[13] E. M. Stein, *Oscillatory integrals in Fourier analysis*, in: Beijing Lectures in Harmonic Analysis, Princeton Univ. Press, 1986, 307–355.

Department of Mathematical Sciences
New Mexico State University
Las Cruces, New Mexico 88003-0001
U.S.A.
E-mail: jalvarez@nmsu.edu

Departamento de Matemática, UFSCar
São Carlos, 13565-905 SP, Brasil
E-mail: hounie@power.ufscar.br

Received May 12, 1997
Revised version July 15, 1998
(3884)

STUDIA MATHEMATICA 133 (1) (1999)

On Sobolev spaces of fractional order and ε-families of operators on spaces of homogeneous type

by

A. EDUARDO GATTO and STEPHEN VÁGI (Chicago, Ill.)

Dedicated to Professor Carlos Segovia Fernández

Abstract. We introduce Sobolev spaces L^p_α for $1 < p < \infty$ and small positive α on spaces of homogeneous type as the classes of functions f in L^p with fractional derivative of order α, $D^\alpha f$, as introduced in [2], in L^p. We show that for small α, L^p_α coincides with the continuous version of the Triebel–Lizorkin space $F^{\alpha,2}_p$ as defined by Y. S. Han and E. T. Sawyer in [4]. To prove this result we give a more general definition of ε-families of operators on spaces of homogeneous type, in which the identity operator is replaced by an invertible operator. Then we show that the family $t^\alpha D^\alpha q(x,y,t)$ is an ε-family of operators in this new sense, where $q(x,y,t) = t\frac{\partial}{\partial t}s(x,y,t)$, and $s(x,y,t)$ is a Coifman type approximation to the identity.

1. Definitions and statement of results. Let (X,δ,μ) be a space of homogeneous type of infinite measure and such that $\mu(\{x\}) = 0$ for every x in X. Without loss of generality it can be assumed that (X,δ,μ) is a normal space of order γ, $0 < \gamma \le 1$. For $0 < \alpha < 1$ let

$$\delta_{-\alpha}(x,y) = \left(\int_0^\infty t^{-\alpha} s(x,y,t)\,\frac{dt}{t}\right)^{1/(-\alpha-1)}$$

where $s(x,y,t)$ is a Coifman type approximation to the identity. In [2] it is shown that $\delta_{-\alpha}(x,y)$ is a quasidistance equivalent to $\delta(x,y)$.

Let C^η_0, $0 < \eta \le \gamma$, be the space of Lipschitz functions of order η with bounded support. The fractional derivative of order α of a function f belonging to C^η_0, $0 < \alpha < \eta$, was defined in [2] by the formula

1991 *Mathematics Subject Classification*: Primary 46E15; Secondary 26A33.

The first author supported in part by a grant from the Faculty Research and Development Fund of the College of Liberal Arts and Sciences of DePaul University. It is also a pleasure for this author to thank Y. S. Han for a helpful discussion of the present subject.

B UW

$$D^{\alpha}f(x) = \int_X \frac{f(y) - f(x)}{\delta_{-\alpha}^{1+\alpha}(x,y)}\, d\mu(y). \tag{1}$$

We extend this definition to functions in L^p, $1 < p < \infty$, as follows.

DEFINITION 1. Let $f \in L^p$, $1 < p < \infty$. If there exists $g \in L^p$ such that for all $\varphi \in C_0^{\eta}$, $0 < \alpha < \eta \le \gamma$, $(f, D^{\alpha}\varphi) = (g, \varphi)$, then we define $D^{\alpha} f = g$.

DEFINITION 2. Let $0 < \alpha < \gamma$ and $1 < p < \infty$. The space L^p_{α} is the set of functions in L^p with fractional derivative of order α in L^p, with the norm $\|f\|_{L^p_\alpha} = \|f\|_{L^p} + \|D^{\alpha} f\|_{L^p}$.

The letter c will denote a constant, not necessarily the same in different occurrences, and the symbol $\simeq$ between two norms indicates that the norms are equivalent.

Triebel–Lizorkin spaces on spaces of homogeneous type have been introduced by Y. S. Han and E. T. Sawyer [4]. Here we will use a continuous version of their definition.

As before, $s(x,y,t)$ will denote a Coifman type approximation to the identity and

$$q(x,y,t) = t\frac{\partial s(x,y,t)}{\partial t}.$$

For the properties of $s(x,y,t)$ see [2].

For $f \in L^p$ we denote by Q_t, $t > 0$, the operator

$$Q_t f(x) = -\int_X q(x,y,t) f(y)\, d\mu(y).$$

DEFINITION 3. For $1 < p < \infty$ and $0 < \alpha < \gamma$ the space $F_p^{\alpha,2}$ is the set of functions $f \in L^p$ for which

$$\|f\|_{\dot F_p^{\alpha,2}} = \left\| \left(\int_0^\infty t^{-2\alpha} |Q_t f|^2 \,\frac{dt}{t}\right)^{1/2} \right\|_p < \infty,$$

and the norm of f in $F_p^{\alpha,2}$ is defined by

$$\|f\|_{F_p^{\alpha,2}} = \|f\|_{L^p} + \|f\|_{\dot F_p^{\alpha,2}}.$$

DEFINITION 4. Let $0 < \varepsilon_2 < \varepsilon_1$. We say that $\{\widetilde Q_t\}_{t>0}$ is an *ε-family of operators* if each operator $\widetilde Q_t f(x) = \int \widetilde q(x,y,t) f(y)\, d\mu(y)$ is given by a continuous kernel $\widetilde q(x,y,t)$ which satisfies the following conditions:

$$|\widetilde q(x,y,t)| \le \frac{c}{t(1+\delta(x,y)/t)^{1+\varepsilon_1}} \tag{2}$$

for all $x, y \in X$ and $t > 0$, and

$$|\widetilde q(x,y,t) - \widetilde q(x,y',t)| \le \left(\frac{\delta(y,y')/t}{1+\delta(x,y)/t}\right)^{\varepsilon_2} \frac{c}{t(1+\delta(x,y)/t)^{1+\varepsilon_1}} \tag{3}$$

for all $t > 0$ and $x, y, y' \in X$ such that

$$\frac{\delta(y,y')}{t} \le \frac{1}{2\kappa}\left(1 + \frac{\delta(x,y)}{t}\right).$$

Here κ is the constant of the "triangle inequality" of δ, and c is a constant independent of x, y, y', t. The ε-families of operators in $\mathbb{R}^n$ were introduced by M. Christ and J. L. Journé [1]. We refer the reader to [3] where these families are studied in the context of Triebel–Lizorkin spaces on $\mathbb{R}^n$. Further, $\widetilde Q'_t$ is defined by $\widetilde Q'_t f(x) = \int_X \widetilde q'(x,y,t) f(y)\, d\mu(y)$, where $\widetilde q'(x,y,t) = \widetilde q(y,x,t)$.

THEOREM 1. *Let $1 < p < \infty$, and let $\{\widetilde Q_t\}_{t>0}$ and $\{\widetilde Q'_t\}_{t>0}$ be ε-families of operators such that $\widetilde Q_t 1 = \widetilde Q'_t 1 = 0$, for all $t > 0$, and*

$$\int_0^\infty \widetilde Q_t \,\frac{dt}{t} = T$$

is an invertible operator in $\dot F_p^{\alpha,2}$. Then there exists $\alpha_1 > 0$ such that for $0 < \alpha < \alpha_1$ and $\varphi \in C_0^{\eta}$, $\alpha < \eta \le \gamma$, we have

$$\|\varphi\|_{\dot F_p^{\alpha,2}} \simeq \left\| \left(\int_0^\infty t^{-2\alpha} |\widetilde Q_t \varphi|^2 \,\frac{dt}{t}\right)^{1/2} \right\|_p.$$

THEOREM 2. *Let $1 < p < \infty$. Then there exists $\alpha_2 > 0$ such that for $0 < \alpha < \alpha_2$, $L^p_\alpha = F_p^{\alpha,2}$, and the corresponding norms are equivalent.*

Proofs

Proof of Theorem 1. Theorem 1 is an extension of Theorem 4.5 of [3]. The main novelty is the replacement of the condition $\sum_k D_k = I$ by $\int_0^\infty \widetilde Q_t \frac{dt}{t} = T$, where T is an invertible operator in $\dot F_p^{\alpha,2}$. The original proof can be adapted to the continuous setting and the new hypotheses, and we refer the reader to [3] for the details.

Proof of Theorem 2. Since L^p_α and $F_p^{\alpha,2}$ are complete, and the space C_0^{η}, $\alpha < \eta \le \gamma$, is dense in both and $0 < \alpha < \alpha_0$, it suffices to show that for $\varphi \in C_0^{\eta}$,

$$\|D^{\alpha}\varphi\|_{L^p} \simeq \|\varphi\|_{\dot F_p^{\alpha,2}},$$

where α_0 will be determined later. By the Littlewood–Paley theory on spaces of homogeneous type [4] we have

$$\|D^{\alpha}\varphi\|_{L^p} \simeq \left\| \left(\int_0^\infty |Q_t(D^{\alpha}\varphi)|^2 \,\frac{dt}{t}\right)^{1/2} \right\|_p. \tag{4}$$

In [2] it is proved that for $f \in \mathrm{Lip}(\eta) \cap L^\infty$, $\alpha < \eta \le \gamma$, and $g \in C_0^\beta$, we have $(D^\alpha f, g) = (f, D^\alpha g)$; therefore

$$(5)\qquad Q_t(D^\alpha \varphi)(x) = \int_X q(x,y,t)(D^\alpha\varphi)(y)\,d\mu(y) = \int_X (D^\alpha_{(y)} q(x,y,t))\varphi(y)\,d\mu(y) = \int_X t^{-\alpha}\widetilde{q}(x,y,t)\varphi(y)\,d\mu(y) = t^{-\alpha}\widetilde{Q}_t\varphi(x)$$

where $\widetilde{q}(x,y,t) = t^\alpha D^\alpha_{(y)} q(x,y,t)$, and $\widetilde{Q}_t$ denotes the operator whose kernel is $\widetilde{q}$.

Using (4) and (5) we have

$$(6)\qquad \|D^\alpha\varphi\|_p \simeq \Big\| \Big(\int_0^\infty t^{-2\alpha}|\widetilde{Q}_t\varphi|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_p .$$

In order to apply Theorem 1 to the right hand side of (6) we must show that $\widetilde{Q}_t$ and $\widetilde{Q}'_t$ are ε-families of operators and satisfy the hypotheses of Theorem 1. In the lemma below we show that $\{\widetilde{Q}_t\}_{t>0}$ and $\{\widetilde{Q}'_t\}_{t>0}$ are ε-families of operators. To show that $\widetilde{Q}_t 1 = \widetilde{Q}'_t 1 = 0$ observe that, as mentioned above, for $f \in \mathrm{Lip}(\eta)\cap L^\infty$, $\alpha < \eta \le \gamma$, and $g \in C_0^\beta$, we have $(D^\alpha f, g) = (f, D^\alpha g)$, and since $D^\alpha 1 = 0$, we have

$$0 = \int (D^\alpha 1) q(x,y,t)\,d\mu(y) = \int 1 D^\alpha_{(y)} q(x,y,t)\,d\mu(y) = t^{-\alpha}\widetilde{Q}_t 1.$$

On the other hand,

$$\widetilde{Q}'_t 1(x) = \int \widetilde{q}(y,x,t)\,d\mu(y) = t^\alpha \int \Big(\int \frac{q(y,z,t) - q(y,x,t)}{\delta_{-\alpha}^{1+\alpha}(z,x)}\,d\mu(z)\Big) d\mu(y) = 0$$

because the double integral is absolutely convergent, and the integral with respect to y is zero for all $t > 0$.

By the representation formulas of Theorem 1.6 of [2], i.e. $\alpha I_\alpha f = \int_0^\infty t^\alpha Q_t(f)\,\frac{dt}{t}$ and $-\alpha D^\alpha f = \int_0^\infty t^{-\alpha} Q_t(f)\,\frac{dt}{t}$, and by (5), for $\varphi \in C_0^\eta$, $0 < \alpha < \eta$, we have

$$S_\alpha \varphi = I_\alpha(D^\alpha\varphi) = \int_0^\infty t^\alpha Q_t(D^\alpha\varphi)\,\frac{dt}{t} = -\alpha^2 \int_0^\infty t^\alpha t^{-\alpha}\widetilde{Q}_t\varphi\,\frac{dt}{t} = -\alpha^2\int_0^\infty \widetilde{Q}_t\varphi\,\frac{dt}{t}.$$

The operator S_α has been considered before in [2], where it was proved to be invertible in L^2. We will show now that S_α is invertible in $\dot{F}_p^{\alpha,2}$ for small $\alpha > 0$. The proof is analogous to the L^2 case except for (7) below. As shown in [2] we have the representation formula

$$(I + \alpha^2 S_\alpha)\varphi = \int_0^\infty (1 - t^\alpha) V_t\varphi\,\frac{dt}{t},$$

where

$$V_t\varphi = \int_0^\infty Q_{st}Q_s\varphi\,\frac{dt}{t}.$$

On the other hand, the following continuous version of the estimates proved in Lemmas (5.21) and (5.24) of [4] holds:

$$(7)\qquad \|V_t\varphi\|_{\dot{F}_p^{\alpha,2}} \le c(t)\|\varphi\|_{\dot{F}_p^{\alpha,2}}$$

where

$$c(t) \le c\begin{cases} t^\beta, & 0 < t \le 1,\\ t^{-\beta}, & t > 1,\end{cases}$$

with $0 < \beta < \gamma$. Therefore

$$\|(I + \alpha^2 S_\alpha)\varphi\|_{\dot{F}_p^{\alpha,2}} \le \int_0^\infty (1 - t^\alpha)c(t)\,\frac{dt}{t}\,\|\varphi\|_{\dot{F}_p^{\alpha,2}}.$$

To estimate the last integral we write it as the sum

$$\int_0^{1/N} |1 - t^\alpha| c(t)\,\frac{dt}{t} + \int_{1/N}^N |1 - t^\alpha| c(t)\,\frac{dt}{t} + \int_N^\infty |1 - t^\alpha| c(t)\,\frac{dt}{t} = I_1 + I_2 + I_3.$$

Using the estimate (7) for $c(t)$ we can find $N = N_0$ sufficiently large so that I_1 and I_3 are less than $1/4$ uniformly with respect to α for α in $(0, \gamma']$ and fixed $\gamma' < \gamma$. Having chosen N_0 we can find α_0 such that for $0 < \alpha < \alpha_0$, I_2 is less than $1/2$. Therefore $\|I + \alpha^2 S_\alpha\|_{\dot{F}_p^{\alpha,2}} < 1$, and hence $-\alpha^2 S_\alpha$ is invertible, and therefore so is S_α. Applying Theorem 1 with $T = -(1/\alpha^2)S_\alpha$ we see that the right hand side of (6) is equivalent to $\|\varphi\|_{\dot{F}_p^{\alpha,2}}$, $0 < \alpha < \alpha_1$ and consequently

$$\|D^\alpha\varphi\|_{L^p} \simeq \|\varphi\|_{\dot{F}_p^{\alpha,2}}.$$

To complete the proof of Theorem 2 we still have to prove the following lemma.

LEMMA. *Let $\widetilde{q}(x,y,t) = t^\alpha D^\alpha_{(y)} q(x,y,t)$ and $\widetilde{q}'(x,y,t) = \widetilde{q}(y,x,t)$. Then the corresponding operators $\widetilde{Q}_t$ and $\widetilde{Q}'_t$ are ε-families of operators.*

Proof. We will use in the proof the fact stated before that $\delta_{-\alpha}$ is equivalent to δ (see [2]). To estimate $\widetilde{q}(x,y,t)$ we need the following known properties of $q(x,y,t)$:

(i) $q(x,y,t) = q(y,x,t)$ for all x, y in X and $t > 0$.
(ii) $q(x,y,t) = 0$ if $\delta(x,y) > b_1 t$, where b_1 is a positive constant.
(iii) $|q(x,y,t)| \le c_1/t$ for all x, y in X and $t > 0$.

(iv) $|q(x,y,t)-q(x',y,t)|\le(c_2/t^{1+\gamma})\delta^\gamma(x,x')$ for all x,x',y in X and $t>0$.
(v) $\int_X q(x,y,t)\,d\mu(y)=0$ for all x in X and $t>0$.

For computing the fractional derivative of $q(x,y,t)$, we use formula (1), and we have

$$\widetilde{q}(x,y,t)=t^\alpha\int\limits_X\frac{q(x,y,t)-q(x,z,t)}{\delta_{-\alpha}^{1+\alpha}(y,z)}\,d\mu(z).$$

To prove (2), consider first $y\in B_{2\kappa b_1 t}(x)$, and write $\widetilde{q}(x,y,t)=t^\alpha\int_{\delta(z,y)\le t}+t^\alpha\int_{\delta(z,y)>t}=I+II$. By (iv) and (i),

$$|I|\le t^\alpha\int\limits_{\delta(y,z)\le t}\frac{c_2\delta^\gamma(y,z)}{t^{1+\gamma}\delta^{1+\alpha}(y,z)}\,d\mu(z)=\frac{c_2t^\alpha}{t^{1+\gamma}}\int\limits_{\delta(y,z)\le t}\frac{d\mu(z)}{\delta^{1+\alpha-\gamma}(y,z)}\le\frac{ct^\alpha}{t^{1+\gamma}}t^{\gamma-\alpha}=\frac{c}{t}.$$

By (iii),

$$|II|\le t^\alpha\int\limits_{\delta(y,z)>t}\frac{2c_1}{t\delta^{1+\alpha}(y,z)}\,d\mu(z)\le t^\alpha\frac{c}{t}t^{-\alpha}=\frac{c}{t}.$$

Since $y\in B_{2\kappa b_1 t}(x)$ it follows that

$$|I|+|II|\le\frac{c}{t(1+\delta(x,y)/t)^{1+\alpha}}.$$

Now consider $y\notin B_{2\kappa b_1 t}(x)$; by (ii) and (iii),

$$|\widetilde{q}(x,y,t)|\le t^\alpha\int\limits_{B_{b_1t}(x)}\frac{|q(x,z,t)|}{\delta^{1+\alpha}(y,z)}\,d\mu(z)\le t^\alpha\int\limits_{B_{b_1t}(x)}\frac{c}{t\delta^{1+\alpha}(y,z)}\,d\mu(z).$$

For $z\in B_{b_1t}(x)$, $\delta(y,z)\ge\delta(x,y)/(2\kappa)$, and the last expression is majorized by $c/\delta^{1+\alpha(y,x)}$; this in turn can be easily seen to be less than or equal to

$$\frac{c}{t}\frac{1}{(1+\delta(x,y)/t)^{1+\alpha}}.$$

This proves (2) with $\varepsilon_1=\alpha$.

We now prove (3). By Theorem 2 of [2] and property (iv) of $q(x,y,t)$,

$$\|D^\alpha q(x,\cdot,t)\|_{\mathrm{Lip}(\gamma-\alpha)}\le c\|q(x,\cdot,t)\|_{\mathrm{Lip}(\gamma)}\le c'/t^{1+\gamma}.$$

Therefore for $y\in B_{2b_1\kappa t}(x)$,

$$|\widetilde{q}(x,y,t)-\widetilde{q}(x,y',t)|\le t^\alpha\frac{c'}{t^{1+\gamma}}\delta^{\gamma-\alpha}(y,y')\le\frac{c''}{t}\left(\frac{\delta(y,y')/t}{1+\delta(x,y)/t}\right)^{\gamma-\alpha}\frac{1}{(1+\delta(x,y)/t)^{1+\varepsilon_1}}$$

where ε_1 is any positive number, and c'' depends on ε_1.

For $y\notin B_{2\kappa b_1 t}(x)$ and

$$\frac{\delta(y,y')}{t}\le\frac{1}{2\kappa}\left(1+\frac{\delta(x,y)}{t}\right),$$

we have $q(x,y,t)=0$, $q(x,y',t)=0$, and hence

$$|\widetilde{q}(x,y,t)-\widetilde{q}(x,y',t)|\le t^\alpha\int\limits_{B_{b_1t}(x)}|q(x,z,t)|\left|\frac{1}{\delta_{-\alpha}^{1+\alpha}(z,y)}-\frac{1}{\delta_{-\alpha}^{1+\alpha}(z,y')}\right|d\mu(z).$$

It also follows that $\delta(y,y')\le c\kappa\delta(y,z)$ with $c>1$, and therefore

$$\left|\frac{1}{\delta_{-\alpha}^{1+\alpha}(z,y)}-\frac{1}{\delta_{-\alpha}^{1+\alpha}(z,y)}\right|\le K\frac{\delta^\gamma(y,y')}{\delta^{1+\alpha+\gamma}(z,y)}$$

(see e.g. Lemma 3 of [2]). Using this estimate and property (iii) of $q(x,y,t)$ we have

$$|\widetilde{q}(x,y,t)-\widetilde{q}(x,y',t)|\le t^\alpha\int\limits_{B_{b_1t}(x)}\frac{c}{t}K\frac{\delta^\gamma(y,y')}{\delta^{1+\alpha+\gamma}(z,y)}\,d\mu(z).$$

For $z\in B_{b_1t}(x)$, we have $\delta(x,y)\le\delta(y,z)$, and hence

$$|\widetilde{q}(x,y,t)-\widetilde{q}(x,y',t)|\le ct^\alpha\frac{\delta^\gamma(y,y')}{\delta^{1+\alpha+\gamma}(x,y)}.$$

As in the proof of (2) it follows that this is less than or equal to

$$\frac{c}{t}\left(\frac{\delta(y,y')/t}{1+\delta(x,y)/t}\right)^{\gamma-\alpha}\frac{1}{(1+\delta(x,y)/t)^{1+\alpha}},$$

because $\gamma-\alpha<\gamma$. This shows that $\{\widetilde{Q}_t\}_{t>0}$ is an ε-family with $\varepsilon_1=\alpha$ and $\varepsilon_2<\min(\alpha,\gamma-\alpha)$.

We now prove that the $\widetilde{Q}_t'$ also form an ε-family. Since $\delta(x,y)=\delta(y,x)$, (2) is already proved. Let $\delta(y,y')=r>0$. Then

$$\begin{aligned}|\widetilde{q}(y,x,t)&-\widetilde{q}(y',x,t)|\\&=\left|t^\alpha\int\limits_X\left(\frac{q(y,z,t)-q(y,x,t)}{\delta_{-\alpha}^{1+\alpha}(x,z)}-\frac{q(y',z,t)-q(y',x,t)}{\delta_{-\alpha}^{1+\alpha}(x,z)}\right)d\mu(z)\right|\\&\le t^\alpha\Bigg|\int\limits_{\delta(x,z)\le r}\Bigg|+t^\alpha\Bigg|\int\limits_{\delta(x,z)>r}\Bigg|=I+II.\end{aligned}$$

Both terms in the integrand of I are estimated the same way using properties (iv) and (i) of $q(x,y,t)$. For the first term we have

$$t^\alpha \int\limits_{\delta(x,z)\le r} \frac{|q(y,z,t)-q(y,x,t)|}{\delta^{1+\alpha}(x,z)}\,d\mu(z) \le t^\alpha \frac{2c_2}{t^{1+\gamma}} \int\limits_{\delta(x,z)\le r} \frac{\delta^\gamma(z,x)}{\delta^{1+\alpha}(z,x)}\,d\mu(z)$$
$$\le \frac{t^\alpha c}{t^{1+\gamma}} r^{\gamma-\alpha} = \frac{c}{t}\left(\frac{\delta(y,y')}{t}\right)^{\gamma-\alpha},$$

and hence

$$I \le \frac{c}{t}\left(\frac{\delta(y,y')}{t}\right)^{\gamma-\alpha}.$$

To estimate II we rearrange the integrand and we have

$$II \le t^\alpha \int\limits_{\delta(x,z)>r} \left| \frac{q(y,z,t)-q(y',z,t)}{\delta_{-\alpha}^{1+\alpha}(x,z)} - \frac{q(y,x,t)-q(y',x,t)}{\delta_{-\alpha}^{1+\alpha}(x,z)} \right| d\mu(z)$$

and, using (iv), we get

$$II \le t^\alpha \frac{2c_2}{t^{1+\gamma}} \int\limits_{\delta(x,z)>r} \frac{\delta^\gamma(y,y')}{\delta^{1+\alpha}(x,z)}\,dz \le c\frac{t^\alpha}{t^{1+\gamma}} \frac{\delta^\gamma(y,y')}{r^\alpha} = \frac{c}{t}\left(\frac{\delta(y,y')}{t}\right)^{\gamma-\alpha}.$$

Adding the estimates of I and II we have

$$|\widetilde{q}(y,x,t)-\widetilde{q}(y',x,t)| \le \frac{c}{t}\left(\frac{\delta(y,y')}{t}\right)^{\gamma-\alpha}$$

for all $x,y,y' \in X$ and $t>0$. On the other hand, for

$$\frac{\delta(y,y')}{t} \le \frac{1}{2\kappa}\left(1+\frac{\delta(x,y)}{t}\right),$$

estimating each term of the difference above using (2), we also have

$$|\widetilde{q}(y,x,t)-\widetilde{q}(y',x,t)| \le |\widetilde{q}(y,x,t)| + |\widetilde{q}(y',x,t)|$$
$$\le \frac{c}{t(1+\delta(x,y)/t)^{1+\alpha}} + \frac{c}{t(1+\delta(x,y)/t)^{1+\alpha}}$$
$$\le \frac{c'}{t(1+\delta(x,y)/t)^{1+\alpha}}.$$

Finally, for $0<\lambda<1$, combining the two estimates we can write

$$|\widetilde{q}'(x,y,t)-\widetilde{q}'(x,y',t)| \le \left(\frac{c}{t}\left(\frac{\delta(y,y')}{t}\right)^{\gamma-\alpha}\right)^\lambda \left(\frac{c'}{t(1+\delta(x,y)/t)^{1+\alpha}}\right)^{1-\lambda}$$
$$\le \left(\frac{\delta(y,y')/t}{1+\delta(x,y)/t}\right)^{\varepsilon_2} \frac{c''}{(1+\delta(x,y)/t)^{1+\varepsilon_1}},$$

where $\varepsilon_1 = \alpha - \lambda(1-\alpha)$ and $\varepsilon_2 = \lambda(\gamma-\alpha)$. Taking λ small enough one has $\varepsilon_2 < \varepsilon_1$.

This concludes the proof of the lemma, and hence also of Theorem 2.

References

[1] M. Christ and J. L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math. 159 (1987), 51–80.

[2] A. E. Gatto, C. Segovia and S. Vági, *On fractional differentiation and integration on spaces of homogeneous type*, Rev. Mat. Iberoamericana 12 (1996), 111–145.

[3] Y. S. Han, B. Jawerth, M. Taibelson and G. Weiss, *Littlewood-Paley theory and ε-families of operators*, Colloq. Math. 60/61 (1990), 321–359.

[4] Y. S. Han and E. T. Sawyer, *Littlewood-Paley theory on spaces of homogeneous type and the classical function spaces*, Mem. Amer. Math. Soc. 530 (1994).

Department of the Mathematical Sciences
DePaul University
2219 North Kenmore Ave.
Chicago, Illinois 60614
U.S.A.
E-mail: aegatto@condor.depaul.edu
svagi@condor.depaul.edu

Received November 17, 1997 (4001)