On a vector-valued local ergodic theorem in $L_{\infty}$

by

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Abstract. Let $T = \{T(u) : u \in \mathbb{R}^d_+\}$ be a strongly continuous $d$-dimensional semigroup of linear contractions on $L_1((\Omega, \Sigma, \mu); X)$, where $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space and $X$ is a reflexive Banach space. Since $L_1((\Omega, \Sigma, \mu); X) \cong L_{\infty}(\Omega, \Sigma, \mu; X^*)$, the adjoint semigroup $T^* = \{T^*(u) : u \in \mathbb{R}^d_+\}$ becomes a weak*–continuous semigroup of linear contractions acting on $L_{\infty}(\Omega, \Sigma, \mu; X^*)$. In this paper the local ergodic theorem is studied for the adjoint semigroup $T^*$. Assuming that each $T(u)$, $u \in \mathbb{R}^d_+$, has a contraction majorant $P(u)$ defined on $L_1((\Omega, \Sigma, \mu); R)$, that is, $P(u)$ is a positive linear contraction on $L_1((\Omega, \Sigma, \mu); R)$ such that $\|P(u)f(\omega)\| \leq P(u)\|f(\omega)\|$ almost everywhere on $\Omega$ for every $f \in L_1((\Omega, \Sigma, \mu); X)$, we prove that the local ergodic theorem holds for $T^*$.

1. Introduction. Define $P_d = \{u = (u_1, \ldots, u_d) : u_i > 0, 1 \leq i \leq d\}$ and $\mathbb{R}^d_+ = \{u = (u_1, \ldots, u_d) : u_i \geq 0, 1 \leq i \leq d\}$, and denote by $I_d$ the class of all bounded intervals in $P_d$ and by $\lambda_d$ the $d$-dimensional Lebesgue measure. Let $X$ be a reflexive Banach space and $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space. We consider a strongly continuous $d$-dimensional semigroup $T = \{T(u) : u \in P_d\}$ of linear contractions on $L_1((\Omega, \Sigma, \mu); X)$, where $L_1((\Omega, \Sigma, \mu); X)$ is the usual Banach space of all $X$-valued strongly measurable functions on $\Omega$ for which the norm is given by

$$\|f\|_1 = \int_{\Omega} \|f(\omega)\| \, d\mu < \infty.$$  

Since $X$ is reflexive by hypothesis, it follows (cf. Chapter IV of [4]) that $L_1((\Omega, \Sigma, \mu); X) \cong L_{\infty}(\Omega, \Sigma, \mu; X^*)$, where $L_{\infty}(\Omega, \Sigma, \mu; X^*)$ is the Banach space of all $X^*$-valued strongly measurable functions on $\Omega$ for which the norm is given by

$$\|f\|_{\infty} = \text{ess sup} \{\|f(\omega)\| : \omega \in \Omega\} < \infty.$$  

Thus the adjoint semigroup $T^* = \{T^*(u) : u \in P_d\}$ becomes a weak*–continuous $d$-dimensional semigroup of linear contractions acting on $L_{\infty}(\Omega, \Sigma, \mu; X^*)$.  

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285
Following Emilion [6], for any \( I \in \mathcal{I}_d \) such that \( \lambda_d(I) > 0 \), we let
\[
\begin{align*}
(1) \quad M_I f &= \lambda_d(I)^{-1} \int_I T(u)f du \quad \text{for } f \in L_1(\Omega; X),
\end{align*}
\]
and
\[
(2) \quad M_I^* g &= (M_I)^* g \quad \text{for } g \in L_\infty(\Omega; X^*).
\]
In particular, for any positive real number \( \alpha \) we put
\[
(3) \quad M_\alpha = M_{I(\alpha)} \quad \text{and} \quad M_\alpha^* = M_{I(\alpha)}^* \quad \text{with} \quad I(\alpha) = (0, \alpha]^d.
\]
If \( f \in L_1(\Omega; X) \) and \( g \in L_\infty(\Omega; X^*) \), then we set
\[
\langle f, g \rangle = \int_\Omega g(\omega)(f(\omega)) d\mu = \int_\Omega (f(\omega), g(\omega)) d\mu
\]
(cf. Chapter IV of [4]). We recall that the mapping \( g \mapsto \langle \cdot, g \rangle \) is the canonical linear isometry from \( L_\infty(\Omega; X^*) \) onto \( L_1(\Omega; X^*) \). It follows that
\[
\langle f, M_I^* g \rangle = \langle M_I f, g \rangle = \lambda_d(I)^{-1} \int \langle T(u)f, g \rangle du,
\]
and hence we have
\[
(4) \quad M_I^* g = \lambda_d(I)^{-1} \int T(u)^* g du,
\]
where the last integral is a weak*-integral (cf. page 74 in [8]).

In this paper we study the convergence a.e. of the averages \( M_\alpha^* g \) as \( \alpha \to 0 \). But this is meaningless if the \( M_\alpha^* g \) denote the equivalence classes and \( \alpha \) ranges through all positive real numbers, and so we either have to select suitable representatives or let \( \alpha \) range through a countable set. As in [2], we introduce the notations
\[
(5) \quad q\text{-lim}_{\alpha \to 0} \quad \text{and} \quad q\text{-lim sup}_{\alpha \to 0}
\]
which mean that these limits are taken as \( \alpha \) tends to zero through a countable dense subset \( Q \) of the positive real numbers. Here we may assume that \( Q \) contains the positive rational numbers.

We are now in a position to state our main result.

**THEOREM 1.** Let \( X \) be a reflexive Banach space. Suppose each \( T(u) \), \( u \in \mathbb{P}_d \), has a contraction majorant \( P(u) \) defined on \( L_1(\Omega; \mathbb{R}) \). If \( T(0) = \text{strong-lim}_{u \to 0} T(u) \) exists then
\[
(6) \quad q\text{-lim}_{\alpha \to 0} M_\alpha^* g = T^*(0)g \quad \text{a.e. on } \Omega
\]
for every \( g \in L_\infty(\Omega; X^*) \). Conversely, if \( q\text{-lim}_{\alpha \to 0} M_\alpha^* g \) exists a.e. on \( \Omega \) for every \( g \in L_\infty(\Omega; X^*) \) then the semigroup \( T \) = \{ T(u) : u \in \mathbb{P}_d \} \) is strongly continuous at the origin \( 0 \in \mathbb{R}_d^+ \).

By the Chacon–Krenkel theorem [3], if \( T \) is a linear contraction on \( L_1(\Omega; X) \), with \( X = \mathbb{R} \) or \( \mathbb{C} \), then \( T \) has a contraction majorant \( P \) defined on \( L_1(\Omega; \mathbb{R}) \). In this sense, Theorem 1 can be regarded as a generalization of the (scalar-valued) local ergodic theorem of Emilion [6]. See also Chapter VII in Krenkel's book [7] for related results. But if \( X \neq \mathbb{R} \) or \( \mathbb{C} \), then we cannot expect in general that \( T \) has a contraction majorant \( P \). An example can be found in [9]. It is known that a necessary and sufficient condition for the existence of a contraction majorant \( P \) is that for every \( 0 \leq h \in L_1(\Omega; \mathbb{R}) \) the function \( h^* \) on \( \Omega \) defined by
\[
h^* = \text{ess sup} \left[ \| T(f(t)) \| : f \in L_1(\Omega; X), \| f(\omega) \| \leq h(\omega) \text{ a.e. on } \Omega \right]
\]
satisfies \( \| h^* \|_1 \leq \| h \|_1 \) (see [9]). It would be desirable to prove Theorem 1 without assuming the condition on the existence of a contraction majorant \( P(u) \) for each \( T(u) \). But the author does not know whether the theorem holds or not when the condition is not assumed.

Suppose for a moment that the semigroup \( T = \{ T(u) : u \in \mathbb{P}_d \} \) is strongly continuous at the origin \( 0 \in \mathbb{R}_d^+ \), i.e., \( T(0) = \text{strong-lim}_{u \to 0} T(u) \) exists. Then \( T \) can be extended continuously to \( \mathbb{R}_d^+ \) in an obvious manner (cf. [1]), and hence we shall use the same symbol \( T \) for the extended semigroup. Thus \( T = \{ T(u) : u \in \mathbb{R}_d^+ \} \). If each \( T(u) \), \( u \in \mathbb{P}_d \), has a contraction majorant \( P(u) \) defined on \( L_1(\Omega; \mathbb{R}) \), then it follows that, for any \( u \in \mathbb{R}_d^+ \), \( T(u) \) possesses a contraction majorant \( P(u) \) (see the proof of Lemma 1 in [9]).

In order to prove Theorem 1 we need the following result, itself of independent interest, which is a generalization of Theorem 3.1 of Emilion [6].

**THEOREM 2.** Let \( X \) be a reflexive Banach space and \( (\Omega, \Sigma, \mu) \) be a finite measure space. Suppose each \( T(u) \), \( u \in \mathbb{P}_d \), has a contraction majorant \( P(u) \) defined on \( L_1(\Omega; \mathbb{R}) \). Then for any \( g \in L_\infty(\Omega; X^*) \) and \( u \in \mathbb{P}_d \) we have
\[
(7) \quad \lim_{t \to 0} \| T^*(t)g - T^*(0)g \|_{L_1(\Omega; X^*)} = 0.
\]
In particular, if \( T(0) = \text{strong-lim}_{u \to 0} T(u) \) exists then (7) holds for every \( g \in L_\infty(\Omega; X^*) \) and \( u \in \mathbb{R}_d^+ \); in this case \( t \) ranges over \( \mathbb{R}_d^+ \).

To prove these theorems we use the fact that, under the condition of the existence of a contraction majorant \( P(u) \) for each \( T(u) \), there exists a sub-semigroup \( \{ \tau(u) \} \) of positive linear contractions on \( L_1(\Omega; \mathbb{R}) \) which dominates the semigroup \( T = \{ T(u) \} \). That is, \( \tau(s + t) \leq \tau(s) \tau(t) \) for any \( s, t \in \mathbb{P}_d \) (or \( s, t \in \mathbb{R}_d^+ \)) and
\[
(8) \quad \| T(u) f(\omega) \| \leq \| \tau(u) f(\omega) \| \quad \text{a.e. on } \Omega
\]
for every \( f \in L_1(\Omega; X) \) and \( u \in P_d \) (or \( u \in \mathbb{R}^d_+ \)). This is basic throughout the paper. Since such a positive sub-semigroup has been used by Emilion [6] to prove the (scalar-valued) local ergodic theorem in \( L_{\infty} \), we can modify his arguments in order to prove our Theorems 1 and 2. This idea is the starting point of our paper. In the next section we establish some preliminary lemmas which enable us to develop a method for proving Theorems 1 and 2, and in the following sections we prove Theorem 2 and then Theorem 1 along the arguments of Emilion [6]. In the last section we note that any bounded additive process \( F : \mathbb{R}^d \to L_{\infty}(\Omega; X^*) \) with respect to the adjoint semigroup \( T^* = \{ T^*(u) : u \in \mathbb{R}^d \} \) has the form \( F(I) = \int \int T^*(u) g \, du, \quad I, J \in \mathbb{R}_d \), for some \( g \in L_{\infty}(\Omega; X^*) \).

2. Lemmas. Without loss of generality we may and do assume \( d = 2^m \) with \( m \geq 1 \). Indeed, if \( 2^m-1 < d < 2^{m+1} \), then we can let \( T'(u, u') = T(u) \) for \( (u, u') \in \mathbb{R}^{2m} \) with \( u \in \mathbb{R}^d \). Let \( g \in L_{\infty}(\Omega; X^*) \). Then for any \( u = (u, u') \in \mathbb{R}^{2m} \) with \( u \in \mathbb{R}^d \), and any \( I \times I' \in \mathbb{R}_d \), we have \( T^*(u, u') g = \int T^*(u, u') g \, du \) and

\[
M_I g = \lambda_2(I)^{-1} \int T^*(u, u') g \, du = \lambda_2(I \times I')^{-1} \int T^*(u, u') g \, du.
\]

It is clear that \( T \) is strongly continuous at the origin if and only if \( T' \) is. Thus it is enough to prove Theorems 1 and 2 for \( T' \).

From now on, \( X \) will denote a reflexive Banach space.

LEMMA 1. If \( g \) is an \( X^* \)-valued strongly measurable function on \( \Omega \), then to each \( \varepsilon > 0 \) there corresponds an \( X \)-valued strongly measurable function \( r \) on \( \Omega \) such that

\[
\| r(w) \| = 1 \quad \text{and} \quad \text{Re} \langle r(w), g(w) \rangle > \| g(w) \| - \varepsilon
\]

for almost all \( w \in \Omega \).

Proof. Since \( g \) is strongly measurable, there exists a set \( A \subseteq \Omega \), with \( \mu(A) = 0 \), and a separable subspace \( Y \) of \( X^* \) such that if \( w \in \Omega \setminus A \) then \( g(w) \in Y \). Hence we can choose a sequence \( (y_n) \) of elements in \( Y \) and a sequence \( (E_n) \) of sets in \( \Sigma \) so that

\[
E_n \cap E_m = \emptyset \quad (n \neq m), \quad \bigcup_{n=1}^{\infty} E_n = \Omega \setminus A \quad \text{and} \quad \| g(w) - y_n \| < \varepsilon/2 \quad \text{for all} \ w \in E_n.
\]

Thus \( \| y_n \| > \| g(w) \| - \varepsilon/2 \) for all \( w \in E_n \). Since \( X \) is reflexive, there exists an \( x_n \in X \) with \( \| x_n \| = 1 \) and \( \langle x_n, y_n \rangle = \| y_n \| \). If \( w \in E_n \), then

\[
\langle x_n, g(w) \rangle - \| y_n \| = \langle x_n, g(w) - y_n \rangle \leq \| g(w) - y_n \| < \varepsilon/2,
\]

and therefore

\[
\text{Re} \langle x_n, g(w) \rangle > \| y_n \| - \varepsilon/2 > \| g(w) \| - \varepsilon.
\]

Thus the function \( r \) on \( \Omega \) defined by

\[
r(w) = \sum_{n=1}^{\infty} \chi_{E_n}(w) \cdot x_n \quad \text{for} \ w \in \Omega,
\]

where \( \chi_A \) denotes the characteristic function of a set \( A \), satisfies (9) for almost all \( w \in \Omega \). The proof is complete.

LEMMA 2. Let \( T \) be a bounded linear operator on \( L_1(\Omega; X) \). If \( \tau \) is a positive linear operator on \( L_1(\Omega; \mathbb{R}) \) such that \( \| Tf(w) \| \leq \tau(\| f \|) \) a.e. on \( \Omega \) for every \( f \in L_1(\Omega; X) \), then

\[
\| T^* g(w) \| \leq \tau^{*}(\| g \|)(w) \quad \text{a.e. on} \ \Omega
\]

for every \( g \in L_{\infty}(\Omega; X^*) \).

Proof. Since \( T^* g \in L_{\infty}(\Omega; X^*) \), we can apply Lemma 1 to infer that for any \( \varepsilon > 0 \) there exists a function \( r \in L_{\infty}(\Omega; X) \) such that

\[
\| r(w) \| = 1 \quad \text{and} \quad \text{Re} \langle r(w), T^* g(w) \rangle > \| T^* g(w) \| - \varepsilon
\]

for almost all \( w \in \Omega \). Let \( 0 \leq h \in L_1(\Omega; \mathbb{R}) \). Then

\[
\int h \cdot \| T^* g \| \, d\mu \leq \int h(w) \cdot \text{Re} \langle r(w), T^* g(w) \rangle + \varepsilon \| h \| \, d\mu
\]

\[
= \int \text{Re} \langle (h \cdot r)(w), T^* g(w) \rangle \, d\mu + \varepsilon \| h \|_1
\]

\[
\leq \| T(h \cdot r)(w), g(w) \| \, d\mu + \varepsilon \| h \|_1
\]

\[
\leq \| T(h \cdot r)(w) \| \cdot \| g \| \cdot d\mu + \varepsilon \| h \|_1
\]

\[
= \int (r h) \cdot g \, d\mu + \varepsilon \| h \|_1 = \int h \cdot (r^* g) \, d\mu + \varepsilon \| h \|_1.
\]

Since \( \varepsilon \) was arbitrary, this implies \( \int h \cdot \| T^* g \| \, d\mu \leq \int h \cdot (r^* g) \, d\mu \) for every \( 0 \leq h \in L_1(\Omega; \mathbb{R}) \), and consequently \( \| T^* g(w) \| \leq \tau^{*}(\| g \|)(w) \) for almost all \( w \in \Omega \). The proof is complete.

LEMMA 3. Let \( T = \{ T(u) : u \in P_d \} \) be a strongly continuous \( d \)-dimensional semigroup of linear contractions on \( L_1(\Omega; X) \). Suppose each \( T(u), u \in P_d \), has a contraction majorant \( P(u) \) defined on \( L_1(\Omega; \mathbb{R}) \). Then there exists a strongly continuous one-dimensional semigroup \( U = \{ U(t) : t > 0 \} \) of positive linear contractions on \( L_1(\Omega; \mathbb{R}) \) such that
(i) for each \( \alpha > 0 \) and \( f \in L_1(\Omega; X) \),
\[
\|M_\alpha f(\omega)\| \leq C_\alpha \cdot \bar{\alpha}^{-1} \left( \int_0^\bar{\alpha} \|U(t)\| dt \right)(\omega) \quad \text{a.e. on } \Omega,
\]
where \( C_\alpha \) is a constant depending only on \( \alpha \) and \( \log \bar{\alpha} = 2^{-m} \log \alpha \) with \( m = 2^m \).

(ii) for each \( \alpha > 0 \) and \( g \in L_\infty(\Omega; X^*) \),
\[
\|M_\alpha^g g(\omega)\| \leq C_\alpha \cdot \bar{\alpha}^{-1} \left( \int_0^\bar{\alpha} \|U^*(t)\| g(t) dt \right)(\omega) \quad \text{a.e. on } \Omega.
\]

In particular, if \( T(0) = \text{strong-limit}_{\omega \rightarrow 0} T(u) \) exists, then so does \( U(0) = \text{strong-limit}_{\omega \rightarrow 0} U(t) \), and
\[
(11) \quad U(0) h = \text{ess sup} \left\{ \sum_{i=1}^h \|T(0) f_i(\cdot)\| : f_i \in L_1(\Omega; X), \sum_{i=1}^h \|f_i(\omega)\| \leq h(\omega) \, \text{a.e. on } \Omega \right\}.
\]

for every \( 0 \leq h \in L_1(\Omega; \mathbb{R}) \).

Proof. Except for (ii), the lemma has been proved in [9], and (ii) is a consequence of Lemma 2. The proof is complete.

3. Proof of Theorem 2. We first consider the case \( d = 1 \). Let \( U = \{ U(t) : t > 0 \} \) be the one-dimensional semigroup in Lemma 3 which corresponds to the semigroup \( T = \{ T(u) : u \in \mathbb{R} \} \). It follows from [9] that \( U \) also satisfies
\[
(12) \quad \|T(t) f(\omega)\| \leq U(\omega) \| f(\cdot) \| (\omega) \quad \text{a.e. on } \Omega
\]
for every \( f \in L_1(\Omega; X) \) and \( t > 0 \). Put
\[
h = \int_0^\infty e^{-U(t)} dt \quad \left( \in L_1^{+}(\Omega; \mathbb{R}) \right), \quad C = \{ \omega : h(\omega) > 0 \}, \quad D = \Omega \setminus C.
\]

Then we have
\[
(13) \quad \begin{cases}
U(u) h = \int_0^\infty e^{-U(u+t)} dt \leq e^u \cdot h \quad \text{for } u > 0, \\
\chi_D \cdot U(u) f = 0 \quad \text{a.e. on } \Omega \text{ for every } f \in L_1(\Omega; \mathbb{R}).
\end{cases}
\]

Thus the adjoint semigroup \( U^* = \{ U^*(t) : t > 0 \} \) acting on \( L_\infty(\Omega; \mathbb{R}) \) satisfies \( U^*(t)(\chi_D \cdot g) = 0 \) a.e. on \( \Omega \) for every \( g \in L_\infty(\Omega; \mathbb{R}) \), and by Lemma 2,
\[
(14) \quad \|U^*(t)(\chi_D \cdot g)\| \leq U^*(t)(\chi_D \cdot \| g(\cdot) \|) = 0 \quad \text{a.e. on } \Omega
\]
for every \( g \in L_\infty(\Omega; X^*) \). Now define
\[
S(t) f = h^{-1} \cdot T(t)(f h) \quad \text{for } f \in L_1((C, h d\mu); X),
\]
so that \( S = \{ S(t) : t > 0 \} \) becomes a strongly continuous semigroup of linear contractions on \( L_1((C, h d\mu); X) \), and the adjoint semigroup \( S^* = \{ S^*(t) : t > 0 \} \) acting on \( L_\infty((C, h d\mu); X^*) = L_\infty(C; X^*) \) satisfies
\[
(15) \quad S^*(t) g = T^*(t) g \quad \text{a.e. on } C \text{ for every } g \in L_\infty(C; X^*).
\]

Let \( f \in L_\infty(C; X) \). Since \( L_\infty(C; X) \subset L_1((C, h d\mu); X) \), we then have
\[
\|S(t) f(\omega)\| = \|h(\omega)^{-1} \cdot T(t)(f h)(\omega)\| \leq h(\omega)^{-1} \cdot U(t) \| f h(\cdot) \| (\omega)
\]
\[
\leq \|f\| \infty h(\omega)^{-1} \cdot U(t)(h\omega) \leq e^\omega \| f \| \infty
\]
for almost all \( \omega \in \Omega \), where the last inequality comes from (13). Hence \( \|S(t)\| \infty \leq e^\omega \).

Similarly, if we set
\[
V(t) f = h^{-1} \cdot U(t)(f h) \quad \text{for } f \in L_1((C, h d\mu); \mathbb{R}),
\]
then \( V = \{ V(t) : t > 0 \} \) becomes a strongly continuous semigroup of positive linear contractions on \( L_1((C, h d\mu); \mathbb{R}) \) satisfying \( \|V(t)\| \infty \leq e^\omega \)
\[
(16) \quad V^*(t) g = U^*(t) g \quad \text{a.e. on } C \text{ for every } g \in L_\infty(C; \mathbb{R}).
\]
Thus \( \|V^*(t)\|_1 \leq e^\omega \) and \( \|V^*(t)\|_\infty \leq 1 \), and by Lemma 2,
\[
\|S^*(t)\|_1 \leq e^\omega \quad \text{and} \quad \|S^*(t)\|_\infty \leq 1.
\]

Consider \( S^* = \{ S^*(t) : t > 0 \} \) to be a locally bounded semigroup of linear operators on \( L_1((C, h d\mu); X^*) \). We then see that \( S^* \) is weakly continuous. Indeed, if \( f \in L_\infty(C; X^*) \subset L_1((C, h d\mu); X^*) \) and \( g \in L_\infty((C, h d\mu); X) = L_\infty(C; X) \) then
\[
\langle g, S^*(t) f \rangle_{h d\mu} = \int_C \langle g(\omega), S^*(t) f(\omega) \rangle h(\omega) d\mu
\]
\[
= \int_C \langle g(\omega), T^*(t) f(\omega) \rangle h(\omega) d\mu
\]
\[
= \int_C \langle h(\omega) g(\omega), T^*(t) f(\omega) \rangle d\mu
\]
\[
= \langle h g, T^*(t) f \rangle_{h d\mu} = \langle T(t)(h g), f \rangle_{h d\mu}.
\]

It follows that the mapping \( t \mapsto \langle g, S^*(t) f \rangle \) is weakly continuous for each \( f \in L_\infty(C; X^*) \). Since \( L_\infty(C; X^*) \) is a dense subspace of \( L_1((C, h d\mu); X^*) \), this yields the weak continuity of the semigroup \( S^* = \{ S^*(t) : t > 0 \} \) by an approximation argument. Thus, by the theory of semigroups of operators (see e.g. Chapter VIII of [5]), \( S^* \) is strongly continuous on the interval \( (0, \infty) \), so that for any \( g \in L_\infty(C; X^*) \subset L_1((2; X^*) \) and \( u > 0 \) we have, by (14) and (15),
\[
\lim_{t \to u} \|T^*(t) g - T^*(u) g\| h d\mu = 0.
\]
Letting \( s = u/2 > 0 \), we then deduce that
\[
\lim_{t \to 0} \int_{\Omega} \left\| T^*(t)g - T^*(u)g \right\| \, d\mu \\
\leq \lim_{t \to 0} \int_{\Omega} U^*(s)(\left\| T^*(t-s)g - T^*(u-s)g \right\|) \, d\mu \\
= \lim_{t \to 0} \int_{\Omega} U(s)1 \cdot \left\| T^*(t-s)g - T^*(u-s)g \right\| \, d\mu = 0,
\]
where the last equality comes from the fact that for any \( \varepsilon > 0 \) there exists an \( N \geq 1 \) such that \( \left\| (U(s)1 - N)h^* \right\| \leq \varepsilon \), together with an approximation argument.

Suppose, in particular, that \( T(0) = \text{strong-limit}_{u \to 0} T(u) \) exists. Then \( U(0) = \text{strong-limit}_{u \to 0} U(t) \) exists by Lemma 3, and \( S^* = \left\{ S^*(t) : t \geq 0 \right\} \) can be considered to be a strongly continuous semigroup of bounded linear operators on \( L_1((C, hd\mu); X^*) \) which is strongly continuous at the origin. Thus
\[
\lim_{t \to 0} \int_{\Omega} \left\| T^*(t)g - T^*(0)g \right\| \, d\mu = 0 \quad \text{for every} \quad g \in L_\infty(\Omega; X^*).
\]
Since
\[
\left\| T^*(t)g - T^*(0)g \right\| = \left\| T^*(0)(T^*(t)g - T^*(0)g) \right\| \\
\leq U^*(0) \left\| T^*(t)g - T^*(0)g \right\| \quad \text{a.e. on} \quad \Omega,
\]
we then apply (18) together with an approximation argument to infer that
\[
\lim_{t \to 0} \int_{\Omega} \left\| T^*(t)g - T^*(0)g \right\| \, d\mu \leq \lim_{t \to 0} \int_{\Omega} U(0)1 \cdot \left\| T^*(t)g - T^*(0)g \right\| \, d\mu = 0,
\]
which completes the proof for the case \( d = 1 \).

We next consider the case \( d > 1 \). If \( u \in \mathbb{P}_d \) is given, choose a sufficiently large \( N > 1 \) and define
\[
T_N(t) = T(t_1 + s_1/N, \ldots, t_d + s_d/N) \quad \text{for} \quad t = (t_1, \ldots, t_d) \in \mathbb{R}_+^d \setminus \{0\},
\]
where
\[
s_i = (t_1 + \ldots + t_d) - t_i \quad \text{for} \quad 1 \leq i \leq d.
\]
It is easily seen that \( T_N = \left\{ T_N(t) : t \in \mathbb{R}_+^d \setminus \{0\} \right\} \) becomes a strongly continuous semigroup of linear contractions on \( L_1(\Omega; X) \), and we may assume that \( u \) has the form
\[
u = (t_1 + s_1/N, \ldots, t_d + s_d/N)
\]
for some \( t = (t_1, \ldots, t_d) \in \mathbb{P}_d \). Thus, without loss of generality, we can assume from the start that \( T \) is a strongly continuous semigroup on \( \mathbb{R}_+^d \setminus \{0\} \).

Let \( T_i = \left\{ T_i(t') : t' > 0 \right\}, 1 \leq i \leq d \), be the one-dimensional strongly continuous semigroup of linear contractions on \( L_1(\Omega; X) \) defined by
\[
T_i(t') = T(t' \cdot e^i) \quad \text{for} \quad t' > 0
\]
where \( e^i \) is the \( i \)-th unit vector in \( \mathbb{R}_+^d \). By the case \( d = 1 \) we see that
\[
\begin{align*}
\lim_{t' \to s'} \int_{\Omega} \left\| T_i(t')g - T_i(s')g \right\| \, d\mu &= 0 \\
\end{align*}
\]
for every \( g \in L_\infty(\Omega; X^*) \) and \( s' > 0 \). If \( u = (u_1, \ldots, u_d) \in \mathbb{P}_d \) is given, then, since
\[
\begin{align*}
T^*(t_1, \ldots, t_d)g &= T^*(u_1, \ldots, u_d)g \\
&= [T^*(t_2, \ldots, t_d)g - T^*(t_1, \ldots, t_{d-1}, u_d)g] \\
&\quad + [T^*(t_1, \ldots, t_{d-1}, u_d)g - T^*(t_1, \ldots, u_{d-1}, u_d)g] + \ldots \\
&\quad + [T^*(t_1, u_2, \ldots, u_d)g - T^*(u_1, \ldots, u_d)g],
\end{align*}
\]
we deduce that
\[
\begin{align*}
\int_{\Omega} \left\| T^*(t_1, \ldots, t_d)g - T^*(u_1, \ldots, u_d)g \right\| \, d\mu \\
&\leq \sum_{i=1}^{d} \int_{\Omega} \left\| T_i^*(t_i) \ldots T_{i-1}^*(t_{i-1})[T_i^*(t_i)g_i - T^*_i(u_i)g_i] \right\| \, d\mu,
\end{align*}
\]
where
\[
g_i = \left\{ \begin{array}{ll}
g & \text{if} \quad i = d, \\
T_i^*(u_{i+1}) \ldots T_i^*(u_d)g & \text{if} \quad 1 \leq i \leq d - 1.
\end{array} \right.
\]
If \( U_i = \left\{ U_i(t') : t' > 0 \right\} \) denotes the one-dimensional semigroup in Lemma 3 corresponding to the semigroup \( T_i \), then we get
\[
\begin{align*}
\int_{\Omega} \left\| T_i^*(t_1) \ldots T_{i-1}^*(t_{i-1})[T_i^*(t_i)g_i - T^*_i(u_i)g_i] \right\| \, d\mu \\
&\leq \int_{\Omega} U_i^*(t_i) \ldots U_{i-1}^*(t_{i-1})[T_i^*(t_i)g_i - T^*_i(u_i)g_i] \, d\mu \\
&= \left[ U_i(t_1) \ldots U_1(t_1) \right] 1 \cdot \left\| T_i^*(t_i)g_i - T^*_i(u_i)g_i \right\| \, d\mu \\
&\leq \left[ U_i(t_i) \ldots U_1(t_1) \right] 1 \cdot \left\| T_i^*(t_i)g_i - T^*_i(u_i)g_i \right\| \, d\mu \\
&\quad + 2\|g_i\| \sum_{i=1}^{d} \left\| U_i(t_i) \ldots U_1(t_1)1 - U_i(t_i) \ldots U_1(t_1)1 - U_i(t_i) \ldots U_1(t_1)1 \right\| \, d\mu \\
&= I(t) + II(t),
\end{align*}
\]
and by (20)

$$\lim_{t \to \infty} I(t) = 0.$$ 

Further, since each $U_i = \{ U_i(t') : t' > 0 \}$ is strongly continuous on the interval $(0, \infty)$, an easy induction argument yields

$$\lim_{t \to \infty} II(t) = 0.$$ 

Hence the first half of the theorem has been proved. The second half can be proved similarly. The proof is complete.

4. Proof of Theorem 1. Since there exists a finite measure on $(\Omega, \Sigma)$ which is equivalent to $\mu$, we may and do assume without loss of generality that $\mu$ is finite. (Cf. e.g. (15).)

To prove the first half of Theorem 1, let $T(0) = \lim_{u \to 0} T(u)$. If $g$ is a function of $L_\infty(\Omega; X^*)$ then, by Theorem 2, the mapping $u \mapsto T^*(u)g$ can be regarded as a strongly continuous function from $\mathbb{R}_+^\times$ to $L_1(\Omega; X^*)$. Hence, as is easily seen, there exists an $X^*$-valued functional $G(u, \omega)$ defined on $\mathbb{R}_+ \times \Omega$, strongly measurable with respect to the product $\sigma$-algebra of the Lebesgue measurable subsets of $\mathbb{R}_+$ and $\Sigma$, such that for each $u \in \mathbb{R}_+$,

$$G(u, \cdot)$$

is a representative of $T^*(u)g$.

Then we have

$$M_\alpha^\alpha g(\omega) = \alpha^{-d} \left( \int_{I(\alpha)} T^*(u)g \, du \right)(\omega) = \alpha^{-d} \int_{I(\alpha)} G(u, \omega) \, du \quad \text{a.e. on } \Omega,$$

where the integral $\int_{I(\alpha)} T^*(u)g \, du$ can be taken in the Bochner sense in place of the weak-$^*$-integral. Since

$$\| T(u)M_\alpha - M_\alpha \|_1 = \| T^*(u)M_\alpha^\alpha - M_\alpha^\alpha \|_\infty \to 0 \quad \text{as } u \to 0 \in \mathbb{R}^\times_+, $$

we obtain

$$\lim_{\beta \to 0} \| M_\alpha^\beta M_\alpha^\alpha - M_\alpha^\alpha \|_\infty = 0. \quad (21)$$

Now put

$$\bar{g}(\omega) = q\lim_{\alpha \to 0} \sup_{\alpha} \| M_\alpha^\alpha g(\omega) - T^*(0)g(\omega) \| \quad \text{for } \omega \in \Omega.$$ 

Then

$$\bar{g}(\omega) \leq q\lim_{\alpha \to 0} \| M_\alpha^\alpha [T^*(0)g - M_\alpha^\beta g](\omega) \|

+ q\lim_{\alpha \to 0} \| M_\alpha^\alpha M_\beta^\alpha g(\omega) - M_\beta^\alpha g(\omega) \|

+ \| M_\beta^\alpha g(\omega) - T^*(0)g(\omega) \| \quad \text{a.e. on } \Omega.$$ 

Since Theorem 2 implies $\lim_{\beta \to 0} \| M_\alpha^\beta g - T^*(0)g \|_{L_1(\Omega; X^*)} = 0$, it follows that for each $\varepsilon > 0$,

$$\lim_{\beta \to 0} \mu(\{ \omega : \| M_\alpha^\beta g(\omega) - T^*(0)g(\omega) \| > \varepsilon \}) = 0. \quad (23)$$

By this, together with (21), it is enough to show that

$$q\lim_{\beta \to 0} \int_{I(\beta)} \bar{g}_\beta \, d\mu = 0 \quad (24)$$

where

$$\bar{g}_\beta(\omega) = q\lim_{\alpha \to 0} \sup_{\alpha} \| M_\alpha^\alpha [T^*(0)g - M_\alpha^\beta g(\omega)] \| \quad \text{for } \omega \in \Omega.$$ 

To prove (24), we use Lemma 3 as follows. Let $U = \{ U(t') : t' \geq 0 \}$ be the one-dimensional strongly continuous semigroup of positive linear contractions on $L_1(\Omega; \mathbb{R})$ in Lemma 3 corresponding to the semigroup $T = \{ T(u) : u \in \mathbb{R}_+ \}$. By the scalar-valued local ergodic theorem in $L_\infty$ (see e.g. Theorem 7.1.14 in [7]), for every $\beta > 0$ the limit

$$h_\beta = q\lim_{\beta \to 0} \alpha^{-1} \int_0^\infty \int_{I(\alpha)} \| T^*(u)g - M_\alpha^\beta g(\omega) \| \, du \, d\mu$$

exists a.e. on $\Omega$, and further

$$h_\beta = U^*(0) \| T^*(0)g - M_\alpha^\beta g(\omega) \| \quad \text{a.e. on } \Omega.$$ 

Since $\bar{g}_\beta \leq C_d \cdot h_\beta$ a.e. on $\Omega$ by Lemma 3(iii), it follows that

$$\int_{\Omega} \bar{g}_\beta \, d\mu \leq C_d \int_{\Omega} h_\beta \, d\mu = C_d \int_{\Omega} U(0) 1 \cdot \| T^*(0)g - M_\alpha^\beta g(\omega) \| \, d\mu,$$

and by (23),

$$\lim_{\beta \to 0} \int_{\Omega} U(0) 1 \cdot \| T^*(0)g - M_\alpha^\beta g(\omega) \| \, d\mu = 0,$$

whence the proof of (24) is complete.

Now suppose that $q\lim_{\alpha \to 0} M_\alpha^\alpha g$ exists a.e. on $\Omega$ for every $g \in L_\infty(\Omega; X^*)$. Then for any $f \in L_1(\Omega; X)$,

$$q\lim_{\alpha \to 0} \langle M_\alpha f, g \rangle = q\lim_{\alpha \to 0} \langle f, M_\alpha^\alpha g(\omega) \rangle = q\lim_{\alpha \to 0} \int_{\Omega} \langle f, M_\alpha^\alpha g(\omega) \rangle \, d\mu$$

by Lebesgue's convergence theorem. Since $X$ is reflexive, $L_1(\Omega; X)$ is weakly sequentially complete (cf. e.g. page 117 of [4]), and thus if $(\alpha_n)$ is a sequence with $\alpha_n \in \mathbb{Q}$ for all $n \geq 1$ and $\lim_n \alpha_n = 0$, then there exists a function $\hat{f}$
in \( L_1(\Omega; X) \) such that
\[
\tilde{f} = \text{weak-lim}_n M_{\alpha_n} f.
\]
Then
\[
T(u)\tilde{f} = \text{weak-lim}_n T(u) M_{\alpha_n} f = \text{strong-lim}_n M_{\alpha_n} (T(u)f) = T(u)f
\]
for every \( u \in \mathbb{P}_d \), and further, by the Hahn–Banach theorem, \( \tilde{f} \) is in the closed linear subspace of \( L_1(\Omega; X) \) generated by the set \( \{ T(u)f : u \in \mathbb{P}_d \} \). It follows from an approximation argument that
\[
\tilde{f} = \text{strong-lim}_{u \to 0} T(u)\tilde{f} = \text{strong-lim}_{u \to 0} T(u)f,
\]
which completes the proof.

5. Additive processes for \( T^* \). We recall that a set function \( F : \mathcal{I}_d \to L_\infty(\Omega; X^*) \) is an additive process with respect to the adjoint semigroup \( T^* = \{ T^*(u) : u \in \mathbb{P}_d \} \) if \( F \) satisfies the following conditions:
(i) \( T^*(u)F(I) = F(u + I) \) for all \( u \in \mathbb{P}_d \) and \( I \in \mathcal{I}_d \),
(ii) If \( I_1, \ldots, I_k \in \mathcal{I}_d \) are pairwise disjoint and \( I = \bigcup_{i=1}^k I_i \in \mathcal{I}_d \) then \( F(I) = \sum_{i=1}^k F(I_i) \).

\( F \) is called bounded if
\[
K(F) := \sup \{ \lambda_d(I)^{-1} \|F(I)\|_\infty : I \in \mathcal{I}_d, \lambda_d(I) > 0 \} < \infty.
\]

In this section we prove the following theorem (cf. Theorem 4.1 in [6]).

THEOREM 3. Let \( X \) be a reflexive Banach space and \( F : \mathcal{I}_d \to L_\infty(\Omega; X^*) \) be a bounded additive process with respect to the adjoint semigroup \( T^* = \{ T^*(u) : u \in \mathbb{P}_d \} \). Then there exists a function \( g \in L_\infty(\Omega; X^*) \) such that \( F(I) = \int_I T^*(u)g \, du \) for all \( I \in \mathcal{I}_d \).

For the proof we need the following lemma.

Lemmata 4. To any \( I \in \mathcal{I}_d \), with \( \lambda_d(I) > 0 \), and an \( \varepsilon > 0 \) there corresponds a \( \gamma > 0 \) such that if \( I \in \mathcal{I}_d \) satisfies \( I \subset (0, \gamma]^d \) and \( \lambda_d(I) > 0 \) then
\[
\left\| F(J) - \int_I T^*(u)[\lambda_d(I)^{-1} F(I)] \, du \right\|_\infty < \varepsilon.
\]

Proof. This is an adaptation of the proof of Lemma 3.2 in [1]. Since
\[
\|F(J) - T^*(u)F(J)\|_\infty = \|F(J) - F(u + J)\|_\infty \leq K(F) \cdot \lambda_d(J \Delta (u + J)),
\]
where \( A \triangle B \) denotes the symmetric difference of two sets \( A \) and \( B \), we have
\[
\lim_{u \to 0} \|F(J) - T^*(u)F(J)\|_\infty = 0.
\]

Hence we can choose a \( \gamma > 0 \) so that if \( I \in \mathcal{I}_d, I \subset (0, \gamma]^d \) and \( \lambda_d(I) > 0 \) then
\[
\left\| F(J) - \lambda_d(I)^{-1} \int_I T^*(u)F(J) \, du \right\|_\infty < \frac{\varepsilon}{3},
\]
and further there exists an \( I' \in \mathcal{I}_d \) which is a disjoint union of intervals \( a_j + I \) with \( a_j \in \mathbb{P}_d, j = 1, \ldots, m \), and satisfies
\[
I' \subset J \quad \text{and} \quad \lambda_d(J \setminus I') < \frac{\varepsilon}{3K(F)}.
\]
Then
\[
\left\| \int_I T^*(u)[\lambda_d(I)^{-1} F(J)] \, du - \int_I T^*(u)[\lambda_d(I)^{-1} F(I') \, du \right\|_\infty = \left\| \int_{J \setminus I'} T^*(u)[\lambda_d(I)^{-1} F(J)] \, du \right\|_\infty \leq \lambda_d(J \setminus I')K(F) < \frac{\varepsilon}{3},
\]
and
\[
\int_I T^*(u)[\lambda_d(I)^{-1} F(I)] \, du = \sum_{j=1}^m \int_{a_j + I} T^*(u)[\lambda_d(I)^{-1} F(I)] \, du
\]
\[
= \sum_{j=1}^m \left[ T^*(a_j + u)[\lambda_d(I)^{-1} F(I)] \, du \right.
\]
\[
= \lambda_d(I)^{-1} \left\{ \int_I T^*(u) \left( \sum_{j=1}^m F(a_j + I) \right) \, du \right.
\]
\[
= \lambda_d(I)^{-1} \int_I T^*(u)F(I') \, du.
\]
Since
\[
\left\| \lambda_d(I)^{-1} \int_I T^*(u)F(J) \, du - \lambda_d(I)^{-1} \int_I T^*(u)F(I') \, du \right\|_\infty < \frac{\varepsilon}{3},
\]
we deduce that
\[
\left\| \lambda_d(I)^{-1} \int_I T^*(u)[\lambda_d(I)^{-1} F(I)] \, du - \lambda_d(I)^{-1} \int_I T^*(u)F(J) \, du \right\|_\infty < \frac{\varepsilon}{3},
\]
which, together with (28) and (29), completes the proof.

Proof of Theorem 3. For a positive number \( \alpha \) let
\[
A(\alpha) = \alpha^{-d}F((0, \alpha]^d).
\]
Since \( \|A(\alpha)\|_\infty \leq K(F) < \infty \) for all \( \alpha > 0 \) and since the closed unit ball of \( L_\infty(\Omega; X^*) \) is compact in the weak* topology, we can choose a net
\{\alpha_i : i \in D\}$, where $D$ is a directed set and $\alpha_i$ is a positive number for each $i \in D$, and a function $g$ in $L_{\infty}(\Omega; X^*)$ such that

\[
\lim_i \alpha_i = 0 \quad \text{and} \quad g = \text{weak}^* \lim_i A(\alpha_i).
\]

If $J \in \mathcal{I}_d$ and $f \in L_1(\Omega; X)$ are given, then, by Lemma 4 together with (31),

\[
\left\langle f, \int \right. \left. T^*(u) g \, du \right\rangle = \left\langle \int \right. \left. T(u) f \, du, g \right\rangle = \lim_i \left\langle \int \right. \left. T(u) f \, du, A(\alpha_i) \right\rangle
\]

\[
= \lim_i \left\langle \int \right. \left. T^*(u)[A(\alpha_i)] \, du \right\rangle = \left\langle f, F(J) \right\rangle.
\]

Hence $F(J) = \int \left. T^*(u) \, g \, du \right\rangle$ for $J \in \mathcal{I}_d$, and the proof is complete.

References


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