Strictly singular operators and the invariant subspace problem

by

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Abstract. Properties of strictly singular operators have recently become of topical interest because the work of Gowers and Maurey in [GM1] and [GM2] gives (among many other brilliant and surprising results, such as those in [G1] and [G2]) Banach spaces on which every continuous operator is of form $A + S$, where $S$ is strictly singular. So if strictly singular operators had invariant subspaces, such spaces would have the property that all operators on them had invariant subspaces. However, in this paper we exhibit examples of strictly singular operators without nontrivial closed invariant subspaces. So, though it may be true that operators on the spaces of Gowers and Maurey have invariant subspaces, yet this cannot be because of a general result about strictly singular operators. The general assertion about strictly singular operators is false.

0. Introduction

0.1. The author would like to thank Prof. A. Pelczyński for suggesting this line of investigation to him.

0.2. Operators without invariant subspaces were first found independently by P. Enflo and this author ([E1], [R1]), on an unknown Banach space. They were found on $l_1$ and $c_0$ by the present author ([R4], [R7]) and various extensions of the method were found ([R2], [R3], [R5], [R6], [R8]), of which the nearest to the present paper is the construction of a quasinilpotent operator without invariant subspaces on $l_1$ in [R8]. A general account of the theory of invariant subspaces, written before all these counterexamples were discovered, will be found in [R1]. A short account of the basic properties of the James space $J$ will be found in roofsinger [S1], pp. 273–279. The original article is [J1].

0.3. A continuous linear map $T : E \to F$, where $E$ and $F$ are normed spaces, is norm increasing if there is an $c > 0$ such that
for all \( x \in E \).

0.4. A continuous linear map \( T : E \to F \), where \( E \) and \( F \) are Banach spaces, is said to be strictly singular if there is no infinite-dimensional subspace \( W \subseteq E \) such that \( T|_W \) is norm increasing.

0.5. The James p-space \( J_p \) (\( 1 < p < \infty \)) is the set of all sequences \( (a_i)_{i=1}^{\infty} \in c_0 \) such that

\[
\|a\| = \sup \left\{ \left( \sum_{i=1}^{n} |a_i - a_{i-1}|^p \right)^{1/p} : i_1 < \ldots < i_n, n \in \mathbb{N} \right\} < \infty.
\]

It is a fact that \( J_p \) is nonreflexive, \( \text{dim}(J_p^* / J_p) = 1 \), but that every infinite-dimensional subspace of \( J_p \) contains a subspace isomorphic to \( l_p \).

1. Strictly singular noncompact operators. It is well known that on \( l_p \) (\( 1 \leq p < \infty \)) or \( c_0 \), any strictly singular operator is compact. On the other hand, the inclusion map \( l_p \to J_q \) (\( 1 < q < \infty \)) is strictly singular but not compact. For our purposes we want something like the inclusion map \( l_p \to J_q \), but which happens between nonreflexive Banach spaces (and which happens, let it be said, in a manner which has respect for the nonreflexivity, in the sense that there is a sequence of unit vectors in the domain space with no weak*-convergent subsequence, which is mapped to a sequence in the image space which also has no weak*-convergent subsequence).

For such a map we look to the James p-spaces \( J_p \).

**Lemma 1.1.** The natural inclusion \( i : J_p \to J_q \) (\( 1 < p < q < \infty \)) is strictly singular.

**Proof.** If not, there is an infinite-dimensional subspace \( E \subseteq J_p \) on which the norms \( \|\cdot\|_p \) and \( \|\cdot\|_q \) are equivalent. Taking a subspace of \( E \) as necessary, this tells us that \( (E, \|\cdot\|_q) \) is isomorphic to \( l_p \) (for every infinite-dimensional subspace of \( J_p \) contains a subspace isomorphic to \( l_p \); see 0.5). Taking a further subspace, we find \( l_q \) embedded up to isomorphism in \( J_p \), which is nonsense. \( \blacksquare \)

**Definition 1.2.** Let us choose, once and for all, a strictly increasing sequence \( (p_i)_{i=1}^{\infty} \) of real numbers strictly greater than 2. The Banach space \( X \) is defined as the \( l_2 \)-direct sum

\[
X = \left( \bigoplus_{i=1}^{\infty} J_{p_i} \right) _{l_2}.
\]

It is on this Banach space \( X \) that we will construct a strictly singular operator without invariant subspaces. We will write \( (x_i)_{i=1}^{\infty} \) for the unit vector basis of \( J_{p_i} \), and \( (f_{ij})_{j=0}^{\infty} \) for the unit vector basis of the space \( l_2 \).

An element \( x \in X \) can be regarded as a sequence \( (x_i)_{i=1}^{\infty} \) with \( x_i \in l_2 \), \( x_i \in J_{p_i} \) (\( i > 0 \)). It can be shown that if \( (\delta_{ij})_{i=0}^{\infty} \) is a sequence of scalars tending to zero, then the "weighted shift" operator

\[
W : (x_0, x_1, x_2, \ldots) \to (0, \delta_0 x_0, \delta_1 x_1, \delta_2 x_2, \ldots)
\]

is strictly singular (see §3.3). We will construct an operator on \( X \) without invariant subspaces, which has a good deal in common with a weighted shift \( W \).

The next few definitions follow [R4].

**Definition 1.3.** Our construction will be built around a strictly increasing sequence \( d = (d_i)_{i=1}^{\infty} \) of positive integers. This sequence will be required to "increase sufficiently rapidly" in the sense of [R1], §1. We will write \( a_i = d_{i-1} \) (\( i = 1, 2, \ldots \)) and \( b_i = d_{2i} \). Thus, \( a_1 < b_1 < a_2 < b_2 < \ldots \)

We define \( a_0 = 1, v_0 = 0, v_n = n(a_n + a_n) \) (\( n > 0 \)). We will use the symbol \( \text{pd} \) to mean, "provided \( d \) increases sufficiently rapidly", as we did in [R4]. We define \( w_n = 1 + \sum_{r=0}^{n-1}(1 + v_r), w_0 = 1 \).

**Definition 1.4.** For \( p \in \mathbb{N} \), denote the dense subspace of \( X \) spanned by the unit vectors \( \{f_{ij} : i \geq 0, j \geq 0\} \). If \( S \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \), then \( F_S \subseteq F \) will denote the linear span of the set \( \{f_{ij} : (i, j) \in S\} \). Further, \( \pi_S \) denotes the projection \( F \to F_S \) such that \( \pi_S(f_{ij}) = f_{ij} \) (\( (i, j) \in S \)) or \( 0 \) (\( (i, j) \not\in S \)).

This \( \pi_S \) is continuous only for certain choices of \( S \); we shall not be using any \( S \) for which it is discontinuous, however. \( f_{ij}^* \) will denote the non-linear functional on \( F \) such that \( f_{ij}^* (f_{kl}) = b_k b_l \).

**Definition 1.5.** Let \( |p| \) denote the sum of the absolute values of the coefficients of the polynomial \( p \). For a finite set \( S \), let \( |S| \) denote the number of elements of \( S \).

2. The main definition. We will now define, in terms of the sequence \( d \) as in 1.3, a sequence \( (e_i)_{i=0}^{\infty} \) whose linear span is the dense subspace \( F \) of \( X \).

We shall begin by rearranging the fundamental set \( (f_{ij})_{j=0}^{\infty} \) into a fundamental sequence \( (f_i)_{i=0}^{\infty} \). Each \( f_{ij} \) is equal to \( f_{i,j}(k) \), where \( T : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+ \) is a suitable bijection (see Definition 2.4). We will write \( F_n \) for the linear span \( \{f_{ij} : i \geq 0, j \geq 0\} \)---a special case of the subspaces \( F_S \) as in Definition 1.4. This particular choice of \( S \) will be called \( S_n \), the unique subset \( \mathbb{Z}^+ \subseteq \{0, n\} \) of \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) such that

\[
\text{lin}\{f_{ij} : i \geq 0, j \geq 0\} = \text{lin}\{f_{ij} : (i, j) \in S_n\}.
\]

We then define linear relationships of general form \( f_i = \sum_{j=0}^{n} \lambda_{ij} f_{ij} \), with \( \lambda_{ij} \neq 0 \), for each \( i \in \mathbb{Z}^+ \) (this is done in Definition 2.5). Because the matrix with entries \( \lambda_{ij} \) is lower triangular with nonzero diagonal entries,
this linear map can be inverted providing us with an alternative vector space basis \((e_i)_{i=0}^{\infty}\) of \(F\), given uniquely as linear combinations of general form \(e_i = \sum_{j=0}^{\infty} c_{ij} J^j f_j\).

So there is a unique linear map \(T : F \to F\) that acts as a right shift operator sending each \(e_i\) to \(e_{i+1}\). It turns out that \(pd, T\) extends to a continuous operator \(X \to X\) that is strictly singular, and has no nontrivial closed invariant subspaces.

**Definition 2.1.** Let the sequence \(d\) be given. Let \(\Omega \subset \mathbb{Z}^+\) be the set
\[
\bigcup_{n=1}^{\infty} \bigcup_{r=1}^{n} ([n-s]a_n, (n-s+a_n+v_r) = \bigcup_{s=0}^{\infty} \bigcup_{n=s+1}^{\infty} \{(n-s)a_n, (n-s+a_n+v_r\}
\]

Provided \(d\) increases sufficiently rapidly, the union (2.1.0) is disjoint, and both \(\Omega\) and \(\mathbb{Z}^+\setminus\Omega\) are infinite sets. If \(d\) does indeed increase sufficiently fast for this to happen, we make the following definitions:

**Definition 2.2.** Let \(\gamma\) be the unique increasing bijection \(\mathbb{Z}^+ \setminus \Omega \to \mathbb{Z}^+\).

**Definition 2.3.** (a) For each \(s \geq 0\), let \(\sigma_s\) be the natural bijection from the set \(\bigcup_{n=s+1}^{\infty} \bigcup_{r=1}^{n} ([n-s]a_n, (n-s+a_n+v_r) \subset \mathbb{Z}^+\) to the set \([0, v_r) \times \mathbb{Z}^+ \subset \mathbb{Z}^+ \times \mathbb{Z}^+\) that sends the integer \((n-s)a_n+i, 0 \leq i \leq v_r\) to the pair \((i, n-s-1)\).

(b) Define maps \(\chi_s, \chi_s(j) = \sigma_s(j) + (v_r, 0)\), so that the image of \(\chi_s\) is equal to \([v_r, v_{r+1}) \times \mathbb{Z}^+ = (v_r, v_{r+1}) \times \mathbb{Z}^+\) (for \(v_{s+1} = v_s + v_{s+1}\), by Definition 1.3).

(c) Let \(\chi : \Omega \to \mathbb{Z}^+ \times \mathbb{Z}^+\) be the unique map whose restriction to each subset \(\bigcup_{s=s+1}^{\infty} \bigcup_{r=1}^{n} ([n-s]a_n, (n-s+a_n+v_r) \subset \Omega\) is equal to \(\chi_s\).

Now \(\chi\) is a bijection from \(\Omega\) onto \([w_0, \infty) \times \mathbb{Z}^+\), that is, onto \(\mathbb{N} \times \mathbb{Z}^+\).

We may obtain a bijection \(\mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+\) by making sure that \(\mathbb{Z}^+ \setminus \Omega\) gets mapped onto \([0, \infty) \times \mathbb{Z}^+\), thus:

**Definition 2.4.** Let us extend \(\chi\) to a map \(\mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+\) by defining \(\chi(i) = (0, \gamma^{-1}(i))\) for each \(i \in \Omega\). Since \(\chi\) is always a bijection, we may also define the map \(\mathcal{I} = \chi^{-1} : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+\).

**Definition 2.5.** Let the sequence \(d\) be given, and let it increase sufficiently fast that the maps \(\chi\) and \(\mathcal{I}\) are defined. For each \(i\), define \(f_i = f_j k\), where \((j, k) = \chi(i)\). We shall show that, \(pd\), there is a unique sequence \((e_i)_{i=0}^{\infty}\) in \(F\) with the following properties. Firstly,
\[
(2.5.0) \quad f_0 = e_0.
\]

Secondly, if integers \(r, n, i\) satisfy \(0 < r \leq n, i \in [0, v_{n-r} + r a_n]\), then
\[
(2.5.1) \quad f_i = \sum_{r=0}^{(r-1) a_n} \left((1+n)^{r n} e_i \right) - \left((1+n)^{r n} e_i \right) = \sum_{r=0}^{(r-1) a_n} \left((1+n)^{r n} e_i \right) - \left((1+n)^{r n} e_i \right).
\]

Thirdly, if \(0 < r < n, i \in (r n + v_{n-r} + r + 1) a_n\) (respectively, if \(1 \leq n, i \in (v_{n-1} - i) a_n\), then
\[
(2.5.2) \quad f_i = (1 + n) i_{(n-1)/\sqrt{\mathcal{I}_n}} a_n e_i
\]

where \(h = (r + 1) a_n\) (respectively, \(h = (1/2) a_n\)). If integers \(r, n, i\) satisfy \(0 < r \leq n, i \in (r n) a_n + v_{n-r} + r a_n\), then
\[
(2.5.3) \quad f_i = (1 + n) i_{(n-1)/\sqrt{\mathcal{I}_n}} a_n e_i
\]

If integers \(r, n, i\) satisfy \(0 \leq r < n, i \in (n a_n + r b_n, (r + 1) a_n + v_{n-r} + r a_n)\), then
\[
(2.5.4) \quad f_i = (1 + n) i_{(n-1)/\sqrt{\mathcal{I}_n}} a_n e_i
\]

where \(h = (r + 1) a_n\).

**Lemma 2.6.** \(pd\), the sequence \((e_i)_{i=0}^{\infty}\) satisfying (2.5.0)–(2.5.4) does indeed exist, is unique, and is a vector space basis of \(F\). There is a unique linear map \(T : F \to F\) such that \(T e_i = e_{i+1}\) for each \(i\).

**Proof.** Each definition is of form \(f_i = \sum_{i=0}^{i} \lambda_{i} e_i\), with \(\lambda_i \neq 0\). The values taken by the index \(i\) in formulae (2.5.0)–(2.5.2) include zero, \([0, v_{n-r} + r a_n, (0 < r \leq n); (r n + v_{n-r} + (r + 1) a_n, (0 < r < n); (v_{n-1} - i) a_n, i \leq n)\). \(pd\), this means each value \(i = 0\) or \(i \in (v_{n-1}, n a_n, n \geq 1)\) is mentioned once and only once.

The remaining values of \(i\) are taken care of by (2.5.3), (2.5.4). These cases cover intervals \([r a_n + v_{n-r}, (r + 1) a_n, (0 < r \leq n); (n a_n + r b_n, (r + 1) \times (a_n + b_n), 0 \leq r < n)\), whose union is \([n a_n, n (a_n + b_n), (n a_n + v_{n-r} + r a_n, i \leq n)\). As the index \(r\) varies, we catch the rest of \(Z^+\).

\(pd\), then, each \(f_i (i \geq 0)\) is defined once and only once, and has the general form \(\sum_{i=0}^{i} \lambda_{i} e_i\).

Because \(\lambda_i \neq 0\) the linear relationship between the \(e_i\) and the \(f_i\) is invertible (we have a lower triangular matrix with nonzero entries in the diagonal) so the \(e_i\) do exist, are unique, and span \(F\). Note by the way that if \(i = \mathcal{I}(j, k)\) then
\[
(2.6.0) \quad f_{j+k}(\lambda_{k} e_i) = 1
\]

since \(f_{j+k}(1) = 1\), and obviously \(f_{j+k}(e_n) = 0\) for \(m < i\). It is then also true that for each \(n,\)
\[
\lim_{n \to \infty} e_n = \lim_{n \to \infty} f_{j+k}(e_n) = 0.
\]

say, where \(S_n = \chi(0, 1, \ldots, n); |S_n| = n + 1\). As we remarked at the beginning of \(\S 2\), we will abbreviate \(F_{S_n}\) to \(F_{S_n}\). \((e_i)_{i=0}^{\infty}\) is an alternative vector space basis for \(F\), so of course there is a unique map \(T\) such that \(T e_i = e_{i+1}\) for all \(i\)--as yet we say nothing about continuity!\n
From now on, we will always assume that the given sequence \(d\) increases sufficiently rapidly for Lemma 2.6 to hold.
Obviously we must now prove that (pd) $T$ is continuous and strictly singular. This is the object of the next section.

3. $T$ is continuous and strictly singular

3.1. Continuity of $T$. The method of achieving this result is to approximate $T$ by an appropriate “weighted shift” operator $W$, and then estimate the norm of the “error term” $T - W$ by ad hoc methods. This also gives us a natural direction to take when proving that $T$ is strictly singular.

**Definition 3.2.** Let $W_0 : l_2 \rightarrow l_2$ be a weighted shift operator with $W_0f_0 = \alpha f_0$, $f_1, \ldots, f_{n-1}$, we define the weights $\alpha_j$ as follows. Writing $i = \gamma^{-1}(j)$, we know that either $\gamma$ is zero, or it lies in one of the intervals of $\{v_n-a_n, (r+1)\} \cap \{v_{n-1}, (r+1)\} \cap \{r, (r+1)\}$ that feature in parts (2.3.2), (2.3.3) and (2.3.4) of Definition 2.5. With an eye on that definition, we set

$$
\alpha_j = \begin{cases} 
\frac{2^j}{\sqrt{n}} & \text{if } i \in \{v_n-a_n, (r+1)\}, \\
\frac{1}{n} & \text{if } i \in \{r, (r+1)\}, \\
0 & \text{otherwise},
\end{cases}
$$

(3.2.0)

for all $j \geq 0$, and $\beta_1, \beta_2, \beta_3, \ldots$ where the coefficients $\beta_j$ are as follows:

$$
\beta_j = \begin{cases} 
\frac{1}{m+1} & \text{if } j \in [w_m, w_m + v_m], \\
0 & \text{if } j \in [w_m + v_m, (m+1)].
\end{cases}
$$

(3.3.0)

Once again, in the cases when $\beta_j \neq 0$, the actions of $T$ on $J_{p_j}$ and of $W_1$ are identical. For $f_j \in \{w_m, w_m + v_m\}$ and $k \geq 0$, let us write $n = k+1$, and let $i = f_j - v_m$. Then Definition 2.3 gives us $\chi((n-s)a_n+i) = (w_n+i, n-s-1) = (j,k)$, and $\chi((n-s)a_n+i+1) = (j+1,k)$. Therefore, Definition 2.1 gives us

$$
f_{j,k} = f_{j+1,k} = f_{i+s} = f_{i+s+1},
$$

and

$$
f_{j+1,k+1} = f_{i+s+1} = f_{i+s+2},
$$

and

$$
Tf_{j+k} = \frac{1}{n+1} f_{j+k+1} = W_0f_{j,k}.
$$

Hence, $W_0$ is a weighted shift operator on $l_2$, obviously of norm $\frac{1}{2} \cdot 2^j \sqrt{n}$. (If we assume the interval $(a_1, a_1 + b_1 - 1)$ is nonempty, a rather mild condition of “rapid increase” on the sequence d). Note it is also compact, for the weights tend to zero.

**Definition 3.3.** Let $W_1 : (\bigoplus_{i=1}^\infty J_{p_i})_1 \rightarrow (\bigoplus_{i=1}^\infty J_{p_i})_1$ be the map such that the sequence $(x_1, x_2, x_3, \ldots)$ with $x_i \in J_{p_i}$ is sent to the sequence $(0, \beta_1 x_1, \beta_2 x_2, \beta_3 x_3, \ldots)$ where the coefficients $\beta_i$ are as follows:

$$
\beta_j = \begin{cases} 
\frac{1}{m+1} & \text{if } j \in [w_m, w_m + v_m], \\
0 & \text{if } j \in [w_m + v_m, (m+1)].
\end{cases}
$$

(3.3.0)

for all $j \geq 0$, and $\beta_1, \beta_2, \beta_3, \ldots$ where the coefficients $\beta_j$ are as follows:

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\frac{1}{m+1} & \text{if } j \in [w_m, w_m + v_m], \\
0 & \text{if } j \in [w_m + v_m, (m+1)].
\end{cases}
$$

(3.3.0)

Once again, in the cases when $\beta_j \neq 0$, the actions of $T$ on $J_{p_j}$ and of $W_1$ are identical. For $f_j \in \{w_m, w_m + v_m\}$ and $k \geq 0$, let us write $n = k+1$, and let $i = f_j - v_m$. Then Definition 2.3 gives us $\chi((n-s)a_n+i) = (w_n+i, n-s-1) = (j,k)$, and $\chi((n-s)a_n+i+1) = (j+1,k)$. Therefore, Definition 2.1 gives us

$$
f_{j,k} = f_{j+1,k} = f_{i+s} = f_{i+s+1},
$$

and

$$
f_{j+1,k+1} = f_{i+s+1} = f_{i+s+2},
$$

and

$$
Tf_{j+k} = \frac{1}{n+1} f_{j+k+1} = W_0f_{j,k}.
$$

Hence, $W_0$ is a weighted shift operator on $l_2$, obviously of norm $\frac{1}{2} \cdot 2^j \sqrt{n}$. (If we assume the interval $(a_1, a_1 + b_1 - 1)$ is nonempty, a rather mild condition of “rapid increase” on the sequence d). Note it is also compact, for the weights tend to zero.

**Definition 3.3.** Let $W_1 : (\bigoplus_{i=1}^\infty J_{p_i})_1 \rightarrow (\bigoplus_{i=1}^\infty J_{p_i})_1$ be the map such that the sequence $(x_1, x_2, x_3, \ldots)$ with $x_i \in J_{p_i}$ is sent to the sequence $(0, \beta_1 x_1, \beta_2 x_2, \beta_3 x_3, \ldots)$ where the coefficients $\beta_i$ are as follows:

$$
\beta_j = \begin{cases} 
\frac{1}{m+1} & \text{if } j \in [w_m, w_m + v_m], \\
0 & \text{if } j \in [w_m + v_m, (m+1)].
\end{cases}
$$

(3.3.0)

for all $j \geq 0$, and $\beta_1, \beta_2, \beta_3, \ldots$ where the coefficients $\beta_j$ are as follows:

$$
\beta_j = \begin{cases} 
\frac{1}{m+1} & \text{if } j \in [w_m, w_m + v_m], \\
0 & \text{if } j \in [w_m + v_m, (m+1)].
\end{cases}
$$

(3.3.0)

for all $j \geq 0$, and $\beta_1, \beta_2, \beta_3, \ldots$ where the coefficients $\beta_j$ are as follows:

$$
\beta_j = \begin{cases} 
\frac{1}{m+1} & \text{if } j \in [w_m, w_m + v_m], \\
0 & \text{if } j \in [w_m + v_m, (m+1)].
\end{cases}
$$

(3.3.0)
Lastly, by (2.5.4), if \( j = \gamma(i) \), \( i = (r+1)(a_n + b_n) - 1 \), \( 0 \leq r < n \), then
\[
(3.3.5) \quad (T - W) f_{0,j} = (1 + n)^{(r+1)(a_n + b_n) - 1} \times 2^{(1-a_n/2)} \sqrt{e_{(r+1)(a_n + b_n)}}.
\]
In all other cases, \( (T - W) f_{i,j} = 0 \).

**Lemma 3.4.** For every \( \eta > 0 \) the following is true: \( pd \), \( T - W \) is a nuclear operator of norm at most \( \eta \).

**Proof.** It is necessary to estimate the sum of the norms of all the vectors in (3.3.1)-(3.3.5), add up the estimates and check that \( pd \) the sum is less than \( \eta \). These sort of details will be very familiar to readers of [R1]-[R8].

Obviously (3.3.1) contributes \( 2^{1+1-(1-a_n/2)/\sqrt{e_{n}}} \) to our sum (which is less than \( \eta/5 \), \( pd \), let us say). Now (2.5.1) gives us (for \( 0 < r < n \))
\[
(3.4.0) \quad (1 + n)^{r-a_n} e_{r,n} = a_{r-1} f_{r-1,n} + n^{(r-2)a_{r-2}} e_{r-2,n-2} + \ldots + a_{n-r} \sum_{a=0}^{n-r} f_{a,n,a} + e_0.
\]
Now the \( J_p \) spaces have the special property—closely related to their non-reflexivity—that for all \( r, j \) we have \( \| \sum_{s=0}^{r-1} f_j \| = 1 \). So,
\[
(3.4.1) \quad \| (1 + n)^{r-a_n} e_{r,n} - e_0 \| = 1/\sqrt{a_{n-r}}
\]
and
\[
(3.4.2) \quad \| (1 + n)^{r-a_n} e_{r,n} \| = \sqrt{1 + a_{n-r}^2}
\]
Hence, (3.3.2) contributes to our sum at most
\[
(3.4.3) \quad \sum_{0 \leq r < n} \sqrt{1 + a_{n-r}^2} 2^{(1-a_n/2)/\sqrt{e_{n}}} < \frac{\eta}{5},
\]
\( pd \). In view of (2.5.2), if \( 0 < r < n \) then
\[
(3.4.4) \quad \| e_1 + (n-1)^{-a_n} - r_n \| = (1 + n)^{-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
If \( r = n > 0 \), (2.5.4) gives
\[
(3.4.5) \quad \| e_1 + (n-1)^{-a_n} - r_n \| = (1 + n)^{-n a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
If \( r = n < n \), (2.5.2) gives
\[
(3.4.6) \quad \| e_1 + (n-1)^{-a_n} - r_n \| = (2 + n)^{-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n+1}}}
\]
Hence the contribution made by (3.3.3) to our sum is at most
\[
(3.4.7) \quad \sum_{n=1}^{\infty} \sum_{r=1}^{n} (n - r + 1)^{r-a_n} (n + 1)^{r-1} \frac{\| e_1 + r_n + a_n \|}{n (n-1)^{a_n} - r_n + 1} \frac{\| e_1 + (r-1)^{a_n} + a_n \|}{n (n-1)^{a_n} - r_n + 1} \frac{\| e_1 + (r-1)^{a_n} + a_n \|}{n (n-1)^{a_n} - r_n + 1}
\]
\[
= \sum_{n=1}^{\infty} \sum_{r=1}^{n} (1 + n)^{-a_n - 1} (1 + n)^{r-a_n} 2^{(1-a_n/2)/\sqrt{e_{n}}} (n - r + 1)^{r-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
\[
= \sum_{n=1}^{\infty} 2^{(1-a_n/2)/\sqrt{e_{n}}} \frac{1}{n+1} \sum_{n=1}^{\infty} \sum_{n=1}^{n} (1 + n)^{-a_n - 2} (1 + n)^{r-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
\[
+ \sum_{n=1}^{\infty} 2^{(1-a_n/2)/\sqrt{e_{n}}} (n - r + 1)^{-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
\[
< \frac{\eta}{5},
\]
\( pd \) (the first two terms on the right hand side of (3.4.7) are summing appropriate multiples of the norms of the vectors \( e_1 + r_n + a_n \) on the left hand side; the last two terms do the same for the vectors \( e_1 + (r-1)^{a_n} + a_n \)).

Then again, (2.5.4) gives us
\[
(3.4.8) \quad \| e_1 + r_n + a_n \| = (1 + n)^{-a_n - 1} (1 + n)^{r-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
when \( 0 \leq r < n \); if \( r = n \) we are looking at \( \| e_1 + e_n \| \), which is given by (3.4.6). Hence the contribution to our sum made by (3.3.4) is at most
\[
(3.4.9) \quad \sum_{n=1}^{\infty} \sum_{r=1}^{n} (1 + n)^{-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
\[
+ \sum_{n=1}^{\infty} (1 + n)^{r-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
\[
+ \sum_{n=1}^{\infty} (1 + n)^{r-a_n - 1} 2^{(1-a_n/2)/\sqrt{e_{n}}}
\]
\[
< \frac{\eta}{5},
\]
\( pd \) (here the first two terms sum the norms of the vectors \( e_1 + r_n + a_n \) appearing in (3.3.4), with appropriate weights; and the last term does the same for the vectors \( e_1 + r_n + (r-1)^{a_n} \)).

Lastly, (2.5.3) gives us (for each \( 0 < r \leq s \leq n \))
\[
\| (1 + n)^{s_n} + r_n e_{s_n + r_s} \| = (1 + b_n)(1 + n)^{s_n + (r-1)^{a_n}} e_{(r-1)^{a_n} + s_n},
\]
hence for \(0 < r \leq n\),
\[
\| (1 + n)^{r(a_n + b_n)} e_{r(a_n + b_n)} \| \leq 1 + b_n + b_n^2 + s + b_n^r \| e_n \| \\
\leq 1 + b_n + b_n^2 + s + b_n^r \cdot 2 \quad \text{(by (3.4.2))} \\
\leq 3b_n^r.
\]

Therefore the contribution to our sum from (3.3.5) is, pd, at most
\[
3b_n^r \sum_{0 \leq r < n} 2^{(1-b_n/2+(r+1)a_n)/\sqrt{b_n}} < \frac{\eta}{5}.
\]

Adding up our estimates ((3.4.7), (3.4.3), (3.4.9), (3.4.11)) and our remark about \(T_{f_{00}}\) we find that pd,
\[
\sum_{i,j} \| (T - W) f_{ij} \| < \frac{\eta}{5} + \frac{\eta}{5} + \frac{\eta}{5} + \frac{\eta}{5} + \frac{\eta}{5} = \eta,
\]
which gives the result. \(\blacksquare\)

**Corollary 3.5.** Pd, \(\|T\| < 1\).

The proof is obvious.

**Corollary 3.6.** Pd, \(T\) is strictly singular.

**Proof.** Strict singularity is not affected if an operator is perturbed by an operator in the norm closure of the finite rank operators. Since \(T - W\) is nuclear pd, it is enough to show that \(W\) is strictly singular. Now with slight abuse of notation, we have
\[
W = W_0 + W_1
\]
where \(W_0\) is a compact operator on \(L_2\). So it is enough to show that \(W_1\) is strictly singular. Now \(W_1\) is the map \((\bigoplus_{i=1}^{\infty} J_{p_i})_{l_2} \to (\bigoplus_{i=1}^{\infty} J_{p_i})_{l_2}\) which sends the sequence \((x_1, x_2, x_3, \ldots)\) to \((0, \beta_1 x_1, \beta_2 x_2, \ldots)\). Furthermore, \(\beta_j \to 0\) as \(j \to \infty\) (see (3.3.0)). All we need then for our corollary is the easy lemma:

**Lemma 3.7.** If \(W_1 : (\bigoplus_{i=1}^{\infty} J_{p_i})_{l_2} \to (\bigoplus_{i=1}^{\infty} J_{p_i})_{l_2}\) is the map sending
\[
(x_1, x_2, x_3, \ldots) \to (0, \beta_1 x_1, \beta_2 x_2, \ldots)
\]
\((x_i \in J_{p_i})\), then \(W_1\) is strictly singular provided \(\beta_i \to 0\) as \(i \to \infty\).

**Proof.** If not, write \(X_1 = (\bigoplus_{i=1}^{\infty} J_{p_i})\) and let \(E \subset X_1\) be an infinite-dimensional subspace, and \(\varepsilon > 0\), such that for all \(x \in E\),
\[
\| W_1 x \| \geq \varepsilon \| x \|.
\]
Let \(P_n\) denote the natural projection onto \(\bigoplus_{i=1}^{n} J_{p_i}\) sending \((x_1, x_2, \ldots)\) to \((x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)\). Let \(P_0 = 0\). Now \(\| W_1 P_n - W_1 \| \to 0\) as \(N \to \infty\) because \(\beta_i \to 0\). Therefore there is an \(N\) such that for all \(x \in E\),
\[
\| W_1 P_N x \| \geq \varepsilon \| x \|/2.
\]

So, \(W_1\) is norm increasing on an infinite-dimensional subspace of \(P_N X_1\) (namely, \(P_N E\)). Let \(S \subset E^+ \times E^+ = \{(M, N) : M < N\}\), and there is an infinite-dimensional subspace of \((P_N - P_M) X_1\) on which \(W_1\) is norm increasing. We have shown \((O, N) \in S\) for large \(N\). Let \((M, N) \in S\) be such that \(N - M\) is minimal, and let \(E\) be a subspace of \((P_N - P_M) X_1\) on which \(W_1\) is norm increasing, spanned by vectors \(x^{(i)} = (0, \ldots, 0, x^{(i)}_{M+1}, \ldots, x^{(i)}_N, 0, \ldots, 0) (i = 1, \ldots, \infty)\). If \(N - M = 1\) we find the inclusion map \(J_{p_M} \to J_{p_{M+1}}\) is norm increasing on \(E\), contradicting Lemma 1.1. If not, then taking a subspace of \(E\) and perturbing slightly as necessary, we can assume that for each \(j\) the \(x^{(i)}_j\) are a block basis in \(p_j\) (here we allow a "block basis" to perhaps include some zero vectors). Now the subspace of \(J_{p_{M+1}}\) spanned by the \(x^{(i)}_{M+1}\) must be infinite-dimensional, or we can remove the \(x^{(i)}_{M+1}\) (and reduce \(N - M\)) by passing to a subspace. So taking a subsequence as necessary we may assume the \(x^{(i)}_{M+1}\) to be independent, and likewise we may assume the \(x^{(i)}_N\) are independent. Consider the two norms
\[
\| \lambda \|_{M+1} = \left\| \sum_{i=1}^{\infty} \lambda_i x^{(i)}_{M+1} \right\|_{J_{p_{M+1}}} \quad \text{and} \quad \| \lambda \|_N = \left\| \sum_{i=1}^{\infty} \lambda_i x^{(i)}_N \right\|_{J_{p_N}}
\]
on the finite nonzero sequences \(\lambda \in c_0\). If on any infinite-dimensional subspace of \(c_0\) they are equivalent, we have a subspace of \(J_{p_{M+1}}\) isomorphic to one of \(J_{p_N}\); this leads to \(J_{p_{M+1}} \subset J_{p_N}\), a contradiction as in Lemma 1.1. If not, there is a normalized block basis \(y^{(i)}\) of the \(x^{(i)}\) such that writing \(y^{(i)} = (0, \ldots, 0, y^{(i)}_{M+1}, \ldots, y^{(i)}_N, 0, \ldots, 0)\) we either have \(\|y^{(i)}_{M+1}\| \to 0\) or \(\|y^{(i)}_N\| \to 0\). Perturbing a subspace of this block basis very slightly, we can obtain an increasing sequence \((\eta_i)\), and vectors \(z^{(i)}\) close to \(y^{(i)}\), spanning a norm increasing subspace of \(X_1\), for which \(z^{(i)}_{M+1} = 0\) (or \(z^{(i)}_N = 0\)). Hence, either \((M, N - 1) \in S\) or \((M + 1, N) \in S\); so \(N - M\) was not minimal. This contradiction shows that \(W_1\) is strictly singular.

4. Estimates concerning \(\|T^{a_n + b_n}\|\). This of course is essentially a repeat of arguments given in [R1]-[R8].

**Lemma 4.1** (cf. [R7], §3.4). Recall that \(F_n = \text{lin}(e_0, \ldots, e_n)\). On \(F_n\), one may consider the two norms
\[
\left\| \sum_{i=0}^{n} \lambda_i e_i \right\| = \sum_{i=0}^{n} |\lambda_i| \quad \text{and} \quad \left\| \sum_{i=0}^{n} \lambda_i e_i \right\|_X = \sum_{i=0}^{n} |\lambda_i| X.
\]
For suitable functions $N_1 : \mathbb{N}^2 \to \mathbb{N}$ and $N_2 : \mathbb{N}^2 \to \mathbb{N}$, we have
\[
(4.1.0) \quad \frac{1}{N_1(n_1, a_n)} |x| \leq \|x\| \leq N_1(n_1, a_n) |x|
\]
for all $x \in F_{n_n}$, and
\[
(4.1.1) \quad \frac{1}{N_2(n_1, b_n)} |x| \leq \|x\| \leq N_2(n_1, b_n) |x|
\]
for all $x \in F_{n_n}$, provided $d$ increases sufficiently rapidly that Definition 2.5 is meaningful.

Proof. For $F_{n_n}$ is $F_{n_n} = \text{lin}\{f_{ij} : (i, j) \in S_{n_n}\}$ where $S_n$ is the set mentioned in (2.0.0). Furthermore, the matrix of the map on $F_{n_n}$ sending $e_i$ to $f_{jk}$ (where $i = I(j, k)$, as in 2.6) is determined by the values $a_1, b_1, \ldots, a_n$ as used in Definition 2.5. If we take the relevant $f_{jk}$ as our basis for $F_{n_n}$ then of course the norm is between the $c_0$ norm and the $l_1$ norm. If we make the change of basis to $e_i$ ($i = 1, \ldots, n_n$) then for a suitable function $M_i(n_1, a_1, b_1, \ldots, a_n)$ the inequality (4.1.0) must hold. Since $d$ is an increasing sequence, we can write $M_i(n_1, a_1, b_1, \ldots, a_n) \leq N_i(n_1, a_n)$ for a suitable function $N_i$. Similarly, a suitable function $N_2$ exists such that (4.1.1) holds.

Definition 4.2. Recall that we write $f_i$ for the vector $f_{j, k, i}$ where $i = I(j, k)$. Let $Q_{n_n}^0 : F \to F_{n_n}$ be the projection such that
\[
Q_{n_n}^0(f_j) = \begin{cases} 
0 & 0 \leq j \leq m_{a_1}, \\
-(m-r+1)^{r-1}a_{n_n}+(r-1)a_{m_{a_1}+1} & j \in [1, m_{a_1}+1], \\
0 & \text{otherwise}.
\end{cases}
\]

We shall establish the following lemma (cf. [R7], Lemma 4.3).

Lemma 4.3. Pd, for all $n$ we have
\[
\|T^{a_1+b_n} (I - Q_n^0)\| \leq 2(n+1)^{-a_1+b_n}.
\]

Later on, we will establish that, for an arbitrary norm-1 vector $x \in l_1$, and for any $\varepsilon > 0$, there is a polynomial $q$ and an integer $n$ such that
\[
\|q(T)^{a_1+b_n}Q_n^0 x - e_0\| < \varepsilon / 2 \quad \text{and} \quad \|q(T)^{a_1+b_n} (I - Q_n^0) x\| < \varepsilon / 2.
\]
This will show that $e_0 \in \text{lim}\{T^n x : n \to 0\}$ and hence that $x$ is cyclic, since $e_0$ obviously is. So $T$ is strictly singular, and has no invariant subspaces.

Proof of Lemma 4.3. As with proving $T$ continuous, we split the operator involved (in this case, $T^{a_1+b_n} (I - Q_n^0)$) into a part that looks like a weighted shift operator, and a nuclear operator. In certain cases, we now find that $T^{a_1+b_n} (I - Q_n^0) f_i$ is of form $\varepsilon f_i + a_1 + b_n$. These cases are as follows:

Case 4.3.1. If $i \in [0, n_{a_1}]$ then $Q_n^0 f_i = f_i$, so of course
\[
(4.3.0) \quad T^{a_1+b_n} (I - Q_n^0) f_i = 0.
\]

Case 4.3.2. If $i \in [r(a_n + b_n), (n-1)a_n + b_n]$ with $0 < r < n$, then by two applications of (2.5.3) we find that
\[
(4.3.1) \quad T^{a_1+b_n} (I - Q_n^0) f_i = T^{a_1+b_n} f_i = (1+n)^{-a_1+b_n} f_i + a_1 + b_n.
\]

Case 4.3.3. If $i \in (n(a_n + b_n), (r+1)(a_n + b_n))$ with $0 \leq r < n - 1$ then two applications of (2.5.4) likewise give us
\[
(4.3.2) \quad T^{a_1+b_n} (I - Q_n^0) f_i = T^{a_1+b_n} f_i = (1+n)^{-a_1+b_n} 2a_n/\sqrt{b_n} f_i + a_1 + b_n.
\]

Case 4.3.4. If $i \in [0, m_{a_1} - a_n - b_n] \cup r_{m_{a_1}}$ with $0 < r < m - n$ then
\[
(4.3.3) \quad T^{a_1+b_n} (I - Q_n^0) f_i = T^{a_1+b_n} f_i = (m-r)^{-a_1+b_n} f_i + a_1 + b_n.
\]

Case 4.3.5. If $i \in (mA + rM_{a_1} + (r+1)(a_n + b_n) - a_n - b_n)$ with $0 < r < m$, $m > n$, or if $i \in (m_{a_1}, m_{a_1} - a_n - b_n)$, $m > n$, then by (2.5.1),
\[
(4.3.4) \quad T^{a_1+b_n} (I - Q_n^0) f_i = T^{a_1+b_n} f_i = (m-r)^{-a_1+b_n} 2a_n/\sqrt{b_n} f_i + a_1 + b_n.
\]

Case 4.3.6. If $i \in [r(a_n + b_n), mA + rM_{a_1} + (r+1)(a_n + b_n) - a_n - b_n]$ with $0 < r < m$, $m > n$, then (2.5.3) gives
\[
(4.3.5) \quad T^{a_1+b_n} (I - Q_n^0) f_i = T^{a_1+b_n} f_i = (m-r)^{-a_1+b_n} f_i + a_1 + b_n.
\]

Case 4.3.7. Finally if $i \in (m_{a_1} + rM_{a_1} + (r+1)(a_n + b_n) - a_n - b_n)$ with $0 < r < m$, $m > n$, then (2.5.4) gives
\[
(4.3.6) \quad T^{a_1+b_n} (I - Q_n^0) f_i = (1 + m)^{-a_1+b_n} 2a_n/\sqrt{b_n} f_i + a_1 + b_n.
\]

Lemma 4.4. Pd, the following is true: The operator $W = \gamma(n) : F \to F$ such that $W f_i$ is as in (4.3.0)-(4.3.6) if the integer $i$ is mentioned in these cases, and $W f_i = 0$ otherwise, has norm
\[
(1+n)^{-a_1+b_n} 2a_n/\sqrt{b_n} < \frac{1}{2} (1+n)^{-a_1+b_n}.
\]

Proof. As earlier, we split up $W$ into an operator $W_0$ acting on $l_2$, and $W_i$ acting on $\bigoplus_{i=1}^n F_i$. The operator $W_i$ covers all cases except Case 4.3.4, and it acts as
\[
W_0(f_i) = \varepsilon_i f_i \gamma(a_n+b_n+1(i))
\]
with $\varepsilon_i \leq (1+n)^{-a_1+b_n} 2a_n/\sqrt{b_n}$, pd, and equality achieved for certain values $i$ such that $\gamma(i)$ is covered by Case 4.3.3. The operator $W_i$ deals with Case 4.3.4. It sends for $i \in [0, m_{a_1} - a_n - b_n] \cup r_{m_{a_1}}$, $0 < r < m - n$ to $(m-r+1)^{-a_1+b_n} f_i + a_1 + b_n$, that is, writing $j = i - rM_{a_1} \in [0, m_{a_1} - a_n - b_n]$,
\[
f_{j, m_{a_1} - a_n - b_n} \to (m-r+1)^{-a_1+b_n} f_{j, m_{a_1} - a_n - b_n}, r-1.
\]
Writing \( m - r = k \geq n \) we find that \( W_1 \) sends \( f_{j + w_{n + s}} \) to
\[ (k + 1)^{-a_n - b_n} f_{j + w_{n + a_n + b_n}} \]
for all \( s \geq 0 \) and \( j \in [0, v_k - a_n - b_n] \).

Otherwise, \( W_1 f_{t, s} = 0 \). So \( W_1 \) acts on each \( J_{p_t} \) space as a multiple \( \beta_1 x^{(n)} \), where \( \beta_1 x^{(n)} \) is the inclusion \( J_{p_t} \hookrightarrow J_{p_t + a_n + b_n} \), and \( \beta_1 \) does not exceed \((n + 1)^{-a_n - b_n} \). So, \( ||W_1|| = (n + 1)^{-a_n - b_n} \). Hence,
\[ ||W|| = \max(||W_0||, ||W_1||) = (n + 1)^{-a_n - b_n}2^{an} \sqrt{\frac{n}{3}}. \]

**Lemma 4.5.** For each \( \eta > 0 \), the following is true: pdf, for every \( n > 0 \) the operator \( T^{a_n + b_n} \circ (I - Q_n^0) - W^{(k)} \) is nuclear, of nuclear norm at most \( \eta (n + 1)^{-a_n - b_n} \).

**Proof.** We must consider the error terms \( T^{a_n + b_n} \circ (I - Q_n^0) f_t - W^{(k)} f_t \), sum all their norms, and obtain at most \( \eta (n + 1)^{-a_n - b_n} \). This is not in fact difficult to do, there are roughly six cases, corresponding to values \( i \) which were "missed out" of Cases 4.3.2-4.3.7 above.

**Case 4.5.1.** If \( i \in [(n - 1)a_n + rb_n, na_n + rb_n] \) with \( 0 < r < n \) (these are some of the "missing values" from Case 4.3.2 above), then (2.5.3) gives us
\[ f_t = (1 + n)^te_t - b_n(1 + n)^{-b_n} e_{1 - b_n} \]
and hence, writing \( j = i + a_n + b_n - na_n - (r + 1)b_n - 1 \geq 0 \), we have
\[
(4.5.0) \quad T^{a_n + b_n}(I - Q_n^0) f_t
= T^{a_n + b_n} f_t = T^{j}(1 + n)^{te_t - a_n - (r + 1)b_n - b_n - 1}\]
Now (3.4.8) and (3.4.6) together give us, for all \( 0 \leq r \leq n \),
\[ ||e_{1 + a_n + b_n}|| \leq (n + 1)^{-1 - n - a_n - b_n} 2^{(a_n - b_n)/3} \leq 2^{-\sqrt{b_n}/3}. \]

**Case 4.5.2.** If \( i = v_n \) (the final value "missed out" of Case 4.3.2 above), then (2.5.3) and (3.4.6), together with the ever-useful fact that \( ||T|| < 1 \), give us
\[
(4.5.3) \quad ||T^{a_n + b_n}(I - Q_n^0) f_t|| = ||T^{a_n + b_n} f_t||
\leq (1 + n)^{r_a + r_m}||T^{a_n + b_n} e_{v_n}||
+ b_n(1 + n)^{ra_n + (n - 1)b_n} ||T^{a_n + b_n} e_{na_n + (n - 1)b_n}||
\leq ((1 + n)^{r_a + r_m} + b_n(1 + n)^{ra_n + (n - 1)b_n}) e_{v_n}
\leq ((1 + n)^{r_a + r_m} + b_n(1 + n)^{ra_n + (n - 1)b_n}) (2 + n)^{-(1 + v_n)}
\times 2^{(1 + v_n - a_n + 1/2)/\sqrt{an}}
\leq 2^{-\sqrt{b_n}/3}.
\]
for all \( n, pd \).

**Case 4.5.3.** If \( i \in (na_n + (r + 1)b_n, na_n + b_n) \) (not covered by Case 4.3.3 above), then by (2.5.4), \( f_t \) is a multiple of \( \lambda_i e_t \) with \( \lambda_i \) (crudely) at most \((1 + n)^{r_a + r_m} \). Therefore, \( T^{a_n + b_n} f_t \) is \( T^k(\lambda_i e_{1 + v_n}) \) for some \( k \geq 0 \). Since \( ||T|| < 1 \), it follows that, pdf,
\[
(4.5.5) \quad ||T^{a_n + b_n} f_t|| \leq ||\lambda_i|| ||e_{1 + v_n}||
\leq (1 + n)^{r_a + r_m} 2^{(1 + v_n - a_n + 1/2)/\sqrt{an}} \quad \text{(by (3.4.6))}
\]
\[
(4.5.6) \quad \leq 2^{-\sqrt{b_n}/3}.
\]

**Case 4.5.4.** If \( i \in (v_{m - r} - a_n + b_n, v_{m - r}) + ram \) with \( 0 < r \leq m - n \) (not covered by Case 4.3.4 above), or if \( i \in [0, v_{m - r}) + ram \) with \( 0 < r \leq m - n \), \( m - r \leq n \), then we see by (2.5.1) that
\[
(4.5.7) \quad f_t = ((1 + m)^{r_a + r_m} e_m - m^{(r_m - 1)a_m - 1} e_{1 + v_m - (1 + r_m - 1)a_m - 1})
\times (m - r + 1)^{r_m a_m - r m}
\]
and in either case, \( i + a_n + b_n > ram + v_m - r \). (In the second case, \( i + a_n + b_n > ram + a_n + b_n > ram + v_m - r \) pdf, since \( m - r < n \). Writing
\[
(4.5.8) \quad j = i + a_n + b_n - v_m - r - t,\]
we have
\[
(4.5.9) \quad ||T^{a_n + b_n} f_t|| = ||(m - r + 1)^{r_m a_m - r m}|| ||(1 + m)^{r_a + r_m + v_m - m}
\times (m - r + 1)^{r_m a_m - r m} - m^{(r_m - 1)a_m - 1} e_{1 + v_m - (1 + r_m - 1)a_m - 1}||
\leq (m - r + 1)^{-r_m a_m - r m} ||e_{1 + v_m - r a_m - r m}||
+ m^{(r_m - 1)a_m - 1} ||e_{m - r m}||
\leq 2^{(1 - a_m - 1/2)/\sqrt{m} - \frac{1}{2} a_m}.
\]
Now (3.4.4), (3.4.5) tell us that pdf, for all \( 0 \leq r \leq n \) we have
\[
(4.5.10) \quad ||e_{1 + r m + v_m - r m}|| \leq (1 + n)^{-r_m a_m - r m - v_m - r} 2^{(1 - a_m - 1/2)/\sqrt{m}}.
\]
So, (4.5.9) is at most
\[
(4.5.11) \quad (m - r + 1)^{-r_m a_m - r m} ((1 + m)^{-r_m a_m - r m - v_m - r} 2^{(1 - a_m - 1/2)/\sqrt{m}})
+ m^{(r_m - 1)a_m - 1} 2^{(1 - a_m - 1/2)/\sqrt{m} - \frac{1}{2} a_m}
\leq 2^{(1 - a_m - 1/2)/\sqrt{m} - \frac{1}{2} a_m} \frac{2}{m} a_m^{-1}.
\]
(the worst case in this estimate is when \( r = 1 \)). Now if we have \( m > n + 1 \), then \((I - Q_n^0) f_i\) is just \( f_i\) (Definition 4.2), and (4.5.13) above is the upper bound we need; thus,

\[
\|T^a n + b n \circ (I - Q_n^0) f_i\| \leq 2^{(1 - a_m - 1/2)/\sqrt{a_m - 1}} \frac{2}{m} \frac{1}{a_n}. \tag{4.5.14}
\]

If, on the other hand, \( m = n + 1 \), then \((I - Q_n^0) f_i\) is in fact not \( f_i\) but just \((1 + m)^{a_m}(m - r + 1)^{a_{m-r}} e_i\) (see Definition 4.2 and (2.5.3)). Our argument then gives, pd, the better estimate

\[
\|T^a n + b n \circ (I - Q_n^0) f_i\| \leq (1 + m)^{a_m} \times (m - r + 1)^{a_{m-r}} \|e_{1 + ra_m + 1 - r}\|
\]

\[
\leq 2^{(1 - a_m + 1/2)/\sqrt{a_m - 1}} \frac{1}{n + 2} \frac{1}{a_n}. \tag{4.5.15}
\]

**Case 4.5.5.** If \( i \in [(r + 1)a_m - a_n - b_n, (r + 1)a_m] \) with \( 0 \leq 0 < r < m \), \( m > n \) (these are the “missing values” from Case 4.3.5 above), then (2.5.3) gives us

\[
T^a n + b n \circ (I - Q_n^0) f_i = T^a n + b n \circ (1 + m)^{i(r + 1/2)a_{m} - i(a_m - 1)/\sqrt{a_m}} e_{i + a_n + b_n}
\]

\[
= (1 + m)^{i(r + 1/2)a_{m} - i(a_m - 1)/\sqrt{a_m}} T^j e_{i + (r + 1)a_m}
\]

for some \( j \geq 0 \). As we have remarked in (3.4.2), \( \|(1 + m)^{i r a_{m} e_{r a_{m}}} \| = \sqrt{1 + a^{-1}_{m - r}} \), so since \( \|T\| \leq 1 \) this is at most

\[
2^{(1 + m)^{i(r + 1/2)a_{m} - i(a_m - 1)/\sqrt{a_m}}}
\]

\[
\leq 2^{(1 + m)^{(r + 1/2)a_{m} - i(a_m - 1)/\sqrt{a_m}}} \quad \text{(since \( i \geq (r + 1)a_{m} - a_n - b_n \))}
\]

\[
\leq 2^{-\sqrt{a_m - 1}/3}
\]

for all \( 0 \leq 0 < r < n \), pd.

**Case 4.5.6.** If \( i \in (ma_m + rb_m - a_n - b_n, ma_m + rb_m] \) with \( 0 < r \leq m > n \), then we have one of the “missing values” from Case 4.3.6 above. Formula (2.5.3) gives us

\[
f_i = (1 + m)^{i} e_i - b_m (1 + i)^{b_{m}} e_{i - b_m}
\]

and hence, writing \( j = i + a_n + b_n - ma_m - rb_m - 1 \geq 0 \), we have

\[
T^a n + b n \circ f_i = T^j \{(1 + m)^{i} e_i + (a_{m} + rb_{m}) e_{i - b_{m}} - b_{m} (1 + i)^{b_{m}} e_{i + a_{m} + rb_{m} + (r - 1)b_{m}}\}
\]

Now (3.4.8) and (3.4.6) together give us, for all \( 0 \leq 0 \leq n \),

\[
\|e_{1 + n a_m + r b_m}\| \leq (1 + n)^{a_m - r b_m} 2^{(1 - a_m - b_m - b_m/2)/\sqrt{a_m - 1}}
\]

\[
= 2^{(1 - a_m - b_m - b_m/2)/\sqrt{a_m - 1}} \frac{2}{m} \frac{1}{a_n}. \tag{4.5.16}
\]

As in (3.4.10),

\[
\|e_{(r + 1)(a_{m} + b_{m})}\| \leq 3 b_m^{r^{1+1}(1 + m) - (r + 1)(a_{m} + b_{m})}
\]

so

\[
2^{(1 + m)^{i(r + 1/2)a_{m} - i(a_m - 1)/\sqrt{a_m}}}
\]

\[
\leq 2^{(1 + m)^{(r + 1/2)a_{m} - i(a_m - 1)/\sqrt{a_m}}} \quad \text{(since \( i \geq (r + 1)a_{m} - a_n - b_n \))}
\]

\[
\leq 2^{-\sqrt{a_m - 1}/3}
\]

in all cases, pd. We now add up all our estimates (4.5.2), (4.5.4), (4.5.6), (4.5.14), (4.5.16), (4.5.17), (4.5.20), and (4.5.23), counting according to the multiplicity of values \( i \) that are involved. We obtain this estimate of the nuclear norm:

\[
\sum_{i} \|T^a n + b n \circ (I - Q_n^0) - W(n)\| f_i\|
\]

\[
\leq (n - 1)a_n 2^{-\sqrt{a_m - 1}/3} + 2^{(1 - a_n - b_n)}
\]

\[
+ b_n 2^{-\sqrt{a_n - 1}/3} + (a_n + b_n) \sum_{m = m+1}^{\infty} \sum_{r = r+1}^{m} (a_{m} - 1/2)/\sqrt{a_m - 1}
\]

\[
+ (a_n + b_n) 2^{(1 - a_n - b_n)/\sqrt{a_m - 1}}
\]

\[
+ \sum_{0 \leq r < m} (a_n + b_n) 2^{(1 - a_n - b_n)/\sqrt{a_m - 1}}
\]

\[
+ \sum_{0 \leq r < m} (a_n + b_n) 2^{-\sqrt{a_m - 1}/3}.
\]
The reader will observe that as a function of \( n \) and \( d \), this sum is, pd, at most \( \eta(n+1)^{-a_n-b_n} \). Thus 4.5 is proved, and 4.3 follows immediately. ■

5. Further estimates concerning \( T \)

**Definition 5.1.** Let \( Q_m \) \(( m \geq 1)\) be the projection \( F \to F_{m,m} \) such that

\[
Q_m(f_j) = \begin{cases} f_j, & 0 \leq j \leq ma_m, \\ 0, & \text{otherwise}. \end{cases}
\]

Note that in terms of what happens to the \( e_j \), this amounts to much the same as (6.1) of [R8], though it does not look the same.

**Lemma 5.2.** \( \|Q_m\| = 1 \) for all \( m \).

**Proof.** We claim that for each \( i \), the vectors \( f_{ik} \) \((k = 0, 1, \ldots)\) appear as a subsequence \((f_{ik})_{i=0}^{\infty}\) of the \( f_j \) in their proper order \((j_0 < j_1 < j_2 < \ldots)\). This is true because \( \gamma \) is an increasing function (Definition 2.1) and \( f_{ij} = f_{i(j+1)} \). and because for \( i > 0 \), say \( i \in [w_m, v_m + v_m) \), we have \( f_{ik} = f_{i(k+1)+i+1-v_m} \) (see 2.3). Hence, for each \( i \) there is a \( k \) such that

\[
Q_m(f_{ij}) = \begin{cases} f_{ik}, & j \leq k, \\ 0, & \text{otherwise}. \end{cases}
\]

The norm of the projection that thus "truncates" a sequence is 1 on \( L_2 \) (of course), and also on any \( L_p \). Hence, \( \|Q_m\| = 1 \). ■

**Definition 5.3.** Let \( P_{n,m} \) \(( m > n \geq 1)\) be the operator \( \tau_{nm} \circ Q_m : F_{m,m} \to F_{m,m} \), where

\[
\tau_{nm}(e_j) = \begin{cases} e_j, & 0 \leq j < (m-n)a_m, \\ 0, & \text{otherwise}. \end{cases}
\]

**Lemma 5.4.** \( \|Q_m^0\| \leq a_m \) for all \( m \), pd.

**Proof.** We know \( \|Q_m\| = 1 \); and 4.2 tells us that \( Q_m - Q_m^0 \) is zero unless \( j \in [0, v_m+1-r] + ra_{m+1} \). Since \( r = 1 \), it is

\[
e_j - ra_{m+1} + (r-1)a_m(m-r+1) - (r-1)a_m a_{m+1} - r.
\]

Hence, crudely,

\[
\|Q_m - Q_m^0\| \leq \sum_j \| (Q_m - Q_m^0) f_j \| \\
\leq \sum_{r=2}^{m+1} (1 + v_m + 1-r)(m+1)(r-1)a_m a_{m+1} - r \\
\times \max \{ \| e_j - ra_{m+1} + (r-1)a_m(m-r+1) - (r-1)a_m a_{m+1} - r \| : j \in [0, v_m+1-r] + ra_{m+1} \}
\]

for all \( n \), pd. ■

\[
\leq \sum_{r=2}^{m+1} (1 + v_m + 1-r)(m+1)(r-1)a_m a_{m+1} - r \\
\leq \sum_{r=2}^{m+1} (1 + v_m + 1-r) \cdot 2 \quad (by \ 3.4.2) \\
\leq 2m(1 + v_m - 1) < a_m
\]

for all \( m, \) pd. ■

**Lemma 5.5.** \( \|P_{nm}\| \leq a_{n+1} \) for all \( n < m \), pd.

**Proof.** \( \|Q_m\| = 1 \) so \( \|P_{nm}\| = \|\tau_{nm}\| \) and (5.3) and (2.5) we find

\[
\tau_{nm} f_i = \begin{cases} f_i, & 0 \leq i < (m-n)a_m, \\ -e_i - ra_{m+1-r} - ra_{m+1} - (m-n)a_m - r, & \text{otherwise}. \end{cases}
\]

Now the projection \( \tau' \) such that

\[
\tau' f_i = \begin{cases} f_i, & 0 \leq i < (m-n)a_m, \\ 0, & \text{otherwise}, \end{cases}
\]

has norm 1, for the same reasons as in 5.2. Therefore

\[
\|\tau_{nm}\| \leq 1 + \sum_{i=0}^{m-n} \| e_i - ra_m + (r-1)a_m - i \| \\
\times (m-r+1)^{r-1} a_m a_{m+1} - r \\
\leq 1 + \sum_{r=m-n}^{m} (1 + v_m - r) \| e_r - ra_m - i \| \\
\times (m-r+1)^{r-1} a_m a_{m+1} - r
\]

since \( \|T\| < 1 \). Recall from (3.4.2) that \( \| (1 + \eta)^{ra_m} e_r \| = \sqrt{1 + a_{n-r}^2} \), Substituting into (5.5.2) we have

\[
\|\tau_{nm}\| \leq 1 + \sum_{r=m-n}^{m} (1 + v_m - r) \\
\times \sqrt{1 + a_{n-r}^2} (m-r+1)^{r-1} a_m a_{m+1} - r \\
= 1 + \sum_{r=m-n}^{m} (1 + v_m - r) \sqrt{1 + a_n^{-2}} (m-r+1)^{r-1} a_m a_{m+1} - r
\]

for all \( n, \) pd. ■
6. $T$ has no invariant subspaces

**Definition 6.1.** For each $1 < n \leq m$, let $K_{n,m} \subset F_{m,m}$ be the set of vectors such that $|x| \leq a_m$ and $\|\tau_{n,m}x\| \geq 1/a_m$. Let $T_m : F_{m,m} \to F_{m,m}$ be the "truncated" version of $T$, i.e. $T_m(e_i) = e_{i+1}$ ($i < m a_m$) or zero ($i = m a_m$).

**Lemma 6.2.** There is a function $N_3 : N^2 \to N$ with the following property: pd, for all $1 < n < m$ and $x \in K_{n,m}$, there is a polynomial $p$ such that $|p| < N_3(m, a_m)$, $p(t)$ is of form $\sum_{i=a_m}^{m a_m} \lambda_i t^i$, and

\[
(6.2.0) \quad \|p(T_m)x - e_0\| < 1/a_m + 1/a_{n-1}.
\]

**Proof.** For any $y \in K_{n,m}$ we can write $y = \sum_{i=a_m}^{m a_m} \lambda_i e_i$ where $\lambda_i \neq 0$. Then

\[
(6.2.1) \quad \lim\{T_m^r y : a_m \leq r \leq m a_m\} = \lim\{e_{a+a_m}, e_{a+a_m+1}, \ldots, e_{m a_m}\}.
\]

Since $\tau_{n,m}y \neq 0$ we know $\alpha < (m - n) a_m$ so certainly $e_{(m-n+1)a_m} \in \lim\{T_m^r e_i : e_m \leq r \leq m a_m\}$. Since $K_{n,m}$ is compact, there are a finite number of polynomials $p_1, \ldots, p_k$ of form $p_j(t) = \sum_{i=a_m}^{m a_m} \lambda_{ij} t^i$ such that for all $x \in K_{n,m}$ there is a $j$ such that

\[
(6.2.2) \quad \|p_j(T_m)x - (m+1)^{m+a_m} e_{(m-n+1)a_m} - e_0\| < 1/a_m.
\]

Writing $N = \max_j |p_j|$, note that $N$ depends only on elements of the underlying sequence $\mathbf{d}$ up to and including $a_m$; so $N < N_3(m, a_m)$ for a suitable function $N_3 : N^2 \to N$. Since in view of (3.4.1) we have

\[
(6.2.3) \quad \|(m+1)^{m+a_m} e_{(m-n+1)a_m} - e_0\| = 1/a_{n-1};
\]

this concludes the proof. $lacksquare$

We now extend the previous lemma as follows.

**Lemma 6.3.** With the notation of (6.2), the polynomial

\[
g(t) = t^{b_m} (m+1)^{b_m} / b_m \cdot p(t)
\]

satisfies $t^{a_m+b_m} \mid g(t)$, $\deg g \leq b_m + a_m$, $|g| \leq N_3(m, a_m) (m+1)^{b_m} / b_m$ and

\[
(6.3.0) \quad \|g(T_m)x - e_0\| \leq 2/a_{n-1} + 3/a_m.
\]

**Proof.** Given $x \in K_{n,m}$ let $p$ be the polynomial as in 6.2, and write $g(t) = t^{b_m} (m+1)^{b_m} / b_m \cdot p(t)$. Let us consider the vector $q(T_m)x$. For all $i \in [a_m + b_m, m a_m + b_m]$ we have $f_i = (m+1)^i e_i - b_m (m+1)^{i-b_m} e_{i-b_m}$; so if we write $p(T_m)x = \sum_{i=a_m}^{m a_m} \lambda_i e_i$ then pd,

\[
(6.3.1) \quad \|\frac{(m+1)^b_m}{b_m} T^{b_m} p(T_m)x - p(T_m)x\|
\]

\[
= \left\| \sum_{i=a_m}^{m a_m} \lambda_i \left( \frac{(m+1)^b_m}{b_m} e_{i+b_m} - e_i \right) \right\|
\]

\[
= \left\| \sum_{i=a_m}^{m a_m} \lambda_i \left( \frac{(m+1)^{-i}}{b_m} f_{i+b_m} \right) \right\| \leq \frac{1}{b_m} \sum_{i=a_m}^{m a_m} |\lambda_i| = \frac{1}{b_m} \|p(T_m)x\|
\]

\[
\leq \frac{1}{b_m} |p| \|x\| \leq \frac{1}{b_m} N_3(m, a_m) \|x\| N_3(m, a_m)
\]

\[
\leq \frac{1}{b_m} N_3(m, a_m) N_1(m, a_m) \quad \text{since} \ x \in K_{n,m} \ \text{so} \ \|x\| \leq a_m
\]

\[
\leq 1/a_m.
\]

Furthermore,

\[
(6.3.2) \quad T^{b_m} (p(T) - p(T_m))x \in T^{b_m} \lim\{e_j : a_m < j \leq 2m a_m\}
\]

\[
(6.3.3) \quad = \lim\{e_j : b_m + a_m < j \leq b_m + 2m a_m\}.
\]

Since $\|T\| \leq 1$, pd, we deduce that

\[
(6.3.4) \quad \frac{(m+1)^b_m}{b_m} \|T^{b_m} (p(T) - p(T_m))x\|
\]

\[
\leq |p| \|x\| \|e_{b_m+a_m}+a_m\| \frac{(m+1)^b_m}{b_m}
\]

\[
\leq \frac{(m+1)^b_m}{b_m} N_3(m, a_m) N_1(m, a_m) \|e_{b_m+a_m+a_m}\| \quad \text{(as above)}
\]

\[
\leq \frac{(m+1)^b_m}{b_m} N_3(m, a_m) N_1(m, a_m)
\]

\[
\times a_m (1 + m) -1 - m a_m - b_m (1+m a_m - b_m/\sqrt{a_m}) \quad \text{(by (3.4.9))}
\]

\[
(6.3.7) \quad \leq \frac{1}{a_m}
\]

for all $m$, pd. Adding up (6.3.7) and (6.3.1) we have

\[
(6.3.8) \quad \|g(T_m)x - p(T_m)x\| \leq 2/a_m.
\]

Using 6.2 we have our result. $lacksquare$

We now have the following very convenient lemma (it corresponds to Lemma 5.17 of [R1]):

**Lemma 6.4.** For all $j \in [0, v_k - r]$, $1 \leq r < k - n$ and $s \geq r$, we have

\[
(6.4.0) \quad f_{j+s+w_{k-r}}^r \circ \tau_{n, k+s-r} \circ Q_{k+s-r}^j = -f_{j+s+w_{k-r}-a} + f_{j+s+w_{k-r}-a-1}^r.
\]
Proof. The vector $f_{j+w_{k-r},s} = f_{j+s,w_{k-r}}$ (Definition 2.3) is in the image of the projection $\tau_{n,k+s} \circ Q_{k+s}^0$ and is fixed by it. The vector $f_{j+w_{k-r},s} = f_{j+(s+1)w_{k-r}}$ is mapped (by 4.2, 5.3) to

$$u = -e_{j+s+w_{k-r}}(k + s - r + 1)^{(s-1)\alpha_{k-r}}(k + r + 1)^r,$$

which by (2.6.0), (2.5.1) satisfies $f_{j+w_{k-r},s}(u) = -1$. It is easily seen that for all other vectors $f_{m}$, $\tau_{n,k+s} \circ Q_{k+s}^0(f_{m})$ is either $f_{m}$, or zero, or another vector similar to $u$ above, being therefore of form $f_{j',s'} + h$ with the pair $(j',s')$ not equal to $(j + w_{k-r}, s)$, and with $h \in P_{n,k+s}$. In all such cases $f_{j+w_{k-r},s} = \tau_{n,k+s} \circ Q_{k+s}^0$ and $f_{j+w_{k-r},s} = 0$, hence the result.  \[ \square \]

Theorem 6.5. Pd. $T$ has no invariant subspace.

Proof. Let $x \in X$, $||x|| = 1$ and $n > 0$. Since $e_0$ is cyclic for $T$, it is enough to show that for all such $x$ and $n$ there is a polynomial $q$ such that $||q(T)x - e_0|| \leq 2/a_{n-1}$. We claim there is an $m > n$ such that

$$||\tau_{n,m} \circ Q_{n}^0 x|| \geq 1/a_m.$$ \[ (6.0) \]

Now $||P_{n,k}|| \leq a_{n+1}$ for all $k$, and certainly for all $k \in F$ we have $P_{n,k}x = x$ for all but finitely many $k$. Therefore, $P_{n,k}x \to x$ as $k \to \infty$ for any vector $x \in X$.

Choose, then, a $k$ so large that $||P_{n,k}x|| = ||\tau_{n,k} \circ Q_{n}^0 x|| > 1/2$. If $||\tau_{n,k} \circ Q_{n}^0 x|| > 1/4$ our assertion is proved; if not then

$$||\tau_{n,k} \circ (Q_{n}^0 x)|| > 1/4.$$ \[ (6.1) \]

For all $j > 0$ we either have

$$||Q_{n}^0 x|| \geq e_{j} \circ (r+1)\alpha_{k} - (r-1)\alpha_{k} - (k - r + 1)^{(r-1)\alpha_{k}}.$$ \[ (6.2) \]

If $j \in [0, w_{k-r}] + \tau_{n,k+1}$, $1 < r < n$, or else $(Q_{n}^0 x)_{j} = 0$. Hence, $\tau_{n,k} \circ (Q_{n}^0 x)_{j}$ is either

$$e_{j} \circ (r+1)\alpha_{k} - (r-1)\alpha_{k} - (k - r + 1)^{(r-1)\alpha_{k}}.$$ \[ (6.3) \]

If $j \in [0, w_{k-r}] + \tau_{n,k+1}$, $1 < r < n$, or else zero. Thus, $\tau_{n,k} \circ (Q_{n}^0 x) = \tau_{n,k} \circ (Q_{n}^0 x) \circ \tau_{n,k}$ where $S$ is the finite set

$$S = \bigcup_{1 \leq r < k} [0, w_{k-r}] + (r+1)\alpha_{k+1} = S_{n,k},$$

say. Crudely, then, we may say that there is an $i \in S_{n,k}$ such that

$$||f_{i}(x)|| \geq \frac{1}{\max(\tau_{n,k}(Q_{n}^0 x)) \cdot |S_{n,k}|}.$$ \[ (6.4) \]

Now if $i = j + (r+1)\alpha_{k+1}$, $j \in [0, w_{k-r}]$, $1 \leq r < k - n$, then we know by (2.5.1) that $f_{i} = f_{j+w_{k-r},r}$. Because any $x \in J_{n,k} \circ w_{k-r}$ is necessarily in $e_0$, we know that as $s \to \infty$, $|f_{j+w_{k-r},r}(x)| \to 0$. Therefore there is an $s > r$ such that

$$|f_{j+w_{k-r},r}(x) - f_{j+w_{k-r},r-1}(x)| \geq \frac{2^{r-s-1}}{\max(\tau_{n,k}(Q_{n}^0 x)) \cdot |S_{n,k}|}.$$ \[ (6.5) \]

If (6.5) holds then we may deduce from (6.5) that, pd,

$$||\tau_{n,k+s} \circ Q_{n}^0 x|| \geq ||f_{j+w_{k-r},r}(x) - f_{j+w_{k-r},r-1}(x)|| \geq 2^{r-s-1}.$$ \[ (6.6) \]

Now $\tau_{n,k+s} \circ Q_{n}^0 x = \tau_{n,k} \circ (Q_{n}^0 x) \circ \tau_{n,k} \circ (Q_{n}^0 x) = P_{n,k} \circ (Q_{n}^0 x) \circ (Q_{n}^0 x),$ and

$$||\tau_{n,k+s} \circ Q_{n}^0 x|| > \frac{1}{a_{k+s-r}}.$$ \[ (6.7) \]

This proves our assertion that there is indeed an $m > n$ such that (6.0) holds. Pick such an $m$, and write $y = Q_{n}^0 x$. We know that $||y|| \leq ||Q_{n}^0 x|| \leq a_{m}$, pd. But $||\tau_{n,m} y|| \geq 1/a_{m}$ so $y \in K_{n,m}$. Therefore by 6.3 there is a polynomial $q$ such that $||q(T)y - e_0|| \leq 1/a_{m-1} + 3/a_{m}, 4^{m-k} |q(t)|, \deg q \leq b_{n} + m_{m}$, and $|q| \leq (m+1)^{b_{n} + m_{m}}$. Using our estimate on $|\tau_{n,m+k} \circ (Q_{n}^0 x)|$ and the fact that $|T| < 1$, we find that

$$||q(T)(I - Q_{n}^0 x)|| \leq \frac{2N_{3}(m, a_{m})(m+1)^{-a_{m}}}{b_{m}}.$$ \[ (6.8) \]

Therefore, pd,

$$||q(T)x - e_0|| \leq ||q(T)y - e_0|| + ||q(T)(y - x)|| \leq \frac{3}{a_{m} + 1} + \frac{2N_{3}(m, a_{m})(m+1)^{-a_{m}}}{b_{m}}.$$ \[ (6.9) \]

This inequality (which can be repeated with different values of $n$ by choosing suitable alternative $q$) shows that in fact $x$ is cyclic; and so we conclude the proof.  \[ \square \]

7. Conclusion. The author would like to emphasize that in spite of this negative result, there is some hope that the spaces of Gowers and Maurey may indeed have no invariant subspace free operators. Though the general result about strictly singular operators is false, yet operators on the spaces of Gowers and Maurey have more properties than just being of form $\lambda I + \lambda^*$ (strictly singular). These spaces do not seem to support many operators like shifted shift operators, whose presence is so important for the construction of operators without invariant subspaces, by our methods at any rate. So
there is some chance that operators on these very special spaces may indeed have invariant subspaces.

References


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A general geometric construction for affine surface area

by

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Abstract. Let K be a convex body in \(\mathbb{R}^n\) and \(B\) be the Euclidean unit ball in \(\mathbb{R}^n\). We show that

\[
\lim_{|B| \to 0} \frac{|K| - |B|}{|B|} = \text{as}(K),
\]

where \(\text{as}(K)\) respectively \(\text{as}(B)\) is the affine surface area of \(K\) respectively \(B\) and \(\{K_t\}_{t \geq 0}\), \(\{B_t\}_{t \geq 0}\) are general families of convex bodies constructed from \(K\), \(B\) satisfying certain conditions. As a corollary we get results obtained in \([M-W]\), \([Schm]\), \([S-W]\) and \([W]\).

The affine surface area \(\text{as}(K)\) was introduced by Blaschke \([B]\) for convex bodies in \(\mathbb{R}^3\) with sufficiently smooth boundary and by Leichtweiss \([L1]\) for convex bodies in \(\mathbb{R}^n\) with sufficiently smooth boundary as follows:

\[
\text{as}(K) = \int \frac{\kappa(x)^{1/(n+1)}}{\partial K} d\mu(x),
\]

where \(\kappa(x)\) is the Gaussian curvature at \(x \in \partial K\) and \(\mu\) is the surface measure on \(\partial K\). As it occurs naturally in many important questions, for example in the approximation of convex bodies by polytopes (see the survey article of Gruber \([G]\) and the paper by Schütt \([S]\)) or in a priori estimates for PDEs \([Lu-O]\), one had to extend the affine surface area to arbitrary convex bodies in \(\mathbb{R}^n\) without any smoothness assumptions on the boundary.

Such extensions were given in recent years by Leichtweiss \([L2]\), Lutwak \([L]\), Meyer and Werner \([M-W]\), Schmuckenschläger \([Schm]\), Schütt and Werner \([S-W]\) and Werner \([W]\).

The extensions of affine surface area to an arbitrary convex body \(K\) in \(\mathbb{R}^n\) in \([L2]\), \([M-W]\), \([Schm]\), \([S-W]\) and \([W]\) have a common feature. First a specific family \(\{K_t\}_{t \geq 0}\) of convex bodies is constructed. This family is different in each of the cited extensions but of course related to the given convex body \(K\). Typically the families \(\{K_t\}_{t \geq 0}\) are obtained from \(K\) through a “geometric” construction. In \([L2]\) respectively \([S-W]\) this construction gives

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