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Facultad de Ciencias
Universidad de Navarra
31080 Pamplona, Spain
E-mail: mjchasco@fisica.unav.es

Facultad de Ciencias Matemáticas
Universidad Complutense de Madrid
28040 Madrid, Spain
E-mail: EM_peinador@mat.ucm.es

Muskhelishvili Institute of Comp. Math.
Georgian Academy of Sciences
Tbilisi 93, Georgia
E-mail: tar@scien.compmath.acnet.ge

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On a vector-valued local ergodic theorem in L_∞

by

RYOTARO SATO (Okayama)

Abstract. Let $T = \{T(u) : u \in \mathbb{R}_d^+\}$ be a strongly continuous d-dimensional semigroup of linear contractions on $L_1((\Omega,\Sigma,\mu);X)$, where (Ω,Σ,μ) is a σ-finite measure space and X is a reflexive Banach space. Since $L_1((\Omega,\Sigma,\mu);X)^* = L_\infty((\Omega,\Sigma,\mu);X^*)$, the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{R}_d^+\}$ becomes a weak*-continuous semigroup of linear contractions acting on $L_\infty((\Omega,\Sigma,\mu);X^*)$. In this paper the local ergodic theorem is studied for the adjoint semigroup T^*. Assuming that each $T(u)$, $u \in \mathbb{R}_d^+$, has a contraction majorant $P(u)$ defined on $L_1((\Omega,\Sigma,\mu);\mathbb{R})$, that is, $P(u)$ is a positive linear contraction on $L_1((\Omega,\Sigma,\mu);\mathbb{R})$ such that $\|T(u)f(\omega)\| \le P(u)\|f(\cdot)\|(\omega)$ almost everywhere on Ω for every $f \in L_1((\Omega,\Sigma,\mu);X)$, we prove that the local ergodic theorem holds for T^*.

1. Introduction. Define $\mathbb{P}_d = \{u=(u_1,\ldots,u_d) : u_i > 0,\ 1 \le i \le d\}$ and $\mathbb{R}_d^+ = \{u = (u_1,\ldots,u_d) : u_i \ge 0,\ 1 \le i \le d\}$, and denote by $\mathcal{I}_d$ the class of all bounded intervals in $\mathbb{P}_d$ and by λ_d the d-dimensional Lebesgue measure. Let X be a reflexive Banach space and (Ω,Σ,μ) be a σ-finite measure space. We consider a strongly continuous d-dimensional semigroup $T = \{T(u) : u \in \mathbb{P}_d\}$ of linear contractions on $L_1(\Omega;X) = L_1((\Omega,\Sigma,\mu);X)$, where $L_1((\Omega,\Sigma,\mu);X)$ is the usual Banach space of all X-valued strongly measurable functions on Ω for which the norm is given by

$$\|f\|_1 = \int_\Omega \|f(\omega)\|\,d\mu < \infty.$$

Since X is reflexive by hypothesis, it follows (cf. Chapter IV of [4]) that $L_1(\Omega;X)^* = L_\infty(\Omega;X^*)$, where $L_\infty(\Omega;X^*)$ is the Banach space of all X^*-valued strongly measurable functions on Ω for which the norm is given by

$$\|f\|_\infty = \operatorname{ess\,sup}\{\|f(\omega)\| : \omega \in \Omega\} < \infty.$$

Thus the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{P}_d\}$ becomes a weak*-continuous d-dimensional semigroup of linear contractions acting on $L_\infty(\Omega;X^*)$.

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Following Emilion [6], for any $I \in \mathcal{I}_d$ such that $\lambda_d(I) > 0$, we let

(1) $$M_I f = \lambda_d(I)^{-1} \int_I T(u) f\, du \quad \text{for } f \in L_1(\Omega; X),$$

and

(2) $$M_I^* g = (M_I)^* g \quad \text{for } g \in L_\infty(\Omega; X^*).$$

In particular, for any positive real number α we put

(3) $$M_\alpha = M_{I(\alpha)} \quad \text{and} \quad M_\alpha^* = M_{I(\alpha)}^* \quad \text{with} \quad I(\alpha) = (0, \alpha]^d.$$

If $f \in L_1(\Omega; X)$ and $g \in L_\infty(\Omega; X^*)$, then we set

$$\langle f, g\rangle = \int_\Omega g(\omega)(f(\omega))\, d\mu = \int_\Omega \langle f(\omega), g(\omega)\rangle\, d\mu$$

(cf. Chapter IV of [4]). We recall that the mapping $g \mapsto \langle \cdot, g\rangle$ is the canonical linear isometry from $L_\infty(\Omega; X^*)$ onto $L_1(\Omega; X)^*$. It follows that

$$\begin{aligned}\langle f, M_I^* g\rangle = \langle M_I f, g\rangle &= \Big\langle \lambda_d(I)^{-1} \int_I T(u) f\, du, g\Big\rangle \\ &= \lambda_d(I)^{-1} \int_I \langle T(u) f, g\rangle\, du = \lambda_d(I)^{-1} \int_I \langle f, T(u)^* g\rangle\, du,\end{aligned}$$

and hence we have

(4) $$M_I^* g = \lambda_d(I)^{-1} \int_I T^*(u) g\, du,$$

where the last integral is a weak*-integral (cf. page 74 in [8]).

In this paper we study the convergence a.e. of the averages $M_\alpha^* g$ as $\alpha \to 0$. But this is meaningless if the $M_\alpha^* g$ denote the equivalence classes and α ranges through all positive real numbers, and so we either have to select suitable representatives or let α range through a countable set. As in [2], we introduce the notations

(5) $$q\text{-}\lim_{\alpha\to 0} \quad \text{and} \quad q\text{-}\limsup_{\alpha\to 0}$$

which mean that these limits are taken as α tends to zero through a countable dense subset $\mathbf{Q}$ of the positive real numbers. Here we may assume that $\mathbf{Q}$ contains the positive rational numbers.

We are now in a position to state our main result.

THEOREM 1. *Let X be a reflexive Banach space. Suppose each $T(u)$, $u \in \mathbb{P}_d$, has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$. If $T(0) = \text{strong-lim}_{u\to 0} T(u)$ exists then*

(6) $$q\text{-}\lim_{\alpha\to 0} M_\alpha^* g = T^*(0) g \quad \text{a.e. on } \Omega$$

for every $g \in L_\infty(\Omega; X^*)$. Conversely, if $q\text{-}\lim_{\alpha\to 0} M_\alpha^* g$ exists a.e. on Ω for every $g \in L_\infty(\Omega; X^*)$ then the semigroup $T = \{T(u) : u \in \mathbb{P}_d\}$ is strongly continuous at the origin $0 \in \mathbb{R}_d^+$.

By the Chacon–Krengel theorem [3], if T is a linear contraction on $L_1(\Omega; X)$, with $X = \mathbb{R}$ or $\mathbb{C}$, then T has a contraction majorant P defined on $L_1(\Omega; \mathbb{R})$. In this sense, Theorem 1 can be regarded as a generalization of the (scalar-valued) local ergodic theorem of Emilion [6]. See also Chapter VII in Krengel's book [7] for related results. But if $X \neq \mathbb{R}$ and $\mathbb{C}$, then we cannot expect in general that T has a contraction majorant P. An example can be found in [9]. It is known that a necessary and sufficient condition for the existence of a contraction majorant P is that for every $0 \le h \in L_1(\Omega; \mathbb{R})$ the function h^* on Ω defined by

$$h^* = \operatorname{ess\,sup}\{\|(Tf)(\cdot)\| : f \in L_1(\Omega; X),\ \|f(\omega)\| \le h(\omega) \text{ a.e. on } \Omega\}$$

satisfies $\|h^*\|_1 \le \|h\|_1$ (see [9]). It would be desirable to prove Theorem 1 without assuming the condition on the existence of a contraction majorant $P(u)$ for each $T(u)$. But the author does not know whether the theorem holds or not when the condition is not assumed.

Suppose for a moment that the semigroup $T = \{T(u) : u \in \mathbb{P}_d\}$ is strongly continuous at the origin $0 \in \mathbb{R}_d^+$, i.e., $T(0) = \text{strong-lim}_{u\to 0} T(u)$ exists. Then T can be extended continuously to $\mathbb{R}_d^+$ in an obvious manner (cf. [1]), and hence we shall use the same symbol T for the extended semigroup. Thus $T = \{T(u) : u \in \mathbb{R}_d^+\}$. If each $T(u)$, $u \in \mathbb{P}_d$, has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$, then it follows that, for any $u \in \mathbb{R}_d^+$, $T(u)$ possesses a contraction majorant $P(u)$ (see the proof of Lemma 1 in [9]).

In order to prove Theorem 1 we need the following result, itself of independent interest, which is a generalization of Theorem 3.1 of Emilion [6].

THEOREM 2. *Let X be a reflexive Banach space and (Ω, Σ, μ) be a finite measure space. Suppose each $T(u)$, $u \in \mathbb{P}_d$, has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$. Then for any $g \in L_\infty(\Omega; X^*)$ and $u \in \mathbb{P}_d$ we have*

(7) $$\lim_{t\to u} \|T^*(t) g - T^*(u) g\|_{L_1(\Omega; X^*)} = 0.$$

In particular, if $T(0) = \text{strong-lim}_{u\to 0} T(u)$ exists then (7) holds for every $g \in L_\infty(\Omega; X^*)$ and $u \in \mathbb{R}_d^+$; in this case t ranges over $\mathbb{R}_d^+$.

To prove these theorems we use the fact that, under the condition of the existence of a contraction majorant $P(u)$ for each $T(u)$, there exists a *sub-semigroup* $\{\tau(u)\}$ of positive linear contractions on $L_1(\Omega; \mathbb{R})$ which dominates the semigroup $T = \{T(u)\}$. That is, $\tau(s+t) \le \tau(s)\tau(t)$ for any $s, t \in \mathbb{P}_d$ (or $s, t \in \mathbb{R}_d^+$) and

(8) $$\|T(u) f(\omega)\| \le \tau(u) \|f(\cdot)\|(\omega) \quad \text{a.e. on } \Omega$$

for every $f \in L_1(\Omega; X)$ and $u \in \mathbb{P}_d$ (or $u \in \mathbb{R}_d^+$). This is basic throughout the paper. Since such a *positive* sub-semigroup has been used by Emilion [6] to prove the (scalar-valued) local ergodic theorem in L_∞, we can modify his arguments in order to prove our Theorems 1 and 2. This idea is the starting point of our paper. In the next section we establish some preliminary lemmas which enable us to develop a method for proving Theorems 1 and 2, and in the following sections we prove Theorem 2 and then Theorem 1 along the arguments of Emilion [6]. In the last section we note that any bounded additive process $F : \mathcal{I}_d \to L_\infty(\Omega; X^*)$ with respect to the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{P}_d\}$ has the form $F(I) = \int_I T^*(u) g\, du$, $I \in \mathcal{I}_d$, for some $g \in L_\infty(\Omega; X^*)$.

2. Lemmas. Without loss of generality we may and do assume $d = 2^m$ with $m \geq 0$. Indeed, if $2^{m-1} < d < 2^m$, then we can let $\widetilde{T}(u, u') = T(u)$ for $(u, u') \in \mathbb{P}_{2^m}$ with $u \in \mathbb{P}_d$. Let $g \in L_\infty(\Omega; X^*)$. Then for any $v = (u, u') \in \mathbb{P}_{2^m}$ with $u \in \mathbb{P}_d$, and any $I \times I' \in \mathcal{I}_{2^m}$ with $I \in \mathcal{I}_d$, we have $T^*(v) g = \widetilde{T}^*(u, u') g$ and

$$M_I^* g = \lambda_d(I)^{-1} \int_I T^*(u) g\, du = \lambda_{2^m}(I \times I')^{-1} \int_{I \times I'} \widetilde{T}^*(v) g\, du.$$

It is clear that T is strongly continuous at the origin if and only if $\widetilde{T}$ is. Thus it is enough to prove Theorems 1 and 2 for $\widetilde{T}$.

From now on, X will denote a reflexive Banach space.

LEMMA 1. *If g is an X^*-valued strongly measurable function on Ω, then to each $\varepsilon > 0$ there corresponds an X-valued strongly measurable function r on Ω such that*

$$\|r(\omega)\| = 1 \quad \text{and} \quad \mathrm{Re}(\langle r(\omega), g(\omega)\rangle) > \|g(\omega)\| - \varepsilon \tag{9}$$

for almost all $\omega \in \Omega$.

Proof. Since g is strongly measurable, there exists a set $A \in \Sigma$, with $\mu(A) = 0$, and a separable subspace Y of X^* such that if $\omega \in \Omega \setminus A$ then $g(\omega) \in Y$. Hence we can choose a sequence (y_n) of elements in Y and a sequence (E_n) of sets in Σ so that

$$E_n \cap E_m = \emptyset \quad (n \neq m), \qquad \bigcup_{n=1}^{\infty} E_n = \Omega \setminus A \quad \text{and}$$
$$\|g(\omega) - y_n\| < \varepsilon/2 \quad \text{for all } \omega \in E_n.$$

Thus $\|y_n\| > \|g(\omega)\| - \varepsilon/2$ for all $\omega \in E_n$. Since X is reflexive, there exists an $x_n \in X$ with $\|x_n\| = 1$ and $\langle x_n, y_n\rangle = \|y_n\|$. If $\omega \in E_n$ then

$$|\langle x_n, g(\omega)\rangle - \|y_n\|\,| = |\langle x_n, g(\omega) - y_n\rangle| \leq \|g(\omega) - y_n\| < \varepsilon/2,$$

and therefore

$$\mathrm{Re}(\langle x_n, g(\omega)\rangle) > \|y_n\| - \varepsilon/2 > \|g(\omega)\| - \varepsilon.$$

Thus the function r on Ω defined by

$$r(\omega) = \sum_{n=1}^{\infty} \chi_{E_n}(\omega) \cdot x_n \quad \text{for } \omega \in \Omega,$$

where χ_A denotes the characteristic function of a set A, satisfies (9) for almost all $\omega \in \Omega$. The proof is complete.

LEMMA 2. *Let T be a bounded linear operator on $L_1(\Omega; X)$. If τ is a positive linear operator on $L_1(\Omega; \mathbb{R})$ such that $\|Tf(\omega)\| \leq \tau\|f(\cdot)\|(\omega)$ a.e. on Ω for every $f \in L_1(\Omega; X)$, then*

$$\|T^* g(\omega)\| \leq \tau^* \|g(\cdot)\|(\omega) \quad \text{a.e. on } \Omega \tag{10}$$

for every $g \in L_\infty(\Omega; X^*)$.

Proof. Since $T^* g \in L_\infty(\Omega; X^*)$, we can apply Lemma 1 to infer that for any $\varepsilon > 0$ there exists a function $r \in L_\infty(\Omega; X)$ such that

$$\|r(\omega)\| = 1 \quad \text{and} \quad \mathrm{Re}(\langle r(\omega), T^* g(\omega)\rangle) > \|T^* g(\omega)\| - \varepsilon$$

for almost all $\omega \in \Omega$. Let $0 \leq h \in L_1(\Omega; \mathbb{R})$. Then

$$\begin{aligned}
\int_\Omega h \cdot \|T^* g\|\, d\mu &\leq \int_\Omega h(\omega) \cdot \mathrm{Re}(\langle r(\omega), T^* g(\omega)\rangle + \varepsilon)\, d\mu \\
&= \int_\Omega \mathrm{Re}(\langle h(\omega) r(\omega), T^* g(\omega)\rangle\, d\mu + \varepsilon \int_\Omega h\, d\mu \\
&\leq \Big| \int_\Omega \langle T(hr)(\omega), g(\omega)\rangle\, d\mu \Big| + \varepsilon \|h\|_1 \\
&\leq \int_\Omega \|T(hr)(\omega)\| \cdot \|g(\omega)\|\, d\mu + \varepsilon \|h\|_1 \\
&\leq \int_\Omega (\tau h) \cdot \|g\|\, d\mu + \varepsilon \|h\|_1 = \int_\Omega h \cdot (\tau^* \|g\|)\, d\mu + \varepsilon \|h\|_1.
\end{aligned}$$

Since ε was arbitrary, this implies $\int_\Omega h \cdot \|T^* g\|\, d\mu \leq \int_\Omega h \cdot (\tau^* \|g\|)\, d\mu$ for every $0 \leq h \in L_1(\Omega; \mathbb{R})$, and consequently $\|T^* g(\omega)\| \leq \tau^* \|g(\cdot)\|(\omega)$ for almost all $\omega \in \Omega$. The proof is complete.

LEMMA 3. *Let $T = \{T(u) : u \in \mathbb{P}_d\}$ be a strongly continuous d-dimensional semigroup of linear contractions on $L_1(\Omega; X)$. Suppose each $T(u)$, $u \in \mathbb{P}_d$, has a contraction majorant $P(u)$ defined on $L_1(\Omega; \mathbb{R})$. Then there exists a strongly continuous one-dimensional semigroup $U = \{U(t) : t > 0\}$ of positive linear contractions on $L_1(\Omega; \mathbb{R})$ such that*

(i) *for each* $\alpha > 0$ *and* $f \in L_1(\Omega; X)$,

$$\|M_\alpha f(\omega)\| \le C_d \cdot \widetilde{\alpha}^{-1}\Big(\int_0^{\widetilde{\alpha}} U(t)\|f\|\,dt\Big)(\omega) \quad \text{a.e. on } \Omega,$$

where C_d *is a constant depending only on* d *and* $\log\widetilde{\alpha} = 2^{-m}\log\alpha$ *with* $d = 2^m$,

(ii) *for each* $\alpha > 0$ *and* $g \in L_\infty(\Omega; X^*)$,

$$\|M^*_\alpha g(\omega)\| \le C_d \cdot \widetilde{\alpha}^{-1}\Big(\int_0^{\widetilde{\alpha}} U^*(t)\|g\|\,dt\Big)(\omega) \quad \text{a.e. on } \Omega.$$

In particular, if $T(0) = \text{strong-lim}_{u\to 0}T(u)$ *exists, then so does* $U(0) = \text{strong-lim}_{t\to 0}U(t)$, *and*

(11) $U(0)h$

$$= \operatorname{ess\,sup}\Big\{\sum_{i=1}^k \|T(0)f_i(\cdot)\| : f_i \in L_1(\Omega; X),\ \sum_{i=1}^k \|f_i(\omega)\| \le h(\omega) \text{ a.e. on } \Omega\Big\}.$$

for every $0 \le h \in L_1(\Omega; \mathbb{R})$.

Proof. Except for (ii), the lemma has been proved in [9], and (ii) is a consequence of Lemma 2. The proof is complete.

3. Proof of Theorem 2. We first consider the case $d = 1$. Let $U = \{U(t) : t > 0\}$ be the one-dimensional semigroup in Lemma 3 which corresponds to the semigroup $T = \{T(u) : u \in \mathbb{P}_1\}$. It follows from [9] that U also satisfies

$$\|T(t)f(\omega)\| \le U(t)\|f(\cdot)\|(\omega) \quad \text{a.e. on } \Omega \tag{12}$$

for every $f \in L_1(\Omega; X)$ and $t > 0$. Put

$$h = \int_0^\infty e^{-t}U(t)1\,dt \quad (\in L_1^+(\Omega;\mathbb{R})), \qquad C = \{\omega : h(\omega) > 0\}, \qquad D = \Omega \setminus C.$$

Then we have

$$\begin{cases} U(u)h = \int_0^\infty e^{-t}U(u+t)1\,dt \le e^u \cdot h & \text{for } u > 0, \\ \chi_D \cdot U(u)f = 0 & \text{a.e. on } \Omega \text{ for every } f \in L_1(\Omega;\mathbb{R}). \end{cases} \tag{13}$$

Thus the adjoint semigroup $U^* = \{U^*(t) : t > 0\}$ acting on $L_\infty(\Omega;\mathbb{R})$ satisfies $U^*(t)(\chi_D \cdot g) = 0$ a.e. on Ω for every $g \in L_\infty(\Omega;\mathbb{R})$, and by Lemma 2,

$$\|T^*(t)(\chi_D \cdot g)\| \le U^*(t)(\chi_D \cdot \|g(\cdot)\|) = 0 \quad \text{a.e. on } \Omega \tag{14}$$

for every $g \in L_\infty(\Omega; X^*)$. Now define

$$S(t)f = h^{-1}\cdot T(t)(fh) \quad \text{for } f \in L_1((C, h\,d\mu); X),$$

so that $S = \{S(t) : t > 0\}$ becomes a strongly continuous semigroup of linear contractions on $L_1((C, h\,d\mu); X)$, and the adjoint semigroup $S^* = \{S^*(t) : t > 0\}$ acting on $L_\infty((C, h\,d\mu); X^*) = L_\infty(C; X^*)$ satisfies

$$S^*(t)g = T^*(t)g \quad \text{a.e. on } C \text{ for every } g \in L_\infty(C; X^*). \tag{15}$$

Let $f \in L_\infty(C; X)$. Since $L_\infty(C; X) \subset L_1((C, h\,d\mu); X)$, we then have

$$\begin{aligned}\|S(t)f(\omega)\| &= \|h(\omega)^{-1}\cdot T(t)(fh)(\omega)\| \le h(\omega)^{-1}\cdot U(t)\|fh(\cdot)\|(\omega) \\ &\le \|f\|_\infty h(\omega)^{-1}\cdot U(t)h(\omega) \le e^t\|f\|_\infty\end{aligned}$$

for almost all $\omega \in \Omega$, where the last inequality comes from (13). Hence $\|S(t)\|_\infty \le e^t$.

Similarly, if we set

$$V(t)f = h^{-1}\cdot U(t)(fh) \quad \text{for } f \in L_1((C, h\,d\mu); \mathbb{R}),$$

then $V = \{V(t) : t > 0\}$ becomes a strongly continuous semigroup of positive linear contractions on $L_1((C, h\,d\mu); \mathbb{R})$ satisfying $\|V(t)\|_\infty \le e^t$ and

$$V^*(t)g = U^*(t)g \quad \text{a.e. on } C \text{ for every } g \in L_\infty(C; \mathbb{R}). \tag{16}$$

Thus $\|V^*(t)\|_1 \le e^t$ and $\|V^*(t)\|_\infty \le 1$, and by Lemma 2,

$$\|S^*(t)\|_1 \le e^t \quad \text{and} \quad \|S^*(t)\|_\infty \le 1.$$

Consider $S^* = \{S^*(t) : t > 0\}$ to be a locally bounded semigroup of linear operators on $L_1((C, h\,d\mu); X^*)$. We then see that S^* is weakly continuous. Indeed, if $f \in L_\infty(C; X^*) \subset L_1((C, h\,d\mu); X^*)$ and $g \in L_\infty((C, h\,d\mu); X) = L_\infty(C; X)$ then

$$\begin{aligned}\langle g, S^*(t)f\rangle_{h\,d\mu} &= \int_C \langle g(\omega), S^*(t)f(\omega)\rangle h(\omega)\,d\mu \\ &= \int_C \langle g(\omega), T^*(t)f(\omega)\rangle h(\omega)\,d\mu \\ &= \int_C \langle h(\omega)g(\omega), T^*(t)f(\omega)\rangle\,d\mu \\ &= \langle hg, T^*(t)f\rangle_\mu = \langle T(t)(hg), f\rangle_\mu.\end{aligned}$$

It follows that the mapping $t \mapsto \langle\cdot, S^*(t)f\rangle$ is weakly continuous for each $f \in L_\infty(C; X^*)$. Since $L_\infty(C; X^*)$ is a dense subspace of $L_1((C, h\,d\mu); X^*)$, this yields the weak continuity of the semigroup $S^* = \{S^*(t) : t > 0\}$ by an approximation argument. Thus, by the theory of semigroups of operators (see e.g. Chapter VIII of [5]), S^* is strongly continuous on the interval $(0, \infty)$, so that for any $g \in L_\infty(\Omega; X^*) \subset L_1(\Omega; X^*)$ and $u > 0$ we have, by (14) and (15),

$$\lim_{t\to u}\int_\Omega \|T^*(t)g - T^*(u)g\|h\,d\mu = 0. \tag{17}$$

Letting $s = u/2 > 0$, we then deduce that

$$\lim_{t\to u}\int_\Omega \|T^*(t)g - T^*(u)g\|\,d\mu$$
$$\le \lim_{t\to u}\int_\Omega U^*(s)(\|T^*(t-s)g - T^*(u-s)g\|)\,d\mu$$
$$= \lim_{t\to u}\int_\Omega U(s)1\cdot\|T^*(t-s)g - T^*(u-s)g\|\,d\mu = 0,$$

where the last equality comes from the fact that for any $\varepsilon > 0$ there exists an $N \ge 1$ such that $\|(U(s)1 - Nh)^+\|_1 < \varepsilon$, together with an approximation argument.

Suppose, in particular, that $T(0) = \text{strong-lim}_{u\to 0}T(u)$ exists. Then $U(0) = \text{strong-lim}_{t\to 0}U(t)$ exists by Lemma 3, and $S^* = \{S^*(t) : t \ge 0\}$ can be considered to be a strongly continuous semigroup of bounded linear operators on $L_1((C, h\,d\mu); X^*)$ which is strongly continuous at the origin. Thus

$$(18)\qquad \lim_{t\to 0}\int_\Omega \|T^*(t)g - T^*(0)g\|h\,d\mu = 0 \quad \text{for every } g \in L_\infty(\Omega; X^*).$$

Since

$$\|T^*(t)g - T^*(0)g\| = \|T^*(0)(T^*(t)g - T^*(0)g)\|$$
$$\le U^*(0)\|T^*(t)g - T^*(0)g\| \quad \text{a.e. on } \Omega,$$

we then apply (18) together with an approximation argument to infer that

$$(19)\quad \lim_{t\to 0}\int_\Omega \|T^*(t)g - T^*(0)g\|\,d\mu \le \lim_{t\to 0}\int_\Omega U(0)1\cdot\|T^*(t)g - T^*(0)g\|\,d\mu = 0,$$

which completes the proof for the case $d = 1$.

We next consider the case $d > 1$. If $u \in \mathbb{P}_d$ is given, choose a sufficiently large $N > 1$ and define

$$T_N(t) = T(t_1 + s_1/N, \ldots, t_d + s_d/N) \quad \text{for } t = (t_1, \ldots, t_d) \in \mathbb{R}_d^+ \setminus \{0\},$$

where

$$s_i = (t_1 + \ldots + t_d) - t_i \quad \text{for } 1 \le i \le d.$$

It is easily seen that $T_N = \{T_N(t) : t \in \mathbb{R}_d^+ \setminus \{0\}\}$ becomes a strongly continuous semigroup of linear contractions on $L_1(\Omega; X)$, and we may assume that u has the form

$$u = (t_1 + s_1/N, \ldots, t_d + s_d/N)$$

for some $t = (t_1, \ldots, t_d) \in \mathbb{P}_d$. Thus, without loss of generality, we can assume from the start that T is a strongly continuous semigroup on $\mathbb{R}_d^+ \setminus \{0\}$.

Let $T_i = \{T_i(t') : t' > 0\}$, $1 \le i \le d$, be the one-dimensional strongly continuous semigroup of linear contractions on $L_1(\Omega; X)$ defined by

$$T_i(t') = T(t'\cdot e^i) \quad \text{for } t' > 0$$

where e^i is the ith unit vector in $\mathbb{R}_d^+$. By the case $d = 1$ we see that

$$(20)\qquad \lim_{t'\to s'}\int_\Omega \|T_i^*(t')g - T_i^*(s')g\|\,d\mu = 0$$

for every $g \in L_\infty(\Omega; X^*)$ and $s' > 0$. If $u = (u_1, \ldots, u_d) \in \mathbb{P}_d$ is given, then, since

$$T^*(t_1, \ldots, t_d)g - T^*(u_1, \ldots, u_d)g$$
$$= [T^*(t_1, \ldots, t_d)g - T^*(t_1, \ldots, t_{d-1}, u_d)g]$$
$$+ [T^*(t_1, \ldots, t_{d-1}, u_d)g - T^*(t_1, \ldots, u_{d-1}, u_d)g] + \ldots$$
$$+ [T^*(t_1, u_2, \ldots, u_d)g - T^*(u_1, \ldots, u_d)g],$$

we deduce that

$$\int_\Omega \|T^*(t_1, \ldots, t_d)g - T^*(u_1, \ldots, u_d)g\|\,d\mu$$
$$\le \sum_{i=1}^d \int_\Omega \|T_1^*(t_1)\ldots T_{i-1}^*(t_{i-1})[T_i^*(t_i)g_i - T_i^*(u_i)g_i]\|\,d\mu,$$

where

$$g_i = \begin{cases} g & \text{if } i = d,\\ T_{i+1}^*(u_{i+1})\ldots T_d^*(u_d)g & \text{if } 1 \le i \le d-1.\end{cases}$$

If $U_i = \{U_i(t') : t' > 0\}$ denotes the one-dimensional semigroup in Lemma 3 corresponding to the semigroup T_i, then we get

$$\int_\Omega \|T_1^*(t_1)\ldots T_{i-1}^*(t_{i-1})[T_i^*(t_i)g_i - T_i^*(u_i)g_i]\|\,d\mu$$
$$\le \int_\Omega U_1^*(t_1)\ldots U_{i-1}^*(t_{i-1})\|T_i^*(t_i)g_i - T_i^*(u_i)g_i\|\,d\mu$$
$$= \int_\Omega U_{i-1}(t_{i-1})\ldots U_1(t_1)1\cdot\|T_i^*(t_i)g_i - T_i^*(u_i)g_i\|\,d\mu$$
$$\le \int_\Omega U_{i-1}(u_{i-1})\ldots U_1(u_1)1\cdot\|T_i^*(t_i)g_i - T_i^*(u_i)g_i\|\,d\mu$$
$$+ 2\|g_i\|_\infty\int_\Omega \|U_{i-1}(t_{i-1})\ldots U_1(t_1)1 - U_{i-1}(u_{i-1})\ldots U(u_1)1\|\,d\mu$$
$$= I(t) + II(t),$$

and by (20)

$$\lim_{t\to u} I(t) = 0.$$

Further, since each $U_i = \{U_i(t') : t' > 0\}$ is strongly continuous on the interval $(0,\infty)$, an easy induction argument yields

$$\lim_{t\to u} II(t) = 0.$$

Hence the first half of the theorem has been proved. The second half can be proved similarly. The proof is complete.

4. Proof of Theorem 1. Since there exists a finite measure on (Ω,Σ) which is equivalent to μ, we may and do assume without loss of generality that μ is finite. (Cf. e.g. (15).)

To prove the first half of Theorem 1, let $T(0) = \text{strong-lim}_{u\to 0} T(u)$. If g is a function of $L_\infty(\Omega; X^*)$ then, by Theorem 2, the mapping $u \mapsto T^*(u)g$ can be regarded as a strongly continuous function from $\mathbb{R}_d^+$ to $L_1(\Omega; X^*)$. Hence, as is easily seen, there exists an X^*-valued function $G(u,\omega)$ defined on $\mathbb{P}_d \times \Omega$, strongly measurable with respect to the product σ-algebra of the Lebesgue measurable subsets of $\mathbb{P}_d$ and Σ, such that for each $u \in \mathbb{P}_d$,

$$G(u,\cdot) \text{ is a representative of } T^*(u)g.$$

Then we have

$$M_\alpha^* g(\omega) = \alpha^{-d}\Big(\int_{I(\alpha)} T^*(u)g\,du\Big)(\omega) = \alpha^{-d}\int_{I(\alpha)} G(u,\omega)\,du \quad \text{a.e. on } \Omega,$$

where the integral $\int_{I(\alpha)} T^*(u)g\,du$ can be taken in the Bochner sense in place of the weak*-integral. Since

$$\|T(u)M_\alpha - M_\alpha\|_1 = \|T^*(u)M_\alpha^* - M_\alpha^*\|_\infty \to 0 \quad \text{as } u \to 0 \in \mathbb{R}_d^+,$$

we obtain

$$\lim_{\beta\to 0} \|M_\beta^* M_\alpha^* - M_\alpha^*\|_\infty = 0. \tag{21}$$

Now put

$$\widehat{g}(\omega) = q\text{-}\limsup_{\alpha\to 0} \|M_\alpha^* g(\omega) - T^*(0)g(\omega)\| \quad \text{for } \omega \in \Omega. \tag{22}$$

Then

$$\begin{aligned}\widehat{g}(\omega) \le\ & q\text{-}\limsup_{\alpha\to 0} \|M_\alpha^*[T^*(0)g - M_\beta^* g](\omega)\| \\ &+ q\text{-}\limsup_{\alpha\to 0} \|M_\alpha^* M_\beta^* g(\omega) - M_\beta^* g(\omega)\| \\ &+ \|M_\beta^* g(\omega) - T^*(0)g(\omega)\| \quad \text{a.e. on } \Omega.\end{aligned}$$

Since Theorem 2 implies $\lim_{\beta\to 0} \|M_\beta^* g - T^*(0)g\|_{L_1(\Omega;X^*)} = 0$, it follows that for each $\varepsilon > 0$,

$$\lim_{\beta\to 0} \mu(\{\omega : \|M_\beta^* g(\omega) - T^*(0)g(\omega)\| > \varepsilon\}) = 0. \tag{23}$$

By this, together with (21), it is enough to show that

$$q\text{-}\lim_{\beta\to 0} \int_\Omega \widehat{g}_\beta\,d\mu = 0 \tag{24}$$

where

$$\widehat{g}_\beta(\omega) = q\text{-}\limsup_{\alpha\to 0} \|M_\alpha^*[T^*(0)g - M_\beta^* g](\omega)\| \quad \text{for } \omega \in \Omega.$$

To prove (24), we use Lemma 3 as follows. Let $U = \{U(t') : t' \ge 0\}$ be the one-dimensional strongly continuous semigroup of positive linear contractions on $L_1(\Omega;\mathbb{R})$ in Lemma 3 corresponding to the semigroup $T = \{T(u) : u \in \mathbb{P}_d\}$. By the scalar-valued local ergodic theorem in L_∞ (see e.g. Theorem 7.1.14 in [7]), for every $\beta > 0$ the limit

$$h_\beta = q\text{-}\lim_{\widetilde{\alpha}\to 0} \widetilde{\alpha}^{-1} \int_0^{\widetilde{\alpha}} U^*(t')(\|T^*(0)g - M_\beta^* g\|)\,dt'$$

exists a.e. on Ω, and further

$$h_\beta = U^*(0)(\|T^*(0)g - M_\beta^* g\|) \quad \text{a.e. on } \Omega.$$

Since $\widehat{g}_\beta \le C_d \cdot h_\beta$ a.e. on Ω by Lemma 3(ii), it follows that

$$\int_\Omega \widehat{g}_\beta\,d\mu \le C_d \int_\Omega h_\beta\,d\mu = C_d \int_\Omega U(0)1 \cdot \|T^*(0)g - M_\beta^* g\|\,d\mu,$$

and by (23),

$$\lim_{\beta\to 0} \int_\Omega U(0)1 \cdot \|T^*(0)g - M_\beta^* g\|\,d\mu = 0,$$

whence the proof of (24) is complete.

Next suppose that $q\text{-}\lim_{\alpha\to 0} M_\alpha^* g$ exists a.e. on Ω for every $g \in L_\infty(\Omega; X^*)$. Then for any $f \in L_1(\Omega; X)$,

$$\begin{aligned} q\text{-}\lim_{\alpha\to 0} \langle M_\alpha f, g\rangle &= q\text{-}\lim_{\alpha\to 0} \langle f, M_\alpha^* g\rangle = q\text{-}\lim_{\alpha\to 0} \int_\Omega \langle f(\omega), M_\alpha^* g(\omega)\rangle\,d\mu \\ &= \int_\Omega \langle f(\omega), q\text{-}\lim_{\alpha\to 0} M_\alpha^* g(\omega)\rangle\,d\mu \end{aligned}$$

by Lebesgue's convergence theorem. Since X is reflexive, $L_1(\Omega; X)$ is weakly sequentially complete (cf. e.g. page 117 of [4]), and thus if (α_n) is a sequence with $\alpha_n \in \mathbf{Q}$ for all $n \ge 1$ and $\lim_n \alpha_n = 0$, then there exists a function $\widetilde{f}$

in $L_1(\Omega;X)$ such that

$$\widetilde{f} = \text{weak-}\lim_n M_{\alpha_n} f. \tag{25}$$

Then

$$T(u)\widetilde{f} = \text{weak-}\lim_n T(u)M_{\alpha_n} f = \text{strong-}\lim_n M_{\alpha_n}(T(u)f) = T(u)f \tag{26}$$

for every $u \in \mathbb{P}_d$, and further, by the Hahn–Banach theorem, $\widetilde{f}$ is in the closed linear subspace of $L_1(\Omega;X)$ generated by the set $\{T(u)f : u \in \mathbb{P}_d\}$. It follows from an approximation argument that

$$\widetilde{f} = \text{strong-}\lim_{u\to 0} T(u)\widetilde{f} = \text{strong-}\lim_{u\to 0} T(u)f, \tag{27}$$

which completes the proof.

5. Additive processes for T^*. We recall that a set function $F : \mathcal{I}_d \to L_\infty(\Omega;X^*)$ is an *additive process* with respect to the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{P}_d\}$ if F satisfies the following conditions:

(i) $T^*(u)F(I) = F(u+I)$ for all $u \in \mathbb{P}_d$ and $I \in \mathcal{I}_d$,

(ii) If $I_1,\ldots,I_k \in \mathcal{I}_d$ are pairwise disjoint and $I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d$ then $F(I) = \sum_i^k F(I_i)$.

F is called *bounded* if

$$K(F) := \sup\{\lambda_d(I)^{-1}\|F(I)\|_\infty : I \in \mathcal{I}_d,\ \lambda_d(I) > 0\} < \infty.$$

In this section we prove the following theorem (cf. Theorem 4.1 in [6]).

THEOREM 3. *Let X be a reflexive Banach space and $F : \mathcal{I}_d \to L_\infty(\Omega;X^*)$ be a bounded additive process with respect to the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{P}_d\}$. Then there exists a function $g \in L_\infty(\Omega;X^*)$ such that $F(I) = \int_I T^*(u)g\,du$ for all $I \in \mathcal{I}_d$.*

For the proof we need the following lemma.

LEMMA 4. *To any $J \in \mathcal{I}_d$, with $\lambda_d(J) > 0$, and an $\varepsilon > 0$ there corresponds a $\gamma > 0$ such that if $I \in \mathcal{I}_d$ satisfies $I \subset (0,\gamma]^d$ and $\lambda_d(I) > 0$ then*

$$\Big\|F(J) - \int_J T^*(u)[\lambda_d(I)^{-1}F(I)]\,du\Big\|_\infty < \varepsilon.$$

Proof. This is an adaptation of the proof of Lemma 3.2 in [1]. Since

$$\|F(J) - T^*(u)F(J)\|_\infty = \|F(J) - F(u+J)\|_\infty \le K(F)\cdot\lambda_d(J \triangle (u+J)),$$

where $A \triangle B$ denotes the symmetric difference of two sets A and B, we have

$$\lim_{u\to 0}\|F(J) - T^*(u)F(J)\|_\infty = 0.$$

Hence we can choose a $\gamma > 0$ so that if $I \in \mathcal{I}_d$, $I \subset (0,\gamma]^d$ and $\lambda_d(I) > 0$ then

$$\Big\|F(J) - \lambda_d(I)^{-1}\int_I T^*(u)F(J)\,du\Big\|_\infty < \frac{\varepsilon}{3}, \tag{28}$$

and further there exists an $I' \in \mathcal{I}_d$ which is a disjoint union of intervals $a_j + I$ with $a_j \in \mathbb{P}_d$, $j = 1,\ldots,m$, and satisfies

$$I' \subset J \quad\text{and}\quad \lambda_d(J \setminus I') < \frac{\varepsilon}{3K(F)}.$$

Then

$$\begin{aligned}\Big\|\int_J T^*(u)[\lambda_d(I)^{-1}F(I)]\,du - \int_{I'} T^*(u)[\lambda_d(I)^{-1}F(I)]\,du\Big\|_\infty \\ = \Big\|\int_{J\setminus I'} T^*(u)[\lambda_d(I)^{-1}F(I)]\,du\Big\|_\infty \le \lambda_d(J\setminus I')K(F) < \frac{\varepsilon}{3},\end{aligned} \tag{29}$$

and

$$\begin{aligned}\int_{I'} T^*(u)[\lambda_d(I)^{-1}F(I)]\,du &= \sum_{j=1}^m \int_{a_j+I} T^*(u)[\lambda_d(I)^{-1}F(I)]\,du \\ &= \sum_{j=1}^m \int_I T^*(a_j+u)[\lambda_d(I)^{-1}F(I)]\,du \\ &= \lambda_d(I)^{-1}\int_I T^*(u)\Big[\sum_{j=1}^m F(a_j+I)\Big]\,du \\ &= \lambda_d(I)^{-1}\int_I T^*(u)F(I')\,du.\end{aligned}$$

Since

$$\begin{aligned}\Big\|\lambda_d(I)^{-1}\int_I T^*(u)F(J)\,du - \lambda_d(I)^{-1}\int_I T^*(u)F(I')\,du\Big\|_\infty \\ = \Big\|\lambda_d(I)^{-1}\int_I T^*(u)[F(J) - F(I')]\,du\Big\|_\infty < \frac{\varepsilon}{3},\end{aligned}$$

we deduce that

$$\Big\|\int_{I'} T^*(u)(\lambda_d(I)^{-1}F(I))\,du - \lambda_d(I)^{-1}\int_I T^*(u)F(J)\,du\Big\|_\infty < \frac{\varepsilon}{3}, \tag{30}$$

which, together with (28) and (29), completes the proof.

Proof of Theorem 3. For a positive number α let

$$A(\alpha) = \alpha^{-d}F((0,\alpha]^d).$$

Since $\|A(\alpha)\|_\infty \le K(F) < \infty$ for all $\alpha > 0$ and since the closed unit ball of $L_\infty(\Omega;X^*)$ is compact in the weak*-topology, we can choose a net

$\{\alpha_\iota : \iota \in D\}$, where D is a directed set and α_ι is a positive number for each $\iota \in D$, and a function g in $L_\infty(\Omega; X^*)$ such that

$$\lim_\iota \alpha_\iota = 0 \quad \text{and} \quad g = \text{weak}^*\text{-}\lim_\iota A(\alpha_\iota). \tag{31}$$

If $J \in \mathcal{I}_d$ and $f \in L_1(\Omega; X)$ are given, then, by Lemma 4 together with (31),

$$\Big\langle f, \int_J T^*(u) g\, du \Big\rangle = \Big\langle \int_J T(u) f\, du,\ g \Big\rangle = \lim_\iota \Big\langle \int_J T(u) f\, du,\ A(\alpha_\iota) \Big\rangle$$
$$= \lim_\iota \Big\langle f, \int_J T^*(u)[A(\alpha_\iota)]\, du \Big\rangle = \langle f, F(J) \rangle.$$

Hence $F(J) = \int_J T^*(u) g\, du$ for $J \in \mathcal{I}_d$, and the proof is complete.

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Department of Mathematics
Faculty of Science
Okayama University
Okayama, 700 Japan
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