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On $\alpha(\cdot)$-monotone multifunctions and differentiability of γ-paraconvex functions

by

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Abstract. Let (X,d) be a metric space. Let Φ be a family of real-valued functions defined on X. Sufficient conditions are given for an $\alpha(\cdot)$-monotone multifunction $\Gamma : X \to 2^{\Phi}$ to be single-valued and continuous on a weakly angle-small set. As an application it is shown that a γ-paraconvex function defined on an open convex subset of a Banach space having separable dual is Fréchet differentiable on a residual set.

Let $(X, \|\cdot\|)$ be a separable real Banach space. Let f be a real-valued convex continuous function defined on an open convex subset $\Omega \subset X$. Mazur [3] proved that there is a subset $A_{\rm G} \subset \Omega$ of the first category such that f is Gateaux differentiable on $\Omega \setminus A_{\rm G}$. Asplund [1] showed that if additionally X has a separable dual, then there is a subset $A_{\rm F} \subset \Omega$ of the first category such that f is Fréchet differentiable on $\Omega \setminus A_{\rm F}$.

Let $1 < \gamma \le 2$. Let f be a real-valued function on X. We say that f is *γ-paraconvex* if there is $C > 0$ such that for all $x, y \in X$ and $0 \le t \le 1$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) + C\|x-y\|^{\gamma}$$

(see [6], [7]).

In this paper we give an extension of Asplund's [1] theorem to γ-paraconvex functions. The subject is treated in the more general framework of metric spaces and we prove a theorem which immediately implies the result mentioned above.

Let (X,d) be a metric space. Let Φ be a family of real-valued functions defined on X. Let $\alpha : [0,\infty) \to [0,\infty]$ be such that $\alpha(0) = 0$ and

$$\lim_{t \to 0} \alpha(t)/t = 0. \tag{1}$$

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A function $\phi \in \Phi$ is called an *$\alpha(\cdot)$-Φ-subgradient* of f at a point x_0 if

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0) - \alpha(d(x, x_0)) \tag{2}$$

for all $x \in X$. If (2) is satisfied only locally we say that $\phi \in \Phi$ is a *local $\alpha(\cdot)$-Φ-subgradient* of f at x_0. For $\alpha(t) \equiv 0$ we obtain the definition of Φ-subgradient and local Φ-subgradient (see for example [4]).

The set of all $\alpha(\cdot)$-Φ-subgradients (local $\alpha(\cdot)$-Φ-subgradients) of f at x_0 is called the *$\alpha(\cdot)$-Φ-subdifferential* (resp. *local $\alpha(\cdot)$-Φ-subdifferential*) of f at x_0 and denoted by $\partial^\alpha_\Phi f|_{x_0}$. For $\alpha(t) \equiv 0$ we obtain the definition of Φ-subdifferential and local Φ-subdifferential (see for example [4], [14]).

In the case when X is a normed space, $\Phi = X^*$ and $\alpha(t) = t^\gamma$ we obtain the definition of (local) γ-subgradient and (local) γ-subdifferential introduced by Jourani [2]. He showed that each local γ-subgradient is in fact a γ-subgradient of the corresponding function at the given point. The converse implication is obvious.

If a real-valued function f has a nonempty $\alpha(\cdot)$-Φ-subdifferential $\partial^\alpha_\Phi f|_x$ for all $x \in X$ we say that f is *$\alpha(\cdot)$-Φ-subdifferentiable.*

Let $\alpha : [0, \infty) \to [0, \infty]$ be as above. We say that a multifunction $\Gamma : X \to 2^\Phi$ is *$\alpha(\cdot)$-monotone* if for all $\phi_x \in \Gamma(x)$, $\phi_y \in \Gamma(y)$ we have

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) + \alpha(d(x, y)) \geq 0. \tag{3}$$

For $\alpha(t) \equiv 0$ we obtain the definition of monotone multifunctions (see for example [4]).

In the case when X is a normed space, $\Phi = X^*$ and $\alpha(t) = t^\gamma$ we obtain the definition γ-monotone multifunctions introduced by Jourani [2].

PROPOSITION 1. *Let (X, d) be a metric space. Let Φ be a family of real-valued functions on X. If a function f on X is $\alpha(\cdot)$-Φ-subdifferentiable, then its $\alpha(\cdot)$-Φ-subdifferential $\partial^\alpha_\Phi f|_x$ considered as a multifunction of x is $2\alpha(\cdot)$-monotone.*

Proof. Take $x, y \in X$. Let $\phi_x \in \partial^\alpha_\Phi f|_x$ and $\phi_y \in \partial^\alpha_\Phi f|_y$. By definition

$$f(y) - f(x) \geq \phi_x(y) - \phi_x(x) - \alpha(d(y, x)) \tag{4}$$

and

$$f(x) - f(y) \geq \phi_y(x) - \phi_y(y) - \alpha(d(y, x)). \tag{5}$$

Adding (4) and (5) we obtain

$$0 \geq \phi_x(y) - \phi_x(x) + \phi_y(x) - \phi_y(y) - 2\alpha(d(x, y)). \tag{6}$$

Thus

$$\phi_x(x) + \phi_y(y) - \phi_x(y) - \phi_y(x) + 2\alpha(d(x, y)) \geq 0. \ \blacksquare \tag{7}$$

Even in the case when $X = \mathbb{R}$ and Φ is the set of linear functions the class of $\alpha(\cdot)$-Φ-subdifferentiable functions does not coincide with the so called DC-functions (functions which can be represented as a difference of convex functions).

EXAMPLE 2. Let $X = \mathbb{R}$ and Φ be the set of linear functions. Let $\alpha(t) = t^{3/2}$. Let

$$t_n = \sum_{k=1}^{n} \frac{1}{k^2}.$$

Let $g(t) = \max[-\sqrt{|t - t_n|}]$ and $f(t) = \int_0^{|t|} g(s)\, ds$. Then f is $\alpha(\cdot)$-Φ-subdifferentiable, i.e. at each point t there is a local $\frac{3}{2}$-subgradient. If $t \neq t_n$ this is obvious since f is locally convex in the neighbourhood of t. At $t = t_n$, 0 is a local $\frac{3}{2}$-subgradient, since $f(t_n + h) - f(t_n) \geq -\int_0^{|h|} \sqrt{s}\, ds = -\frac{2}{3}|h|^{3/2}$. Recall that by Jourani [2] this shows that f is $t^{3/2}$-subdifferentiable.

On the other hand, since g is not of bounded variation, f is not a DC-function.

In [4] we have given conditions for a monotone multifunction to be single-valued and continuous on a residual set. In this note we show that the same holds for $\alpha(\cdot)$-monotone multifunctions.

Let (X, d) be a metric space. Let $\mathcal{L}$ be the space of all Lipschitz functions defined on X. We define on $\mathcal{L}$ a quasinorm

$$\|\phi\|_{\mathrm{L}} = \sup_{\substack{x_1, x_2 \in X \\ x_1 \neq x_2}} \frac{|\phi(x_1) - \phi(x_2)|}{d(x_1, x_2)}. \tag{8}$$

Observe that if $\|\phi_1 - \phi_2\|_{\mathrm{L}} = 0$ then the difference of ϕ_1 and ϕ_2 is a constant function, i.e., $\phi_1(x) = \phi_2(x) + c$. Thus we consider the quotient space $\widetilde{\mathcal{L}} = \mathcal{L}/\mathbb{R}$. The quasinorm $\|\phi\|_{\mathrm{L}}$ induces a norm in $\widetilde{\mathcal{L}}$. Since this will not lead to any misunderstanding, this norm will also be denoted by $\|\phi\|_{\mathrm{L}}$.

Let Φ be a family of Lipschitz functions on X. If there is a constant k, $0 < k < 1$, such that for all $x \in X$, $\phi \in \Phi$ and $t > 0$, there is a $y \in X$ such that $0 < d(x, y) < t$ and

$$\phi(y) - \phi(x) \geq k\|\phi\|_{\mathrm{L}} d(y, x), \tag{9}$$

we say that Φ has the *monotonicity property* with constant k ([9], see also [4]). It is obvious that the linear continuous functionals on a Banach space have the monotonicity property with any constant smaller than 1.

For any $\phi \in \Phi$, $0 < \beta < 1$, $x \in X$, write ([4], Sec. 2.4, cf. Preiss and Zajíček [5] for the linear case)

$$K(\phi, \beta, x) = \{y \in X : \phi(y) - \phi(x) \geq \beta\|\phi\|_{\mathrm{L}} d(y, x)\}. \tag{10}$$

The set $K(\phi, \beta, x)$ will be called the *β-cone* with vertex x and direction ϕ. Of course, it may happen that $K(\phi, \beta, x) = \{x\}$. However, if $\beta < k$, it is obvious that the set $K(\phi,\beta,x)$ has a nonempty interior and, even more,

(11) $$x \in \overline{\operatorname{Int} K(\phi,\beta,x)}.$$

Now we extend this definition a little. Namely the set

(12) $$K(\phi,\beta,x,\varrho) = K(\phi,\beta,x) \cap \{y : d(x,y) < \varrho\}$$

will be called the *(β,ϱ)-cone* with vertex x and direction ϕ.

Observe that just from the definition it follows that if $\beta_1 < \beta_2$, then $K(\phi,\beta_1,x,\varrho) \supset K(\phi,\beta_2,x,\varrho)$.

We recall that $M \subset X$ is said to be *β-cone meagre* if for all $x \in M$ and $\varepsilon > 0$ there are $z \in X$ with $d(x,z) < \varepsilon$ and $\phi \in \Phi$ such that

(13) $$M \cap \operatorname{Int} K(\phi,\beta,z) = \emptyset$$

([9], see also [4]).

A set $M \subset X$ is said to be *(β,ϱ)-cone meagre* if for all $x \in M$ and $\varepsilon > 0$ there are $z \in X$ with $d(x,z) < \varepsilon$ and $\phi \in \Phi$ such that

(14) $$M \cap \operatorname{Int} K(\phi,\beta,z,\varrho) = \emptyset.$$

The arbitrariness of ε and (14) implies that a (β,ϱ)-cone meagre set M is nowhere dense.

There is a simple example showing that the two notions do not coincide.

EXAMPLE 3. Let $X = \mathbb{R}^2$ and let $M = \{(x,0) : x \in \mathbb{R}\} \cup \{(x,1) : x \in \mathbb{R}\} \cup \{(x,-1) : x \in \mathbb{R}\}$. It is easy to see that M is (β,ϱ)-cone meagre for all $\beta > 0$ and $0 < \varrho < 1$. By a simple observation, it is not β-cone meagre for any β.

We recall that a set $M \subset X$ is called *angle-small* if it can be represented as a union of a countable number of β-cone meagre sets M_n,

(15) $$M = \bigcup_{n=1}^{\infty} M_n,$$

for some $\beta > 0$. We say that $M \subset X$ is *weakly angle-small* if it can be represented as a union (15) of a countable number of (β,ϱ_n)-cone meagre sets M_n for some $\beta,\varrho_n > 0$.

Of course, every weakly angle-small set M is of the first category.

Adapting the method of Preiss and Zajíček [5] and the proof of [9] (see also proof of Theorem 2.4.11 of [4]) we obtain

THEOREM 4. *Let (X,d) be a metric space. Let Φ be a family of Lipschitz functions on X which is a group with respect to addition and has the monotonicity property with a constant k, $0 < k \le 1$. Assume that Φ is separable in the metric d_{L}. Let $\Gamma : X \to 2^{\Phi}$ be an $\alpha(\cdot)$-monotone multifunction such that $\operatorname{dom}\Gamma = X$ (i.e., $\Gamma(x) \neq \emptyset$ for all $x \in X$). Then there exists a weakly angle-small set A such that Γ is single-valued and continuous on $X \setminus A$.*

Proof. It is sufficient to show that the set

(16) $$A = \{x \in X : \lim_{\delta\to 0} \operatorname{diam}\Gamma(B(x,\delta)) > 0\},$$

where diam is the diameter in the Lipschitz metric d_{L}, is weakly angle-small. Of course, we can represent A as the union of the sets

(17) $$A_n = \{x \in X : \lim_{\delta\to 0} \operatorname{diam}\Gamma(B(x,\delta)) > 1/n\}.$$

Let $\{\phi_m\}$ be a dense sequence in Φ in the metric d_{L}. Suppose that $0 < \beta < k$. Let

(18) $$A_{n,m} = \{x \in A_n : \operatorname{dist}(\phi_m,\Gamma(x)) < \beta/(4n)\},$$

where as usual $\operatorname{dist}(\phi_m,\Gamma(x)) = \inf\{\|\phi_m - \phi\| : \phi \in \Gamma(x)\}$ (we write $\|\cdot\|$ for $\|\cdot\|_{\mathrm{L}}$). By the density of $\{\phi_m\}$ in Φ,

$$\bigcup_{m=1}^{\infty} A_{n,m} = A_n.$$

We now show that the sets $A_{n,m}$ are (β,ϱ)-meagre for sufficiently small ϱ.

Indeed, let $x \in A_{n,m}$ and $\varepsilon > 0$. Since $x \in A_n$, by (17), there are $0 < \delta < \varepsilon$ and $z_1, z_2 \in X$, $\phi_1 \in \Gamma(z_1)$, $\phi_2 \in \Gamma(z_2)$ such that $d(z_1,x) < \delta$, $d(z_2,x) < \delta$ and

(19) $$\|\phi_1 - \phi_2\| > 1/n.$$

Thus by the triangle inequality, for every $\phi \in \Gamma(x)$ either $\|\phi_1 - \phi\| > 1/(2n)$ or $\|\phi_2 - \phi\| > 1/(2n)$. By the definition of $A_{n,m}$, we can find $\phi_x \in \Gamma(x)$ such that $\|\phi_x - \phi_m\| < \beta/(4n)$. Therefore choosing for z either z_1 or z_2, we can say that there are $z \in X$ and $\phi_z \in \Gamma(z)$ such that $d(z,x) < \delta$ and

(20) $$\|\phi_z - \phi_m\| \ge \|\phi_z - \phi_x\| - \|\phi_x - \phi_m\| > \frac{1}{2n} - \frac{\beta}{4n} > \frac{1}{4n}.$$

By (1) there is ϱ_n such that

(21) $$\frac{1}{2n} - \frac{\beta}{4n} - \frac{1}{\beta} \sup_{0<t<\varrho_n} \frac{\alpha(t)}{t} > \frac{1}{4n}.$$

We show that

$$A_{n,m} \cap K(\phi_z - \phi_m, \beta, z) \cap \{y : d(z,y) < \varrho_n\} = \emptyset.$$

Indeed, suppose that $y \in K(\phi_z - \phi_m,\beta,z)$. This means that

$$\phi_z(y) + \phi_m(z) - \phi_m(y) - \phi_z(z) \ge \beta\|\phi_z - \phi_m\| d(y,z).$$

Suppose that $\phi_y \in \Gamma(y)$. Since Γ is $\alpha(\cdot)$-monotone we have

$$\phi_y(y) - \phi_y(z) - \phi_z(y) + \phi_z(z) \ge -\alpha(d(y,z)).$$

Adding these two inequalities we get

$$\phi_y(y) - \phi_m(y) - \phi_y(z) + \phi_m(z) \ge \beta\|\phi_z - \phi_m\| d(y,z) - \alpha(d(y,z)),$$

and if additionally $d(y,z) < \varrho_n$ we have, by (21),

$$\begin{aligned}\phi_y(y) - \phi_m(y) - \phi_y(z) + \phi_m(z) \\ &\geq \beta\left(\frac{1}{2n} - \frac{\beta}{4n}\right)d(y,z) - \alpha(d(y,z)) \\ &= \beta\left(\frac{1}{2n} - \frac{\beta}{4n} - \frac{1}{\beta}\frac{\alpha(d(y,z))}{d(y,z)}\right)d(y,z) > \frac{\beta}{4n}d(y,z).\end{aligned}$$

This implies that

$$\|\phi_y - \phi_m\| \geq \frac{\beta}{4n}$$

and by the definition of $A_{n,m}$, $y \notin A_{n,m}$. Thus

$$A_{n,m} \cap K(\phi_z - \phi_m, \beta, z) \cap \{y : d(z,y) < \varrho_n\} = \emptyset$$

and the set $A_{n,m}$ is (β, ϱ_n)-meagre. Therefore A is weakly angle-small. ∎

Since the subdifferential $\partial_\Phi^\alpha f|_x$ of an $\alpha(\cdot)$-Φ-differentiable function is a $2\alpha(\cdot)$- monotone multifunction of x, we immediately obtain

COROLLARY 5. *Let (X,d) be a metric space. Let Φ be a family of Lipschitz functions on X which is a group with respect to addition and has the monotonicity property with a constant k, $0 < k \leq 1$. Suppose that Φ is separable in the metric d_{L}. Let f be an $\alpha(\cdot)$-Φ-subdifferentiable function on X. Then there is a weakly angle-small set A such that outside A the $\alpha(\cdot)$-Φ-subdifferential $\partial_\Phi^\alpha f|_x$ is single-valued and continuous in the metric d_{L}.*

We recall that a set B of second category is called *residual* if its complement is of the first category. Since weakly angle-small sets are always of the first category we immediately obtain

THEOREM 6. *Let (X,d) be a metric space of the second category in itself (in particular, let X be a complete metric space). Let Φ be a family of Lipschitz functions on X which is a group with respect to addition and has the monotonicity property with a constant k, $0 < k \leq 1$. Assume that Φ is separable in the metric d_{L}. Let $\Gamma : X \to 2^\Phi$ be an $\alpha(\cdot)$-monotone multifunction such that $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then there is a residual set B such that Γ is single-valued and continuous on B.*

COROLLARY 7. *Let (X,d) be a metric space which is of the second category in itself (in particular, let X be a complete metric space). Let Φ be a class of Lipschitz functions on X which is a group with respect to addition and has the monotonicity property with a constant k, $0 < k \leq 1$. Suppose that Φ is separable in the metric d_{L}. Let f be a function on X having an $\alpha(\cdot)$-Φ-subgradient at each point. Then there is a residual set B such that the $\alpha(\cdot)$-Φ-subdifferential $\partial_\Phi^\alpha f|_x$ is single-valued and continuous on B in the metric d_{L}.*

We say that a function $f : X \to \mathbb{R}$ is *Fréchet Φ-differentiable* at a point x_0 if there is a function $\sigma : [0,\infty) \to [0,\infty]$ such that

$$\lim_{t\to 0} \sigma(t)/t = 0$$

and a function $\phi \in \Phi$ such that

$$|[f(x) - f(x_0)] - [\phi(x) - \phi(x_0)]| \leq \sigma(d(x,x_0))$$

for all x in a neighbourhood of x_0. The function ϕ will be called a *Fréchet Φ-gradient* of f at x_0. The function σ will be called a *modulus of smoothness*.

Recall that in normed spaces Gateaux differentiability of a convex continuous function f at a point x is equivalent to the subdifferential $\partial f|_x$ consisting of one point only. Moreover, the continuity of Gateaux differentials in the norm operator topology implies that these differentials are Fréchet differentials. Similarly we have an extension of this fact to metric spaces ([12], [13]). Here we extend this result to $\alpha(\cdot)$-monotone operators.

We recall that the $\alpha(\cdot)$-Φ-subdifferential $\partial_\Phi^\alpha f|_x$ is lower semicontinuous at x_0 in the Lipschitz norm if for any $\phi_{x_0} \in \partial_\Phi^\alpha f|_{x_0}$ there is a function μ such that $\mu(0) = 0$, $\mu(t) > 0$ for $t > 0$ and

$$\lim_{t\to 0} \mu(t) = 0 \tag{22}$$

and such that for all $x \in X$ there is $\phi_x \in \partial_\Phi^\alpha f|_x$ such that

$$\|\phi_x - \phi_{x_0}\|_{\mathrm{L}} \leq \mu(d(x,x_0)). \tag{23}$$

PROPOSITION 8. *Let (X,d) be a metric space. Let Φ be a class of Lipschitz functions on X which is a group with respect to addition. Let f be an $\alpha(\cdot)$-Φ-subdifferentiable function on X. Let ϕ_{x_0} be an $\alpha(\cdot)$-Φ-subgradient of f at a point x_0. Suppose that the $\alpha(\cdot)$-Φ-subdifferential $\partial_\Phi^\alpha f|_x$ is lower semicontinuous at x_0 in the Lipschitz norm. Then ϕ_{x_0} is a Fréchet Φ-gradient of f at x_0.*

Proof. Let

$$F(x) = [f(x) - f(x_0)] - [\phi_{x_0}(x) - \phi_{x_0}(x_0)]. \tag{24}$$

It is easy to see that $F(x_0) = 0$. Since ϕ_{x_0} is an $\alpha(\cdot)$-Φ-subgradient of f at x_0,

$$F(x) \geq -\alpha(d(x,x_0)). \tag{25}$$

Let ϕ_x be such that (23) holds. Since ϕ_x is an $\alpha(\cdot)$-Φ-subgradient of f at x, $\psi_x = \phi_x - \phi_{x_0}$ is an $\alpha(\cdot)$-Φ-subgradient of F at x. Thus

$$F(y) - F(x) \geq \psi_x(y) - \psi_x(x) - \alpha(d(x,y)).$$

In particular, if $y = x_0$, then

$$F(x_0) - F(x) \geq \psi_x(x_0) - \psi_x(x) - \alpha(d(x,x_0)). \tag{26}$$

Taking into account (25), we find that for $x \in V_\varepsilon$,

$$(27)\qquad -\alpha(d(x,x_0)) \le F(x) \le \psi_x(x) - \psi_x(x_0) + \alpha(d(x,x_0)) \le 2\mu(d(x,x_0))d(x,x_0) + \alpha(d(x,x_0)).$$

Putting $\sigma(t) = \max[\alpha(t), 2\mu(t)t + \alpha(t)]$ we obtain

$$|F(x)| \le \sigma(d(x,x_0)).$$

Thus 0 is a Fréchet Φ-gradient of F at x_0. This immediately implies that ϕ_{x_0} is a Fréchet Φ-gradient of f at x_0. ∎

As a consequence we obtain

THEOREM 9. *Let (X,d) be a metric space which is of the second category in itself (in particular, let X be a complete metric space). Let Φ be a family of Lipschitz functions on X which is a group with respect to addition and has the monotonicity property with a constant k. Suppose that Φ is separable in the metric d_{L}. Let f be a continuous $\alpha(\cdot)$-Φ-subdifferentiable function. Then there is a weakly angle-small set A such that f is Fréchet Φ-differentiable at every point $x_0 \in X \setminus A$. Moreover, the Fréchet Φ-subgradient is unique and it is continuous in the metric d_{L}.*

Proof. Since f is $\alpha(\cdot)$-Φ-subdifferentiable, there is a weakly angle-small set A such that on $Y = X\setminus A$ the $\alpha(\cdot)$-Φ-subdifferential $\partial^\alpha_\Phi f|_x$ is single-valued and continuous in the metric d_{L}. Thus by Proposition 8 the restriction $f|_Y$ is Fréchet Φ-differentiable at every $x_0 \in Y$. The continuity of f and the density of Y in X imply that f is Fréchet Φ-differentiable at every $x_0 \in Y$. ∎

In Theorem 9 we can weaken the monotonicity assumption to its local version in a similar way to [4], Section 2.4.

Applying Theorem 9 to Banach spaces and γ-paraconvex functions and using Jourani's [2] results we get the following extension of the Asplund [1] theorem.

THEOREM 10. *Let $(X, \|\cdot\|)$ be a real Banach space which has separable dual X^*. Let f be a γ-paraconvex function, $1 < \gamma \le 2$, defined on an open convex subset $\Omega \subset X$. Then there is a subset $A_f \subset \Omega$ of the first category such that f is Fréchet differentiable on $\Omega \setminus A_f$.*

Proof. Put $\Phi = X^*$. Of course X^* has the monotonicity property with any constant $k < 1$. By Proposition 2.2 of [2] the function f is locally Lipschitz, i.e. for each $x_0 \in X$ there are a convex neighbourhood V_{x_0} and a constant L_{x_0} such that f satisfies the Lipschitz condition with constant L_{x_0} on V_{x_0}. This implies that for every $y \in V_{x_0}$ the Clarke subdifferential $\partial f|_y$ at y is not empty. By Theorem 3.1 of [2] this subdifferential is equal to the γ-subdifferential, hence f is a continuous t^γ-X^*-subdifferentiable function on V_{x_0}. Since we can cover the whole set Ω by such neighbourhoods V_{x_0}, f is continuous and t^γ-X^*-subdifferentiable on the whole Ω. Then by Theorem 9 there is a subset $A_f \subset \Omega$ of the first category such that f is Fréchet differentiable on $\Omega \setminus A_f$. ∎

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