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Transitivity for linear operators on a Banach space

by

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Abstract. Let G be the multiplicative group of invertible elements of $E(X)$, the algebra of all bounded linear operators on a Banach space X. In 1945 Mackey showed that if $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are any two sets of linearly independent elements of X with the same number of items, then there exists $T\in G$ so that $T(x_k)=y_k$, $k=1,\ldots,n$. We prove that some proper multiplicative subgroups of G have this property.

1. Introduction. Throughout, X is an infinite-dimensional Banach space and $E(X)$ is the algebra of all bounded linear operators on X. A subset S of $E(X)$ is called *l. i. transitive* if, given two sets $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ of linearly independent elements of X, there exists $T\in S$ such that $T(x_k)=y_k$, $k=1,\ldots,n$. In [5, Theorem II-3] Mackey showed that the set G of invertible elements of $E(X)$ is l. i. transitive. Our results show that smaller subgroups of the multiplicative group G suffice. We show the following in §2.

THEOREM 1. *Let A be any closed subalgebra of $E(X)$ containing the identity I and all $T\in E(X)$ with finite-dimensional range. Let $\mathfrak{G}$ be the set of invertible elements of A. Then any open multiplicative subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is l. i. transitive.*

As is well known, $\mathfrak{G}$ is open.

Next let ψ be the set of all elements of $\mathfrak{G}$ of the form $I+T$ where T has finite-dimensional range. If we write its inverse as $I+V$, $V\in E(X)$, we see that $T+V+TV=0$ so that V also has finite-dimensional range. It follows that ψ is a multiplicative subgroup of $\mathfrak{G}$, and

THEOREM 2. *ψ is l. i. transitive.*

2. On transitivity. Our aim is to prove Theorems 1 and 2 given above. We shall use an easy lemma.

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LEMMA 1. *Given two sets $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of linearly independent elements of X there is a set $z_1, \ldots, z_n$ in X so that $x_1, \ldots, x_n, z_1, \ldots, z_n$ and $y_1, \ldots, y_n, z_1, \ldots, z_n$ are sets of $2n$ linearly independent elements each.*

Proof. Let L be the linear subspace of X generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$. As X/L is infinite-dimensional we can choose a linearly independent set $z_1 + L, \ldots, z_n + L$, where each $z_k \in X$, in X/L. Then $x_1, \ldots, x_n, z_1, \ldots, z_n$ is a linearly independent set.

We turn to the proof of Theorem 1. Let n be any fixed positive integer. Let K denote the cartesian product of X with itself for n factors. Let Γ be the collection of all n-tuples $(v_1, \ldots, v_n)$ in K where $v_1, \ldots, v_n$ is a linearly independent set in X. We fix $\xi = (x_1, \ldots, x_n)$ in Γ and consider the continuous linear mapping ϕ_ξ of A to K defined by

$$\phi_\xi(T) = (T(x_1), \ldots, T(x_n)).$$

We claim that the image of ϕ_ξ is all of K. For let $(z_1, \ldots, z_n) \in K$. As $\xi \in \Gamma$ we can choose $x_1^*, \ldots, x_n^*$ in X^* so that $x_i^*(x_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. If we set $T(x) = \sum_{k=1}^n x_k^*(x) z_k$ we see that $\phi_\xi(T) = (z_1, \ldots, z_n)$.

We invoke Banach's open mapping theorem (see, for example, [4, p. 215]) to see that $\phi_\xi(\mathfrak{H})$ is an open subset of K. As each $T \in \mathfrak{H}$ is one-to-one, $\phi_\xi(\mathfrak{H})$ is contained in Γ. As $I \in \mathfrak{H}$, $\alpha \in \phi_\alpha(\mathfrak{H})$ for each α in Γ. Thus Γ is the union of all the sets $\phi_\alpha(\mathfrak{H})$. We will have established Theorem 1 if we show that $\phi_\xi(\mathfrak{H}) = \Gamma$.

To this end we show first that Γ is a connected set. For let $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_n)$ be two elements of Γ. By Lemma 1 we have $(z_1, \ldots, z_n) \in \Gamma$ where $v_1, \ldots, v_n, z_1, \ldots, z_n$ and $w_1, \ldots, w_n, z_1, \ldots, z_n$ are linearly independent sets. Consider the set of elements of K of the form

$$(tv_1 + (1-t)z_1, \ldots, tv_n + (1-t)z_n)$$

for $0 \le t \le 1$. Each of these lies in Γ. For if, for some $0 < t < 1$,

$$\sum_{k=1}^{m} C_k[tv_k + (1-t)z_k] = 0$$

we would have each $C_k t = 0$ and $C_k(1-t) = 0$ so each $C_k = 0$. Thus we have a connected path in Γ joining $(v_1, \ldots, v_k)$ and $(z_1, \ldots, z_n)$ as well as one joining $(z_1, \ldots, z_n)$ to $(w_1, \ldots, w_n)$.

Suppose we had some $\eta = (y_1, \ldots, y_n)$ in Γ which is not in $\phi_\xi(\mathfrak{H})$. We claim that $\phi_\xi(\mathfrak{H}) \cap \phi_\eta(\mathfrak{H})$ is empty. For if $(z_1, \ldots, z_n) = \phi_\xi(T_1) = \phi_\eta(T_2)$, where $T_1, T_2 \in \mathfrak{H}$, then $z_k = T_1(x_k) = T_2(y_k)$, $k = 1, \ldots, n$. Hence $T_2^{-1}T_1(x_k) = y_k$, $k = 1, \ldots, n$, contrary to $\eta \notin \phi_\xi(\mathfrak{H})$.

So, assuming that $\eta \notin \phi_\xi(\mathfrak{H})$ exists, we see that Γ is the union of two non-empty disjoint open sets, namely $\phi_\xi(\mathfrak{H})$ and the union of the sets $\phi_\alpha(\mathfrak{H})$ where $\alpha \in \Gamma$ and $\alpha \notin \phi_\xi(\mathfrak{H})$. This contradicts the connectivity of Γ. Consequently, given $\xi \in \Gamma$, we have $\phi_\xi(\mathfrak{H}) = \Gamma$ and so $\mathfrak{H}$ is l. i. transitive.

LEMMA 2. *To each set $x_1, \ldots, x_n$ of linearly independent elements of X there corresponds $\varepsilon > 0$ with the following property. If $y_1, \ldots, y_n$ is a set of elements in X where $\|x_k - y_k\| < \varepsilon$, $k = 1, \ldots, n$, then, for some $V \in \psi$, we have $V(x_k) = y_k$, $k = 1, \ldots, n$.*

Proof. Choose $x_1^*, \ldots, x_n^* \in X^*$ so that $x_i^*(x_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. We choose ε so that $0 < \varepsilon < 1/(n \max \|x_k^*\|)$. Suppose $\|y_k - x_k\| < \varepsilon$, $k = 1, \ldots, n$. We define $U \in E(X)$ by $U(x) = \sum_{k=1}^n x_k^*(x)(y_k - x_k)$. Then $(I + U)(x_k) = y_k$, $k = 1, \ldots, n$. Inasmuch as $\|U\| \le \sum_{k=1}^n \|x_k^*\| \varepsilon < 1$ we have $I + U \in \psi$.

Of course, $y_1, \ldots, y_n$ is also a linearly independent set. Also, as ψ is a group, there is $V \in \psi$ so that $V(y_k) = x_k$, $k = 1, \ldots, n$.

We now turn to the proof of Theorem 2. We use the notation employed in the proof of Theorem 1. By Lemma 2, for each $(x_1, \ldots, x_n)$ in Γ there is a neighborhood Γ_0 in Γ so that if $(z_1, \ldots, z_n) \in \Gamma_0$ we have some $T \in \psi$ where $T(x_k) = z_k$, $k = 1, \ldots, n$. If also $(w_1, \ldots, w_n) \in \Gamma_0$ and $V \in \psi$ with $V(x_k) = w_k$, $k = 1, \ldots, n$, then $VT^{-1}(z_k) = w_k$, $k = 1, \ldots, n$, with $VT^{-1} \in \psi$.

Now let $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ be in Γ. As in the proof of Theorem 1 there is a connected linear path P joining $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$. For each $(x_1, \ldots, x_n)$ in P we have a neighborhood as described above. As P is compact a finite number of these neighborhoods $\Gamma_1, \ldots, \Gamma_r$ cover P. By listing these so that each $\Gamma_i \cap \Gamma_{i+1}$ is non-empty we can find $W_1, \ldots, W_r$ in ψ so that $(W_r \ldots W_1)(v_k) = w_k$, $k = 1, \ldots, n$.

3. On semigroups in $E(X)$. The arguments of Theorem 1 applied to A can be readily adapted to show the following generalization of that result. We omit details.

THEOREM 3. *Let $\mathfrak{H}$ be an open multiplicative semigroup in A where each $T \in \mathfrak{H}$ is one-to-one. Then $\mathfrak{H}$ is l. i. transitive if and only if given $T \in \mathfrak{H}$ and a set $x_1, \ldots, x_n$ of linearly independent elements there exists $V \in \mathfrak{H}$ such that $VT(x_k) = x_k$, $k = 1, \ldots, n$.*

We shall point out that there are non-trivial examples satisfying Theorem 3 which are disjoint from G, the set of invertible elements of $E(X)$.

For an infinite-dimensional Banach space X an open problem of Banach [1, p. 245] is whether X and the null-space of a non-zero linear functional must have the same linear dimension. In [6, p. 502] it is pointed out that this is the case if and only if there exists $T \in E(X)$ which is an isomorphism of X onto a proper closed linear subspace. We consider X where such T exist.

Let $\mathfrak{H}$ be the multiplicative subgroup of $E(X)$ of all $T \in E(X)$ which are isomorphisms of X onto a proper closed linear subspace. By [2, Th. 2.5.6], $\mathfrak{H}$ is open in $E(X)$. We check that $\mathfrak{H}$ is l. i. transitive and so see that $\mathfrak{H}$ satisfies all the requirements of Theorem 3.

Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be two sets of linearly independent elements of X. Take $T \in \mathfrak{H}$. Then $T(x_1), \ldots, T(x_n)$ is a linearly independent set. By Mackey's theorem there exists V, an invertible element of $E(X)$, where $VT(x_k) = y_k$, $k = 1, \ldots, n$. But one sees that VT is an isomorphism of X onto the proper closed linear subspace $VT(X)$. Hence $VT \in \mathfrak{H}$.

4. Banach spaces as Banach algebras. In this section we apply a special case of the Mackey theorem cited above. Let X be a Banach space and let Δ be the set of all $x^* \in X^*$ such that $x^* \neq 0$ and $\|x^*\| \leq 1$. For each $x^* \in \Delta$ we define a multiplication in X via $x \cdot y = x^*(x)y$. This multiplication makes X into an associative algebra and, as $\|x \cdot y\| \leq \|x\|\,\|y\|$, a Banach algebra as well. We say that this is the Banach algebra *induced* by x^*.

THEOREM 4. *Suppose that X is reflexive. Then the Banach algebras induced by any two elements of Δ are equivalent Banach algebras.*

Proof. Let x_1^* and x_2^* be in Δ and set $x \cdot y = x_1^*(x)y$ and $x \# y = x_2^*(x)y$. Let V be a continuous isomorphism of X onto X. We seek now the requirement for V to be an algebra isomorphism of the Banach algebra induced by x_1^* onto that induced by x_2^*. We must have $V(x \cdot y) = V(x) \# V(y)$ for all $x, y \in X$ or $x_1^*(x)V(y) = x_2^*[V(x)]V(y)$. As $V(X) = X$ this requires that $x_1^* = V^*(x_2^*)$ in terms of the adjoint operation $T \to T^*$ of $E(X)$ into $E(X^*)$. For X reflexive the mapping $T \to T^*$ is a conjugate-linear isomorphism of $E(X)$ onto $E(X^*)$. If G (G^*) denotes the set of invertible elements of $E(X)$ ($E(X^*)$) then G maps onto G^* via $T \to T^*$. We then apply Mackey's theorem to G^* to see that for some $V \in G$ we have $V^*(x_2^*) = x_1^*$.

In [3] an example is given of a Banach algebra with no involution. This example is semisimple and commutative but is rather complicated. The Banach algebras of this section furnish simpler examples at the expense of the loss of semisimplicity and commutativity. We assume that X is at least two-dimensional.

THEOREM 5. *The Banach algebra for X induced by $x^* \in \Delta$ has no involution.*

Proof. Suppose that $x \to x'$ is an involution on X. Let $x \cdot y = x^*(x)y$ be the multiplication for X. Then $(x \cdot y)' = \overline{x^*(x)}y'$ and $y' \cdot x' = x^*(y')x'$ for all $x, y \in X$. Setting $x = y$ we see that $x^*(x') = \overline{x^*(x)}$ for all $x \in X$. We select x_0 so that $x^*(x_0) = 1$. This says that $y' = \overline{x^*(y)}x_0'$ for all y, which is impossible as X is not one-dimensional.

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