h_{\vec{v}}(\Phi) = 0$ for every $\vec{v} \in \mathbb{R}^2$.*

Proof. It follows from our comment in Example 4.1 that it is enough to show the above equality for rotation actions on a compact abelian group.

First, assume that Φ is a rotation action on the one-dimensional torus $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ determined by algebraically independent numbers $a, b \in \mathbb{T}$.

We consider the two-element partition $P = \{P_1, P_2\}$ of $\mathbb{T}$ where $P_1 = \{e^{2\pi it} : 0 \le t < \pi\}$ and $P_2 = \{e^{2\pi it} : \pi \le t < 2\pi\}$. The density of the set $\{a^k b^l : (k,l) \in \mathbb{Z}^2\}$ implies P is a generator. It is easy to see that for every finite subset $A \subset \mathbb{Z}^2$ the partition $P(A)$ has at most $2\#A$ elements.

Fix now $m > 0$ and $t > 0$. From the above remark it follows that

$$H(P(R(\vec{v}, m, t))) \le \log(4m([t]+1)).$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} H(P(R(\vec{v}, m, t))) = 0$$

for every $m > 0$ and so $h_{\vec{v}}(\Phi, P) = 0$. Since P is a generator we have $h_{\vec{v}}(\Phi) = 0$.

Further, we proceed similarly to [18], p. 101. Below we only give the necessary comments concerning the two-dimensional situation.

The next step is to show the result for rotation actions on the torus $\mathbb{T}^s$ where s is an arbitrary positive integer. In this case the action has the form

$$\Phi^{(m,n)}(z_1, \ldots, z_s) = (a_1^m b_1^n z_1, \ldots, a_s^m b_s^n z_s),$$

i.e.

$$\Phi = \Phi_1 \times \ldots \times \Phi_s$$

where $\Phi_k^{(m,n)} z_k = a_k^m b_k^n z_k, 1 \le k \le s$. Now, the desired property follows from the previous case and from Corollary 2.4.

Let now Φ be a rotation $\mathbb{Z}^2$-action on a compact abelian group G determined by algebraically independent elements $a, b \in G$. Let H_n and F_n be the groups defined in [18], p. 101.

We consider a $\mathbb{Z}^2$-action Φ_n on G/H_n generated by two rotations T_n and S_n defined as follows:

$$T_n(gH_n) = agH_n, \quad S_n(gH_n) = bgH_n.$$

Hence, as in [18] we have $T_n = T_{n,1} \times T_{n,2}$, $S_n = S_{n,1} \times S_{n,2}$, i.e. $\Phi_n = \Phi_{n,1} \times \Phi_{n,2}$ where $\Phi_{n,k}$ is generated by $T_{n,k}$ and $S_{n,k}$, $k = 1, 2$. From the previous two cases and Corollary 2.4, we have $h(\Phi_n) = 0$. The desired result now follows from Proposition 2.2.

THEOREM 4.2. *There exists a rigid Gaussian $\mathbb{Z}^2$-action Φ such that*

$$h_{\vec{v}}(\Phi) = \begin{cases} 0 & \text{if } \vec{v} = (x, 0) \text{ for } x \in \mathbb{R}, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Let λ denote the Lebesgue measure on $\mathbb{T}$ and σ a continuous symmetric finite measure concentrated on a set $D \cup D^{-1}$ where D is a Kronecker subset of $\mathbb{T}$ (cf. [7]). We put $\varrho = \sigma \times \lambda$.

Let Φ be the Gaussian $\mathbb{Z}^2$-action with spectral measure ϱ. Let T and S be the standard generators of Φ, i.e.

$$(Tx)(m,n) = x(m+1,n), \quad (Sx)(m,n) = x(m,n+1), \quad (m,n) \in \mathbb{Z}^2.$$

For a fixed $n \in \mathbb{Z}$ we denote by $\mathcal{A}_n$ the σ-algebra generated by $\xi_{m,n}$, $m \in \mathbb{Z}$.

It easily follows from the definition of ϱ that

(3) the σ-algebras $\mathcal{A}_n, n \in \mathbb{Z}$, are independent and $\sigma(\mathcal{A}_n, n \in \mathbb{Z}) = \mathcal{B}$.

It follows from Theorem 14 of [17] that for every $n \in \mathbb{Z}$, T is rigid in $(X, \mathcal{A}_n, \mu)$. Using (3) and Theorem 6 of [17] we conclude that T is rigid. Hence Φ is rigid.

Let $a \in \mathbb{R}$ and let $Q = Q_a$ be the natural two-element partition of $\mathbb{R}$ determined by a:

$$Q = \{Q_1, Q_2\}, \quad Q_1 = (-\infty, a), \quad Q_2 = [a, \infty).$$

Let $P = P_a = \xi_{(0,0)}^{-1}(Q)$, i.e. P is the zero-time partition of X determined by Q. Fix $m > 0$ and $t > 0$. It follows from (3) that the partitions

$$\bigvee_{-m+jx<i<m+jx} T^i S^j P, \quad j = 0, 1, \ldots, [t],$$

are independent. Hence,

$$H(P(R(\vec{v}, m, t))) = H\Big(\bigvee_{j=0}^{[t]} \bigvee_{i=-m+[jx]+1}^{m+[jx]} T^i S^j P\Big) = \sum_{j=0}^{[t]} H\Big(\bigvee_{i=-m+[jx]+1}^{m+[jx]} T^i P\Big)$$

$$= \sum_{j=0}^{[t]} H\Big(\bigvee_{i=-m+1}^{m} T^i P\Big) = ([t]+1) H\Big(\bigvee_{i=-m+1}^{m} T^i P\Big).$$

The automorphism T is Gaussian in the space $(X, \mathcal{A}_n, \mu)$, $n \ge 1$, and its spectral measure is σ. Since σ is continuous, T is weakly mixing (cf. [4]). Applying again (3) we see that T is weakly mixing. Hence,

$$\lim_{m\to\infty} H\Big(\bigvee_{i=-m+1}^{m} T^i P\Big) = \infty.$$

Therefore, for $\vec{v} \ne (x,0)$, $x \in \mathbb{R}$, we have

$$h_{\vec{v}}(\Phi, P) = \lim_{m\to\infty}\lim_{t\to\infty} \frac{1}{t} H(P(R(\vec{v}, m, t))) = \infty$$

and so $h_{\vec{v}}(\Phi) = \infty$. Since σ is singular, for $\vec{v}_1 = (1,0)$ we have $h_{\vec{v}_1}(\Phi) = h(T) = 0$. Therefore the property (iii) implies $h_{\vec{v}}(\Phi) = 0$ for $\vec{v} = (x,0)$, $x \in \mathbb{R}$.

REMARK. Let Φ be the Gaussian $\mathbb{Z}^2$-action defined above. The function

$$P \to h_{\vec{v}}(\Phi, P), \quad P \in \mathcal{Z},\ \vec{v} \ne (x,0),\ v \in \mathbb{R},$$

is not continuous.

Indeed, if we take a sequence $(a_n) \subset \mathbb{R}$ such that $a_n \nearrow \infty$ then the corresponding sequence $(P_n) = (P_{a_n})$ converges to the trivial partition ν. The desired result follows at once from the equalities

$$h_{\vec{v}}(\Phi, P_n) = \infty \quad (n \ge 1), \quad h_{\vec{v}}(\Phi, \nu) = 0.$$

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