there is some chance that operators on these very special spaces may indeed have invariant subspaces.

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A general geometric construction for affine surface area

by

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Abstract. Let K be a convex body in $\mathbb{R}^n$ and B be the Euclidean unit ball in $\mathbb{R}^n$. We show that
$$\lim_{t\to 0}\frac{|K|-|K_t|}{|B|-|B_t|}=\frac{\operatorname{as}(K)}{\operatorname{as}(B)},$$
where $\operatorname{as}(K)$ respectively $\operatorname{as}(B)$ is the affine surface area of K respectively B and $\{K_t\}_{t\ge 0}$, $\{B_t\}_{t\ge 0}$ are general families of convex bodies constructed from K, B satisfying certain conditions. As a corollary we get results obtained in [M-W], [Schm], [S-W] and [W].

The affine surface area $\operatorname{as}(K)$ was introduced by Blaschke [B] for convex bodies in $\mathbb{R}^3$ with sufficiently smooth boundary and by Leichtweiss [L1] for convex bodies in $\mathbb{R}^n$ with sufficiently smooth boundary as follows:
$$\operatorname{as}(K)=\int_{\partial K}\kappa(x)^{1/(n+1)}\,d\mu(x),$$
where $\kappa(x)$ is the Gaussian curvature at $x\in\partial K$ and μ is the surface measure on ∂K. As it occurs naturally in many important questions, for example in the approximation of convex bodies by polytopes (see the survey article of Gruber [Gr] and the paper by Schütt [S]) or in a priori estimates for PDEs [Lu-O], one wanted to have extensions of the affine surface area to arbitrary convex bodies in $\mathbb{R}^n$ without any smoothness assumptions on the boundary.

Such extensions were given in recent years by Leichtweiss [L2], Lutwak [Lu], Meyer and Werner [M-W], Schmuckenschläger [Schm], Schütt and Werner [S-W] and Werner [W].

The extensions of affine surface area to an arbitrary convex body K in $\mathbb{R}^n$ in [L2], [M-W], [Schm], [S-W] and [W] have a common feature. First a specific family $\{K_t\}_{t\ge 0}$ of convex bodies is constructed. This family is different in each of the cited extensions but of course related to the given convex body K. Typically the families $\{K_t\}_{t\ge 0}$ are obtained from K through a "geometric" construction. In [L2] respectively [S-W] this construction gives

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as $\{K_t\}_{t\geq 0}$ the family of floating bodies respectively convex floating bodies. In [M-W] the construction gives the family of Santaló regions, in [Schm] the convolution bodies and in [W] the illumination bodies.

The affine surface area is then obtained by using expressions involving volume differences $|K|-|K_t|$ respectively $|K_t|-|K|$.

Therefore it seemed natural to ask whether there are completely general conditions on a family $\{K_t\}_{t\geq 0}$ of convex bodies in $\mathbb{R}^n$ that (by means of volume difference expressions) will give us affine surface area. We give a positive answer to this question which was asked—among others—by A. Pełczyński.

Throughout the paper we use the following notations.

$B(a,r) = B^n(a,r)$ is the n-dimensional Euclidean ball with radius r centered at a. We put $B = B(0,1)$. We denote by $\|\cdot\|$ the standard Euclidean norm on $\mathbb{R}^n$, and by $\langle\cdot,\cdot\rangle$ the standard inner product on $\mathbb{R}^n$. For x and y in $\mathbb{R}^n$, $[x,y] = \{\alpha x + (1-\alpha)y : 0 \leq \alpha \leq 1\}$ denotes the line segment from x to y. For a convex set C in $\mathbb{R}^n$ and a point $x \in \mathbb{R}^n \setminus C$, $\mathrm{co}[x,C]$ is the convex hull of x and C.

$\mathcal{K}$ denotes the set of convex bodies in $\mathbb{R}^n$. For $K \in \mathcal{K}$, $\mathrm{int}(K)$ is the interior of K and ∂K is the boundary of K. For $x \in \partial K$, $N(x)$ is the outer unit normal vector to ∂K at x. We denote the n-dimensional volume of K by $\mathrm{vol}_n(K) = |K|$.

Let $K \in \mathcal{K}$ and $x \in \partial K$ with unique outer unit normal vector $N(x)$. We say that ∂K is *approximated at x by a ball from the inside* (respectively *from the outside*) if there exists a hyperplane H orthogonal to $N(x)$ such that $H \cap \mathrm{int}(K) \neq \emptyset$ and a Euclidean ball $B(r) = B(x - rN(x), r)$ (respectively $B(R) = B(x - RN(x), R)$) such that

$$B(r) \cap H^+ \subseteq K \cap H^+$$

respectively

$$K \cap H^+ \subseteq B(R) \cap H^+.$$

Here H^+ is one of the two half-spaces determined by H.

Definition 1. For $t \geq 0$, let $\mathcal{F}_t : \mathcal{K} \to \mathcal{K}$, $K \mapsto \mathcal{F}_t(K) = K_t$, be a map with the following properties:

(i) $K_0 = K$ and either $K_t \subseteq K$ for all $t \geq 0$ and $\mathcal{F}_t$ is decreasing in t (that is, $K_{t_1} \subseteq K_{t_2}$ if $t_1 \geq t_2$), or $K \subseteq K_t$ for all $t \geq 0$ and $\mathcal{F}_t$ is increasing in t.

(ii) For all affine transformations A with $\det A \neq 0$, and all t,

$$(A(K))_{|\det A| t} = A(K_t).$$

(iii) For all $t \geq 0$, B_t is a Euclidean ball with center 0 and radius $f_1(t)$ and

$$\lim_{t\to 0} \left| \frac{|B| - |B_t|}{t^{2/(n+1)}} \right| = c,$$

where c is a constant (depending on n only).

(iv) Let ∂K be approximated at x from the inside by a ball $B(r)$. If $H^+ \cap \partial(K_t) \cap \partial(B(r))_s \neq \emptyset$ for some s and t, then $s \leq Ct$ where C is a constant (depending only on n).

(v) Let $\varepsilon > 0$ be given and $x \in \partial K$ be such that ∂K is approximated at x from the inside by a ball $B(\varrho - \varepsilon)$ and from the outside by a ball $B(\varrho + \varepsilon)$. There exists a hyperplane H orthogonal to $N(x)$ and t_0 such that whenever

$$H^+ \cap \partial(K_t) \cap \partial(B(\varrho - \varepsilon))_s \neq \emptyset \quad \text{for } t \leq t_0,\ s = s(t),$$

respectively

$$H^+ \cap \partial(K_t) \cap \partial(B(\varrho + \varepsilon))_s \neq \emptyset \quad \text{for } t \leq t_0,\ s = s(t),$$

then

$$s \leq (1+\varepsilon)t$$

respectively

$$s \geq (1-\varepsilon)t.$$

Remarks 2. (i) Note that the maps $\mathcal{F}_t$ are essentially determined by the invariance property of Definition 1(ii) and by their values at Euclidean balls.

(ii) Let $f_r(t)$ be the radius of $B(0,r)_t$. Then it follows immediately from Definition 1(ii), (iii) that

$$\lim_{t\to 0} \frac{r - f_r(t)}{1 - f_1(t)} = r^{(n-1)/(n+1)}.$$

(iii) For some examples the following Definition 1′ is easier to check than Definition 1.

Definition 1′. (i)–(iii) as in Definition 1.

(iv)′ If $s < t$, then $K_t \subseteq \mathrm{int}(K_s)$.

(v)′ If $K \subset L$ where L is a convex body in $\mathbb{R}^n$, then $K_t \subseteq L_t$ for all $t \geq 0$.

However, not all the examples mentioned below satisfy (iv)′ and (v)′. For instance the illumination bodies (defined below) do not satisfy (v)′.

Examples for Definitions 1 and 1′

1. *The (convex) floating bodies* [S-W]. Let K be a convex body in $\mathbb{R}^n$ and $t \geq 0$. F_t is a *(convex) floating body* if it is the intersection of all half-spaces whose defining hyperplanes cut off a subset of volume t of K. More precisely, for $u \in S^{n-1}$ let a_t^u be defined by

$$t = |\{x \in K : \langle u, x\rangle \geq a_t^u\}|.$$

Then

$$F_t = \bigcap_{u\in S^{n-1}} \{x \in K : \langle u, x\rangle \le a_t^u\}$$

is a (convex) floating body. The family $\{F_t\}_{t\ge 0}$ satisfies Definitions 1 and 1′.

2. *The convolution bodies* [K], [Schm]. Let K be a symmetric convex body in $\mathbb{R}^n$ and $t \ge 0$. Let

$$C(t) = \{x \in \mathbb{R}^n : |K \cap (K + x)| \ge 2t\} \quad \text{and} \quad C_t = \tfrac{1}{2}C(t).$$

Then $\{C_t\}_{t\ge 0}$ satisfies Definitions 1 and 1′.

3. *The Santaló regions* [M-W]. For $t \in \mathbb{R}$ and a convex body K in $\mathbb{R}^n$ the *Santaló region* $S(K,t)$ of K is defined as

$$S(K,t) = \left\{x \in K : \frac{|K|\cdot|K^x|}{|B|^2} \le t\right\},$$

where $K^x = (K - x)^\circ = \{z \in \mathbb{R}^n : \langle z, y - x\rangle \le 1 \text{ for all } y \in K\}$ is the polar of K with respect to x. (We consider only those t for which $S(K,t) \ne \emptyset$.) Put

$$S_t = S\left(K, \frac{|K|}{t|B|^2}\right) = \left\{x \in K : |K^x| \le \frac{1}{t}\right\}.$$

Then the family $\{S_t\}_{t\ge 0}$ satisfies Definitions 1 and 1′.

4. *The Illumination bodies* [W]. Let K be a convex body in $\mathbb{R}^n$ and $t \ge 0$. The *illumination body* I_t is the convex body defined as

$$I_t = \{x \in \mathbb{R}^n : |\mathrm{co}[x, K]\backslash K| \le t\}.$$

Then the family $\{I_t\}_{t\ge 0}$ satisfies Definition 1.

THEOREM 3. *Let K be a convex body in $\mathbb{R}^n$. For all $t \ge 0$ let K_t respectively B_t be convex bodies obtained from K respectively B by Definition 1 or 1′. Then*

$$\lim_{t\to 0} \frac{|K| - |K_t|}{|B| - |B_t|} = \frac{\mathrm{as}(K)}{\mathrm{as}(B)}.$$

REMARK. Note that

$$\mathrm{as}(B) = \mathrm{vol}_{n-1}(\partial B) = n|B|.$$

COROLLARY 4. (i) [S-W] *Let K be a convex body in $\mathbb{R}^n$ and for $t \ge 0$ let F_t be a floating body. Then*

$$\lim_{t\to 0} c_n \frac{|K| - |F_t|}{t^{2/(n+1)}} = \mathrm{as}(K) \quad \textit{where} \quad c_n = 2\left(\frac{|B^{n-1}|}{n+1}\right)^{2/(n+1)}.$$

(ii) [Schm] *Let K be a symmetric convex body in $\mathbb{R}^n$ and for $t \ge 0$ let C_t be a convolution body. Then*

$$\lim_{t\to 0} c_n \frac{|K| - |C_t|}{t^{2/(n+1)}} = \mathrm{as}(K)$$

where c_n is as in (i).

(iii) [M-W] *Let K be a convex body in $\mathbb{R}^n$ and for $t \ge 0$ let S_t be a Santaló region. Then*

$$\lim_{t\to 0} e_n \frac{|K| - |S_t|}{t^{2/(n+1)}} = \mathrm{as}(K) \quad \textit{where} \quad e_n = \frac{2}{|B|^{2/(n+1)}}.$$

(iv) [W] *Let K be a convex body in $\mathbb{R}^n$ and for $t \ge 0$ let I_t be an illumination body. Then*

$$\lim_{t\to 0} d_n \frac{|I_t| - |K|}{t^{2/(n+1)}} = \mathrm{as}(K) \quad \textit{where} \quad d_n = 2\left(\frac{|B^{n-1}|}{n(n+1)}\right)^{2/(n+1)}.$$

For the proof of Theorem 3 we need several lemmas. The basic idea of the proof is as in [S-W].

LEMMA 5. *Let K and L be two convex bodies in $\mathbb{R}^n$ such that $0 \in \mathrm{int}(L)$ and $L \subseteq K$. Then*

(i) $$|K| - |L| = \frac{1}{n}\int_{\partial K} \langle x, N(x)\rangle\left(1 - \left(\frac{\|x_L\|}{\|x\|}\right)^n\right) d\mu(x),$$

where $x_L = [0,x] \cap \partial L$ and μ is the usual surface measure on ∂K.

(ii) $$|K| - |L| = \frac{1}{n}\int_{\partial L} \langle x, N(x)\rangle\left(\left(\frac{\|x_K\|}{\|x\|}\right)^n - 1\right) d\mu(x),$$

where x_K is the intersection of the half-line from 0 through x with ∂K and μ is the usual surface measure on ∂L.

The proof of Lemma 5 is standard.

For $x \in \partial K$ denote by $r(x)$ the radius of the largest Euclidean ball contained in K that touches ∂K at x. More precisely,

$$r(x) = \max\{r : x \in B(y,r) \subseteq K \text{ for some } y \in K\}.$$

REMARK. It was shown in [S-W] that:

(i) If $B \subseteq K$, then $\mu\{x \in \partial K : r(x) \ge \beta\} \ge (1-\beta)^{n-1}\mathrm{vol}_{n-1}(\partial K)$.

(ii) $\int_{\partial K} r(x)^{-\alpha}\, d\mu(x) < \infty$ for all α with $0 \le \alpha < 1$.

LEMMA 6. *Suppose* $0 \in \mathrm{int}(K)$. *Then for all* x *with* $r(x) > 0$ *and for all* $t \ge 0$ *we have*

$$0 \le \frac{\langle x, N(x)\rangle (1 - (\|x_t\|/\|x\|)^n)}{n(|B| - |B_t|)} \le g(x),$$

where $\int_{\partial K} g(x)\,d\mu(x) < \infty$. *Here* $x_t = [0, x] \cap \partial K$ *if* $K_t \subseteq K$, *and* x_t *is the intersection of the half-line from* 0 *through* x *with* ∂K_t *if* $K \subseteq K_t$.

LEMMA 7. *Let* x_t *be as in Lemma* 6. *Then*

$$\lim_{t\to 0} \frac{\langle x, N(x)\rangle (1 - (\|x_t\|/\|x\|)^n)}{n(|B| - |B_t|)} \quad \textit{exists a.e.}$$

and is equal to

(i) $\varrho(x)^{-(n-1)/(n+1)}/(n|B|)$ *if the Dupin indicatrix at* $x \in \partial K$ *is an* $(n-1)$*-dimensional sphere with radius* $\sqrt{\varrho(x)}$,

(ii) 0 *if the Dupin indicatrix at* x *is an elliptic cylinder.*

REMARK. (i) $r(x) > 0$ a.e. [S-W] and the Dupin indicatrix exists a.e. [L2] and is an ellipsoid or an elliptic cylinder.

(ii) If the indicatrix is an ellipsoid, we can reduce this case to the case of a sphere by an affine transformation with determinant 1 (see [S-W]).

Proof of Theorem 3. We may assume that 0 is in the interior of K. By Lemma 5 and with the notations of Lemma 6 we have

$$\frac{|K| - |K_t|}{|B| - |B_t|} = \frac{1}{n} \int_{\partial K} \frac{\langle x, N(x)\rangle (1 - (\|x_t\|/\|x\|)^n)}{|B| - |B_t|}\, d\mu(x).$$

By Lemma 6 and the Remark preceding it, the integrands are bounded uniformly in t by an L^1-function and by Lemma 7 they are pointwise convergent a.e. We apply Lebesgue's convergence theorem.

Proof of Lemma 6. Let $x \in \partial K$ be such that $r(x) > 0$. We consider the proof in the case of Definition 1′ and of Definition 1 in the case where $K_t \subseteq K$ for all $t \ge 0$. The case of Definition 1 where $K \subseteq K_t$ for all $t \ge 0$ is treated in a similar way.

As $\|x\| \ge \|x_t\|$, we have, for all t,

$$\text{(1)} \qquad \frac{1}{n}\langle x, N(x)\rangle \left(1 - \left(\frac{\|x_t\|}{\|x\|}\right)^n\right) \le \left\langle \frac{x}{\|x\|}, N(x)\right\rangle \|x - x_t\|.$$

Put $r(x) = r$, $x - r(x)N(x) = z$ and $\langle \frac{x}{\|x\|}, N(x)\rangle = \cos\theta$. We can assume that there is an $\alpha > 0$ such that

$$\text{(2)} \qquad B(0, \alpha) \subseteq K \subseteq B(0, 1/\alpha),$$

and hence

$$\cos\theta\, \|x - x_t\| \le 2/\alpha.$$

Let $\varepsilon > 0$ be given. By Remark 2(ii) there exists t_1 such that for all $t \le t_1$,

$$\text{(3)} \qquad r\left(1 - \frac{1 - f_1(t)}{r^{2n/(n+1)}}(1 + \varepsilon)\right) \le f_r(t) \le r\left(1 - \frac{1 - f_1(t)}{r^{2n/n+1}}(1 - \varepsilon)\right).$$

Let t_0 be such that $Ct_0 < t_1$. By Definition 1(i), $f_1(t)$ is decreasing in t, hence for all $t \ge t_0$ we have

$$f_1(t) \le f_1(t_0)$$

and thus for all $t \ge t_0$, by (1) and (2),

$$\frac{\langle x, N(x)\rangle (1 - (\|x_t\|/\|x\|)^n)}{n(|B| - |B_t|)} \le \frac{2}{\alpha |B| (1 - f_1(t_0)^n)}.$$

Therefore the expression in question is bounded by a constant in this case and hence is integrable. It remains to consider the case when $t < t_0$.

(a) Suppose first that

$$\|x - x_t\| < r\cos\theta.$$

For $B(z, r)$ we construct the corresponding inner body $(B(z, r))_s$ such that x_t is a boundary point of $(B(z, r))_s$. By Definition 1(iii), $(B(z, r))_s$ is the Euclidean ball with center z and radius $f_r(s)$. As x_t is a boundary point of $(B(z, r))_s$,

$$\text{(4)} \qquad f_r(s) = r\left(1 - \frac{2\|x - x_t\|\cos\theta}{r} + \frac{\|x - x_t\|^2}{r^2}\right)^{1/2} \le r\left(1 - \frac{\|x - x_t\|\cos\theta}{2r}\right).$$

The last inequality holds by assumption (a).

So far the arguments are the same for Definitions 1 and 1′. From now on they differ slightly.

By Definition 1(iv), $s \le Ct$, hence by monotonicity $f_r(s) \ge f_r(Ct)$ and thus, as $Ct < t_1$, from (3) we have

$$f_r(Ct) \ge r\left(1 - (1 + \varepsilon)\frac{1 - f_1(Ct)}{r^{2n/(n+1)}}\right),$$

which, using Definition 1(iii), can be shown to be

$$\text{(5)} \qquad \ge r\left(1 - (1 + \varepsilon)(C^{2/(n+1)} + \varepsilon)\frac{1 - f_1(t)}{r^{2n/(n+1)}}\right).$$

From (4) and (5) we get

$$\text{(6)} \qquad 1 - f_1(t) \ge \frac{\|x - x_t\|\cos\theta\, r^{(n-1)/(n+1)}}{2(1 + \varepsilon)(C^{2/(n+1)} + \varepsilon)}.$$

Observe also that

$$|B| - |B_t| = |B|(1 - f_1(t)^n) \geq |B|(1 - f_1(t)).$$

This inequality together with (1) and (6) shows that

$$\frac{\langle x, N(x)\rangle(1 - (\|x_t\|/\|x\|)^n)}{n(|B| - |B_t|)} \leq \frac{2(1+\varepsilon)(C^{2/(n+1)} + \varepsilon)}{|B|} r^{-(n-1)/(n+1)},$$

and the latter is integrable by the Remark preceding Lemma 6.

In the case of Definition 1′ it follows from (iv)′ and (v)′ that $s \leq t$. For if $s > t$, then $(B(z,r))_s \subset \operatorname{int}(B(z,r))_t$ by (iv)′ and $\operatorname{int}(B(z,r))_t \subset \operatorname{int}(K_t)$ by (v)′, which contradicts the fact that $x_t \in \partial K_t \cap \partial(B(z,r))_s$. Therefore $f_r(s) \geq f_r(t)$ and thus, as $t < t_1$, (3) yields

$$f_r(t) \geq r\left(1 - (1+\varepsilon)\frac{1 - f_1(t)}{r^{2n/(n+1)}}\right).$$

We then conclude as above.

(b) Now we consider the case when

$$\|x - x_t\| \geq r\cos\theta.$$

We choose α so small that $x_t \notin B(0,\alpha)$. Let H be the hyperplane through 0 orthogonal to x. Then the spherical cone $C = [x, H \cap B(0,\alpha)]$ is contained in K and $x_t \in C$. Let $d = \operatorname{dist}(x_t, C)$. Then

$$d = \|x - x_t\| \frac{\alpha}{(\alpha^2 + \|x\|^2)^{1/2}}. \tag{7}$$

Let $w \in [0, x_t]$ be such that $\|x_t - w\| = d/2$. Let $B(w,R) \subseteq K$ be the largest Euclidean ball with center w such that $B(w,R) \subseteq K$. Then $\partial B(w,R) \cap \partial K \neq \emptyset$. Moreover $R \geq d$, which implies that $x_t \in B(w,R)$. Let $(B(w,R))_s$ be the corresponding inner ball such that $x_t \in \partial(B(w,R))_s$.

Now we have to distinguish between Definitions 1 and 1′.

By Definition 1(iv), $s \leq Ct$. By monotonicity $f_R(s) \geq f_R(Ct)$, which, as above, is

$$\geq R\left(1 - (1+\varepsilon)(C^{2/(n+1)} + \varepsilon)\frac{1 - f_1(t)}{R^{2n/(n+1)}}\right).$$

As $R \geq d$, the latter is

$$\geq d\left(1 - (1+\varepsilon)(C^{2/(n+1)} + \varepsilon)\frac{1 - f_1(t)}{d^{2n/(n+1)}}\right).$$

On the other hand, by construction $f_R(s) = d/2$. Therefore

$$1 - f_1(t) \geq \frac{d^{2n/(n+1)}}{2(1+\varepsilon)(C^{2/(n+1)} + \varepsilon)}.$$

Note also that (2) implies that $\cos\theta \geq \alpha^2$. Hence by (1), (2), (7) and assumption (b) we get

$$\frac{\langle x, N(x)\rangle(1 - (\|x_t\|/\|x\|)^n)}{n(|B| - |B_t|)} \leq \frac{2(1+\alpha^4)^{n/(n+1)}(1+\varepsilon)(C^{2/(n+1)} + \varepsilon)}{|B|\alpha^{(6n-2)/(n+1)}} r^{-(n-1)/(n+1)}.$$

The case of Definition 1′ is treated similarly and the above inequalities hold true with $C = 1$ and $C^{2/(n+1)} + \varepsilon = 1$.

Proof of Lemma 7. We again consider the case when $K_t \subseteq K$ for all $t \geq 0$ in Definition 1. The case $K \subseteq K_t$ for all $t \geq 0$ in Definition 1 and the case of Definition 1′ are done in a similar way (compare the proof of Lemma 6).

As in the proof of Lemma 6 we can choose $\alpha > 0$ such that

$$B(0,\alpha) \subseteq K \subseteq B(0, 1/\alpha).$$

Therefore

$$1 \geq \langle x/\|x\|, N(x)\rangle \geq \alpha^2. \tag{8}$$

We put again $\cos\theta = \langle x/\|x\|, N(x)\rangle$. (1) holds, that is,

$$\frac{1}{n}\langle x, N(x)\rangle(1 - (\|x_t\|/\|x\|)^n) \leq \langle x/\|x\|, N(x)\rangle\|x - x_t\|.$$

Since x and x_t are collinear, $\|x\| = \|x_t\| + \|x - x_t\|$ and hence

$$\frac{1}{n}\langle x, N(x)\rangle\left(1 - \left(\frac{\|x_t\|}{\|x\|}\right)^n\right) = \frac{1}{n}\langle x, N(x)\rangle\left(1 - \left(1 - \frac{\|x - x_t\|}{\|x\|}\right)^n\right) \geq \left\langle \frac{x}{\|x\|}, N(x)\right\rangle\|x - x_t\|\left(1 - k_1\frac{\|x - x_t\|}{\|x\|}\right) \tag{9}$$

for some constant k_1, if we choose t sufficiently large.

(i) *Case where the indicatrix is an ellipsoid.* We have seen that then we can assume that the indicatrix is a Euclidean sphere. Let $\sqrt{\varrho(x)}$ be its radius. We put $\varrho(x) = \varrho$ and we introduce a coordinate system such that $x = 0$ and $N(x) = (0, \ldots, 0, -1)$. H_0 is the tangent hyperplane to ∂K at $x = 0$ and $\{H_\alpha : \alpha \geq 0\}$ is the family of hyperplanes parallel to H_0 that have non-empty intersection with K and are at distance α from H_0. For $\alpha > 0$, H_α^+ is the half-space generated by H_α that contains $x = 0$. For $a \in \mathbb{R}$, let $z_a = (0, \ldots, 0, a)$ and $B_a = B(z_a, a)$ be the Euclidean ball with center z_a and radius a. As in [W], for $\varepsilon > 0$ we can choose α_0 so small that for all $\alpha \leq \alpha_0$,

$$B_{\varrho-\varepsilon} \cap H_\alpha^+ \subseteq K \cap H_\alpha^+ \subseteq B_{\varrho+\varepsilon} \cap H_\alpha^+.$$

We choose t so small that $x_t \in \operatorname{int}(B_{\varrho-\varepsilon} \cap H_\alpha^+)$ $(\subseteq \operatorname{int}(B_{\varrho+\varepsilon} \cap H_\alpha^+))$. For $B_{\varrho+\varepsilon}$ we construct the corresponding inner body $(B_{\varrho+\varepsilon})_s$ such that x_t is a boundary point of $(B_{\varrho+\varepsilon})_s$. $(B_{\varrho+\varepsilon})_s$ is the Euclidean ball with center $z_{\varrho+\varepsilon}$ and radius $f_{\varrho+\varepsilon}(s)$. We have

$$f_{\varrho+\varepsilon}(s) = ((\varrho+\varepsilon)^2 + \|x-x_t\|^2 - 2(\varrho+\varepsilon)\|x-x_t\|\cos\theta)^{1/2}, \geq (\varrho+\varepsilon)\left(1 - \frac{\|x-x_t\|\cos\theta}{\varrho+\varepsilon}\right).$$

Definition 1(v) implies that $s \geq (1-\varepsilon)t$, hence by monotonicity $f_{\varrho+\varepsilon}(s) \leq f_{\varrho+\varepsilon}((1-\varepsilon)t)$, which for t small enough is (cf. the proof of Lemma 6)

$$\leq (\varrho+\varepsilon)\left(1 - (1-k_2\varepsilon)\frac{1-f_1(t)}{(\varrho+\varepsilon)^{2n/(n+1)}}\right),$$

where k_2 is a constant. Thus

$$1 - f_1(t) \leq \frac{\|x-x_t\|\cos\theta\,(\varrho+\varepsilon)^{(n-1)/(n+1)}}{1-k_2\varepsilon}.$$

Note that

$$|B| - |B_t| = |B|(1-f_1(t)^n) \leq n|B|(1-f_1(t)).$$

Therefore by (9),

$$\frac{\langle x, N(x)\rangle(1-(\|x_t\|/\|x\|)^n)}{n(|B|-|B_t|)} \geq (1-k_2\varepsilon)\left(1 - k_1\frac{\|x-x_t\|}{\|x\|}\right)\frac{(\varrho+\varepsilon)^{-(n-1)/(n+1)}}{n|B|}.$$

This is the lower bound for the expression in question.

To get an upper bound we proceed similarly. For $B_{\varrho-\varepsilon}$ we construct the corresponding inner body $(B_{\varrho-\varepsilon})_s$ such that x_t is a boundary point of $(B_{\varrho-\varepsilon})_s$. $(B_{\varrho-\varepsilon})_s$ is the Euclidean ball with center $z_{\varrho-\varepsilon}$ and radius $f_{\varrho-\varepsilon}(s)$. We have

$$\begin{aligned} f_{\varrho-\varepsilon}(s) &= ((\varrho-\varepsilon)^2 + \|x-x_t\|^2 - 2(\varrho-\varepsilon)\|x-x_t\|\cos\theta)^{1/2} \\ &\leq (\varrho-\varepsilon)\left(1 - \frac{\|x-x_t\|\cos\theta}{\varrho-\varepsilon}\left(1 - \frac{\|x-x_t\|}{2(\varrho-\varepsilon)\cos\theta}\right)\right. \\ &\quad \times \left.\left(1 + k_3\frac{\|x-x_t\|\cos\theta}{\varrho-\varepsilon}\left(1 - \frac{\|x-x_t\|}{2(\varrho-\varepsilon)\cos\theta}\right)\right)\right) \end{aligned}$$

for some constant k_3, if t is small enough. Again by Definition 1(v), $s \leq (1+\varepsilon)t$ and therefore $f_{\varrho-\varepsilon}(s) \geq f_{\varrho-\varepsilon}((1+\varepsilon)t)$, which by arguments similar to those before is

$$\geq (\varrho-\varepsilon)\left(1 - (1+k_4\varepsilon)\frac{1-f_1(t)}{(\varrho-\varepsilon)^{2n/(n+1)}}\right)$$

with a suitable constant k_4. Thus

$$\begin{aligned} (10)\qquad 1 - f_1(t) &\geq \frac{\|x-x_t\|\cos\theta}{1+k_4\varepsilon}\left(1 - \frac{\|x-x_t\|}{2(\varrho-\varepsilon)\cos\theta}\right) \\ &\quad \times \left(1 + \frac{k_3\|x-x_t\|\cos\theta}{\varrho-\varepsilon}\left(1 - \frac{\|x-x_t\|}{2(\varrho-\varepsilon)\cos\theta}\right)\right)(\varrho-\varepsilon)^{(n-1)/(n+1)}. \end{aligned}$$

Observe now that

$$(11)\qquad |B| - |B_t| = |B|(1-f_1(t)^n) \geq n|B|(1-f_1(t))\left(1 - \frac{n-1}{2}(1-f_1(t))\right).$$

We choose t so small that $1 - f_1(t) < 2\varepsilon/(n-1)$. This together with (1), (10) and (11) implies that

$$\begin{aligned} &\frac{\langle x, N(x)\rangle(1-(\|x_t\|/\|x\|)^n)}{n(|B|-|B_t|)} \\ &\leq \frac{1+k_4\varepsilon}{(1-\varepsilon)\left(1 - \frac{\|x-x_t\|}{2(\varrho-\varepsilon)\cos\theta}\right)\left(1 + k_3\frac{\|x-x_t\|\cos\theta}{\varrho-\varepsilon}\left(1 - \frac{\|x-x_t\|}{2(\varrho-\varepsilon)\cos\theta}\right)\right)} \\ &\quad \times \frac{(\varrho-\varepsilon)^{-(n-1)/(n+1)}}{n|B|}. \end{aligned}$$

Note that $\cos\theta \geq \alpha^2$ by (8). This finishes the proof of Lemma 7 in the case where the indicatrix is an ellipsoid.

(ii) *Case where the indicatrix is an elliptic cylinder*. Recall that then we have to show that

$$\lim_{t\to 0}\frac{\langle x, N(x)\rangle(1-(\|x_t\|/\|x\|)^n)}{n(|B|-|B_t|)} = 0.$$

We can again assume (see [S-W]) that the indicatrix is a spherical cylinder, i.e. the product of a k-dimensional plane and an $(n-k-1)$-dimensional Euclidean sphere of radius ϱ. We can moreover assume that ϱ is arbitrarily large (see also [S-W]).

By Lemma 9 of [S-W] we then have for sufficiently small α and some $\varepsilon > 0$,

$$B_{\varrho-\varepsilon} \cap H_\alpha^+ \subseteq K \cap H_\alpha^+.$$

Using similar methods, this implies that

$$\lim_{t\to 0}\frac{\langle x, N(x)\rangle(1-(\|x_t\|/\|x\|)^n)}{n(|B|-|B_t|)} = 0.$$

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Transitivity for linear operators on a Banach space

by

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Abstract. Let G be the multiplicative group of invertible elements of $E(X)$, the algebra of all bounded linear operators on a Banach space X. In 1945 Mackey showed that if $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are any two sets of linearly independent elements of X with the same number of items, then there exists $T \in G$ so that $T(x_k) = y_k$, $k = 1, \ldots, n$. We prove that some proper multiplicative subgroups of G have this property.

1. Introduction. Throughout, X is an infinite-dimensional Banach space and $E(X)$ is the algebra of all bounded linear operators on X. A subset S of $E(X)$ is called *l. i. transitive* if, given two sets $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of linearly independent elements of X, there exists $T \in S$ such that $T(x_k) = y_k$, $k = 1, \ldots, n$. In [5, Theorem II-3] Mackey showed that the set G of invertible elements of $E(X)$ is l. i. transitive. Our results show that smaller subgroups of the multiplicative group G suffice. We show the following in §2.

THEOREM 1. *Let A be any closed subalgebra of $E(X)$ containing the identity I and all $T \in E(X)$ with finite-dimensional range. Let $\mathfrak{G}$ be the set of invertible elements of A. Then any open multiplicative subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is l. i. transitive.*

As is well known, $\mathfrak{G}$ is open.

Next let ψ be the set of all elements of $\mathfrak{G}$ of the form $I + T$ where T has finite-dimensional range. If we write its inverse as $I + V$, $V \in E(X)$, we see that $T + V + TV = 0$ so that V also has finite-dimensional range. It follows that ψ is a multiplicative subgroup of $\mathfrak{G}$, and

THEOREM 2. *ψ is l. i. transitive.*

2. On transitivity. Our aim is to prove Theorems 1 and 2 given above. We shall use an easy lemma.

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