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L^p-improving properties of measures supported on curves on the Heisenberg group

by

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Abstract. L^p-L^q boundedness properties are obtained for operators defined by convolution with measures supported on certain curves on the Heisenberg group. We find the curvature condition for which the type set of these operators can be the full optimal trapezoid with vertices $A = (0,0)$, $B = (1,1)$, $C = (2/3,1/2)$, $D = (1/2,1/3)$. We also give notions of right curvature and left curvature which are not mutually equivalent.

1. Introduction. Let $\mathbb{H}_1$ be the *Heisenberg group*, that is, $\mathbb{R}^3$ with the product

$$(x,y,t)\cdot(x',y',t') = \left(x+x', y+y', t+t'+\tfrac{1}{2}(xy'-x'y)\right).$$

We consider a curve γ in $\mathbb{H}_1$ whose tangent vector at any point is not parallel to the center of $\mathbb{H}_1$. Without loss of generality we can define γ by

$$\gamma : I \to \mathbb{H}_1, \quad s \mapsto \gamma(s) = (s, \phi_1(s), \phi_2(s)), \tag{1.1}$$

where $I = [a,b]$ is a bounded interval in $\mathbb{R}$ and $\phi_1(s)$, $\phi_2(s)$ are smooth real-valued functions.

Consider the singular measure μ on $\mathbb{H}_1$ supported on the curve (1.1) and given by

$$\langle \mu, f\rangle = \int_I f(\gamma(s))\,ds. \tag{1.2}$$

We define the right convolution operator by μ:

$$Tf(w) = f * \mu(w) = \int_I f\left(w\cdot(\gamma(s))^{-1}\right)\,ds, \quad w\in\mathbb{H}_1. \tag{1.3}$$

We are interested in studying the *type set* $\mathcal{T}$ of T, that is, the set of points $(1/p, 1/q)\in[0,1]\times[0,1]$ such that T is bounded from $L^p(\mathbb{H}_1)$ to $L^q(\mathbb{H}_1)$. We say that the measure μ defined in (1.2) is *L^p-improving* if $\mathcal{T}$ is not reduced to the diagonal $1/p = 1/q$.

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It is well known that if in (1.3) we replace the Heisenberg group convolution with the ordinary convolution in $\mathbb{R}^3$, then the L^p-improving properties of T are closely related to the curvature and torsion properties of γ. In particular, if $\gamma'(s), \gamma''(s), \gamma'''(s)$ are linearly independent for all $s \in I$ (or equivalently, if

$$\begin{vmatrix} \phi_1''(s) & \phi_2''(s) \\ \phi_1'''(s) & \phi_2'''(s) \end{vmatrix} \neq 0$$

for all $s \in I$) then $\mathcal{T}$ is the closed trapezoid $ABCD$ with vertices

$$A = (0,0), \quad B = (1,1), \quad C = \left(\frac{2}{3}, \frac{1}{2}\right), \quad D = \left(\frac{1}{2}, \frac{1}{3}\right)$$

(see [3], [4]). If either the curvature or the torsion degenerates at some point, the type set becomes smaller in general [1], [5], [6], [8]. We study the L^p-L^q boundedness properties of T on $\mathbb{H}_1$ adapting the "T^*T method" which was already used by Oberlin in [3] to reduce the problem to an $L^p \to L^{p'}$ estimate on $\mathbb{R}^2$. In this way, in Section 3, we prove that the operator T is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^2(\mathbb{H}_1)$ if

$$\text{(1.4)} \qquad \begin{vmatrix} \phi_1''(s) & \phi_2''(s) \\ \phi_1'''(s) & \phi_2'''(s) \end{vmatrix} \neq -\frac{(\phi_1''(s))^2}{2} \quad \text{for all } s \in I$$

and it is bounded from $L^2(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$ if

$$\text{(1.5)} \qquad \begin{vmatrix} \phi_1''(s) & \phi_2''(s) \\ \phi_1'''(s) & \phi_2'''(s) \end{vmatrix} \neq \frac{(\phi_1''(s))^2}{2} \quad \text{for all } s \in I.$$

Conditions (1.4) and (1.5) are better understood if we give a group-invariant notion of higher order derivatives of γ. We must regard $\gamma'(s)$ as an element of the tangent space $T_{\gamma(s)}\mathbb{H}_1$ at the point $\gamma(s)$. Before further differentiating, we must transport the various $\gamma'(s)$ to the same tangent space. Hence we consider the right translation

$$R_{\gamma(s)} : \mathbb{H}_1 \to \mathbb{H}_1$$

which maps (x,y,t) to $(x,y,t)\cdot\gamma(s)$. Its differential $dR_{\gamma(s)}$ maps the tangent space $T_0\mathbb{H}_1$ onto $T_{\gamma(s)}\mathbb{H}_1$. So we define

$$\gamma_R'(s) = dR_{\gamma(s)}^{-1}\gamma'(s).$$

At this point we can define

$$\gamma_R''(s) = \frac{d}{ds}\gamma_R'(s)$$

and $\gamma_R'''(s)$ similarly.

Condition (1.4) is then equivalent to saying that $\gamma_R'(s), \gamma_R''(s), \gamma_R'''(s)$ are linearly independent for each $s \in I$.

Replacing right translations with left translations gives the corresponding interpretation of (1.5). Therefore, analogously to the case of a curve in $\mathbb{R}^3$, when conditions (1.4) and (1.5) fail at some point $s \in I$, we get a curve γ which has a sort of degenerate curvature and/or torsion at that point.

As an example of the curve (1.1) we consider in Section 5 the curve

$$\text{(1.6)} \qquad \gamma_\alpha : [0,1] \to \mathbb{H}_1, \quad s \mapsto \gamma_\alpha(s) = (s, s^2, \alpha s^3),$$

where α is a real-valued parameter.

Notice that γ_α satisfies condition (1.4) for all values of α except $\alpha = -1/6$, in which case $(\gamma_\alpha)_R'''$ is identically zero. Similarly, (1.5) is satisfied for all values of α except $\alpha = 1/6$. Denoting by μ_α the singular measure supported on γ_α and by T_α the right convolution operator by μ_α, i.e.

$$\text{(1.7)} \qquad T_\alpha f(w) = \int_0^1 f(w\cdot(\gamma_\alpha(s))^{-1})\,ds, \quad w \in \mathbb{H}_1,$$

we are interested in studying the type set $\mathcal{T}_\alpha$ of T_α.

If $\alpha \neq \pm 1/6$ we prove that $\mathcal{T}_\alpha$ is the closed trapezoid $ABCD$, hence we restrict our further attention to the degenerate cases $\alpha = \pm 1/6$. We prove that the type set $\mathcal{T}_{-1/6}$ is the closed triangle ABD and $\mathcal{T}_{1/6}$ is the closed triangle ABC (the fact that $\mathcal{T}_{\pm 1/6}$ are not reduced to the diagonal $1/q = 1/p$ is due to the fact that $\gamma_{\pm 1/6}$ are analytic curves that generate the full group [7]). This situation contrasts with the abelian case, i.e. when the Heisenberg convolution in (1.7) is replaced by ordinary convolution. Here the curve degenerates for $\alpha = 0$, in which case it becomes completely flat and consequently there is no L^p-improving. Another interesting remark is that the type sets of $\gamma_{\pm 1/6}$ are not symmetric with respect to the diagonal $1/p + 1/q = 1$. This is a phenomenon which cannot occur in the abelian case. We remark that asymmetry of convolution operators has been extensively studied in the case $p = q$ and some explicit examples of asymmetric operators have been given on the Heisenberg group [2]. As far as we know, no such simple examples of asymmetric operators are known in the case $p < q$.

In the last section of this paper we consider a family of curves $\Gamma_\sigma(s)$, $s \in J \subset \mathbb{R}$, whose vector $(\Gamma_\sigma)_R''(s)$ does not vanish at any point $s \in J$, while the vector $(\Gamma_\sigma)_R'''(s)$ is zero at an isolated point $s_0 \in J$. Up to group automorphisms, we can suppose that s_0 is the origin and that the curve is of the form

$$\text{(1.8)} \qquad \Gamma_\sigma : J \to \mathbb{H}_1, \quad s \mapsto \Gamma_\sigma(s) = (s, s^2, -s^3/6 + s^\sigma),$$

where $\sigma > 3$ and $J = [0,\delta]$ for a sufficiently small $\delta > 0$ so that the curve (1.8) has only the origin as a singular point. We define the operator U_σ as

$$\text{(1.9)} \qquad U_\sigma f(w) = f * \nu_\sigma(w) = \int_J f(w\cdot(\Gamma_\sigma(s))^{-1})\,ds, \quad w \in \mathbb{H}_1,$$

where ν_σ is a singular measure supported on the curve (1.8), and we study the type set $\mathcal{U}_\sigma$ of U_σ. We will see that $\mathcal{U}_\sigma$ is contained in the closed quadrilateral with vertices

$$A=(0,0),\quad B=(1,1),\quad P=\left(\frac{\sigma+1}{\sigma+3},\frac{\sigma}{\sigma+3}\right),\quad D=\left(\frac{1}{2},\frac{1}{3}\right)$$

with the only possible exception of the open segment DP and the point P. This result also holds when the operator U_σ is defined on a curve more general than (1.8), that is, on the curve

$$\mathcal{C}:J\to\mathbb{H}_1,\qquad s\mapsto\mathcal{C}(s)=(s,s^2,-s^3/6+\phi(s)),$$

where $J=[0,\delta]$ for a sufficiently small $\delta>0$, ϕ is a smooth real-valued function which satisfies the following three conditions:

$$\phi(0)=0,\qquad \phi'(0)=0,\qquad \phi''(s)\asymp Cs^{\sigma-2}\quad\text{as } s\to 0,$$

and $\sigma>3$.

2. Preliminaries. We begin with a lemma on L^p-L^q estimates for convolution operators with a singular measure supported on a curve in $\mathbb{R}^2$, which will be used in the proof of Theorems 3.1 and 6.1.

Let $I=[a,b]$ be a bounded interval of $\mathbb{R}$; define the curve

$$\Psi:I\to\mathbb{R}^2,\qquad s\mapsto\Psi(s)=(s,\psi(s)),\tag{2.1}$$

where $\psi(s)$ is a smooth real-valued function.

For any test function f on $\mathbb{R}^2$, consider the measure λ on $\mathbb{R}^2$ given by

$$\langle\lambda,f\rangle=\int_I f(\Psi(s))\eta(s)\,ds,\tag{2.2}$$

where η is a smooth real-valued function, and define the operator

$$Sf(x_1,x_2)=f*\lambda(x_1,x_2)=\int_I f(x_1-s,x_2-\psi(s))\eta(s)\,ds.\tag{2.3}$$

Lemma 2.1. *Let Ψ be the curve* (2.1), *suppose that $|\psi''(s)|\ge K>0$ for all $s\in I$, and let S be the operator defined in* (2.3). *Then*

$$\|Sf\|_{L^3(\mathbb{R}^2)}\le cK^{-1/3}(\|\eta\|_\infty+\|\eta'\|_1)^{2/3}\|\eta\|_\infty^{1/3}\|f\|_{L^{3/2}(\mathbb{R}^2)}\tag{2.4}$$

where c is a positive constant.

The proof of this lemma is based on complex interpolation and follows standard lines; therefore we omit the details.

Remark 2.2. Suppose that the curve (2.1) has the form

$$\Psi(s)=(\theta(s),\zeta(s)),\qquad s\in I,$$

where θ,ζ are smooth real-valued functions satisfying the following two conditions:

$$|\theta'(s)\zeta''(s)-\theta''(s)\zeta'(s)|\ge C_1>0,\tag{2.5}$$

$$|\theta'(s)|\ge C_2>0\tag{2.6}$$

for all $s\in I$, and define the operator

$$Sf(x_1,x_2)=\int_a^b f(x_1-\theta(s),x_2-\zeta(s))\,ds.\tag{2.7}$$

If we change variable in (2.7), namely if we put $\tau=\theta(s)$, we get the operator

$$Sf(x_1,x_2)=\int_{\theta(a)}^{\theta(b)} f(x_1-\tau,x_2-(\zeta\circ\theta^{-1})(\tau))(\theta^{-1})'(\tau)\,d\tau.$$

Therefore to estimate the $L^{3/2}(\mathbb{R}^2)$-$L^3(\mathbb{R}^2)$ norm of S we can apply Lemma 2.1 where in this case we have

$$\eta(\tau)=(\theta^{-1})'(\tau)=\frac{1}{\theta'(s)}.\tag{2.8}$$

By (2.5) and putting $M=\max_{s\in I}|\theta'(s)|$ we get

$$\begin{aligned}|(\zeta\circ\theta^{-1})''(\tau)|&=\left|\frac{\zeta''(\theta^{-1}(\tau))\theta'(\theta^{-1}(\tau))-\zeta'(\theta^{-1}(\tau))\theta''(\theta^{-1}(\tau))}{(\theta'(\theta^{-1}(\tau)))^3}\right|\\&=\left|\frac{\zeta''(s)\theta'(s)-\zeta'(s)\theta''(s)}{(\theta'(s))^3}\right|\ge\frac{C_1}{M^3}\end{aligned}\tag{2.9}$$

for all $s\in I$, and by (2.8) and (2.6) we have

$$\|\eta\|_\infty\le\frac{1}{C_2}.\tag{2.10}$$

Moreover, since θ'' is a continuous function on I, there exists a non-negative constant C_3 such that $|\theta''(s)|\le C_3$ for all $s\in I$. Therefore by (2.6) we get

$$\|\eta'\|_1=\int_I\left|\frac{\theta''(s)}{(\theta'(s))^3}\right|ds\le\frac{C_3}{C_2^3}|I|.\tag{2.11}$$

Hence by (2.9), (2.10), (2.11), inequality (2.4) becomes

$$\|Sf\|_{L^3(\mathbb{R}^2)}\le c\left(\frac{C_1C_2}{M^3}\right)^{-1/3}\left(\frac{1}{C_2}+\frac{C_3}{C_2^3}|I|\right)^{2/3}\|f\|_{L^{3/2}(\mathbb{R}^2)}.\tag{2.12}$$

Let T be the operator defined in (1.3). We want to determine the operator T^*.

Let $f,g\in C_c^\infty(\mathbb{H}_1)$. By definition,

$$\langle T^*f,g\rangle=\langle f,Tg\rangle$$

where
$$\langle f,g\rangle=\int f(w)\overline{g(w)}\,dw,\qquad w\in\mathbb{H}_1.$$
For $w\in\mathbb{H}_1$ and $v_s=(s,\phi_1(s),\phi_2(s))$, using Fubini's Theorem and the right invariance of the Lebesgue measure we have
$$\langle f,Tg\rangle=\int f(w)\overline{Tg(w)}\,dw=\int f(w)\int_I\overline{g(w\cdot v_s^{-1})}\,ds\,dw$$
$$=\int\Big(\int_I f(w\cdot v_s)\,ds\Big)\overline{g(w)}\,dw=\int f*\check{\mu}(w)\overline{g(w)}\,dw=\langle f*\check{\mu},g\rangle$$
where
$$\langle\check{\mu},f\rangle=\int_I f((s,\phi_1(s),\phi_2(s))^{-1})\,ds.$$
Therefore we have $T^*f(w)=f*\check{\mu}(w)$ for $w\in\mathbb{H}_1$.

3. Convolution estimates for the operator T. In this section we prove the following theorem.

THEOREM 3.1. *Let T be the operator defined in* (1.3).

(i) *If condition* (1.4) *holds then T is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^2(\mathbb{H}_1)$.*

(ii) *If condition* (1.5) *holds then T is bounded from $L^2(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$.*

Proof. (i) We decompose the interval I into n disjoint subintervals $I_k=[a_k,a_k+\delta]$ of length δ, with $\delta>0$ to be determined.

Define μ_k by
$$\langle\mu_k,f\rangle=\int_{I_k} f(\gamma(s))\,ds$$
for any test function f on $\mathbb{H}_1$, and $T_kf=f*\mu_k$. Then T is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^2(\mathbb{H}_1)$ if and only if $T_k^*T_k$ is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$ for all k.

We write
$$T_k^*T_kf(x,y,t)=\int_{I_k}\int_{I_k} f((x,y,t)\cdot(s,\phi_1(s),\phi_2(s))\cdot(r,\phi_1(r),\phi_2(r))^{-1})\,dr\,ds$$
$$=\int_{I_k}\int_{I_k} f\bigg((x,y,t)\cdot\Big(r-s,\phi_1(r)-\phi_1(s),\phi_2(r)-\phi_2(s)+\frac{s\phi_1(r)-r\phi_1(s)}{2}\Big)^{-1}\bigg)\,dr\,ds$$
for any test function f on $\mathbb{H}_1$. Making the change of variables $r-s=u$, $s=v$ we get

(3.1)
$$T_k^*T_kf(x,y,t)$$
$$=\int_{-\delta}^{\delta}\int_{I_k(u)} f\bigg((x,y,t)\cdot\Big(u,\phi_1(v+u)-\phi_1(v),\phi_2(v+u)-\phi_2(v)+\frac{v\phi_1(v+u)-(v+u)\phi_1(v)}{2}\Big)^{-1}\bigg)\,dv\,du$$
$$=\int_{-\delta}^{\delta}\int_{I_k(u)} f\bigg(x-u,y-(\phi_1(v+u)-\phi_1(v)),t-(\phi_2(v+u)-\phi_2(v))-\frac{v\phi_1(v+u)-(v+u)\phi_1(v)}{2}-\frac{x(\phi_1(v+u)-\phi_1(v))-yu}{2}\bigg)\,dv\,du$$
where
$$I_k(u)=[a_k,a_k+\delta]\cap[a_k-u,a_k+\delta-u]=\begin{cases}[a_k,a_k+\delta-u], & 0\le u\le\delta,\\ [a_k-u,a_k+\delta], & -\delta\le u<0.\end{cases}$$

For $|u|\le\delta$ and a fixed $x\in\mathbb{R}$, consider the curve $\gamma_{x,u}(v)$ on $\mathbb{R}^2$ given by
$$\gamma_{x,u}(v)=\Big(\phi_1(v+u)-\phi_1(v),\phi_2(v+u)-\phi_2(v)+\frac{(v+x)\phi_1(v+u)}{2}-\frac{(v+x+u)\phi_1(v)}{2}\Big),\qquad v\in I_k(u).$$
For any continuous compactly supported function g on $\mathbb{R}^2$ we will prove

(3.2)
$$\|g*_{\mathbb{R}^2}\gamma_{x,u}\|_{L^3(\mathbb{R}^2)}\le C|u|^{-2/3}\|g\|_{L^{3/2}(\mathbb{R}^2)},$$
where C is a positive constant which is independent of x and u. Assume for a moment that (3.2) holds and put
$$f_{x-u}(y,t)=f(x-u,y,t).$$
Then (3.1) becomes
$$T_k^*T_kf(x,y,t)=\int_{-\delta}^{\delta}(f_{x-u}*_{\mathbb{R}^2}\gamma_{x,u})\Big(y,t+\frac{1}{2}uy\Big)\,du$$

and we get

$$\|T_k^*T_kf\|_{L^3(\mathbb{H}_1)} = \Big\|\Big\|\int_{-\delta}^{\delta}(f_{x-u} *_{\mathbb{R}^2} \gamma_{x,u})\Big(y,t+\frac{1}{2}uy\Big)\,du\Big\|_{L^3(\mathbb{R}^2)}\Big\|_{L^3(\mathbb{R})}$$
$$\le \Big\|\int_{-\delta}^{\delta}\Big\|(f_{x-u} *_{\mathbb{R}^2} \gamma_{x,u})\Big(y,t+\frac{1}{2}uy\Big)\Big\|_{L^3(\mathbb{R}^2)}\,du\Big\|_{L^3(\mathbb{R})}$$
$$\le C\Big\|\int_{-\delta}^{\delta}|u|^{-2/3}\|f_{x-u}\|_{L^{3/2}(\mathbb{R}^2)}\,du\Big\|_{L^3(\mathbb{R})}.$$

By the boundedness of the Riesz potential of order $1/3$ as a mapping from $L^{3/2}(\mathbb{R})$ to $L^3(\mathbb{R})$, this last term is bounded by

$$C\|\,\|f_x\|_{L^{3/2}(\mathbb{R}^2)}\|_{L^{3/2}(\mathbb{R},dx)} = C\|f\|_{L^{3/2}(\mathbb{H}_1)}.$$

So, to prove that the operator $T_k^*T_k$ is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$ we need to establish inequality (3.2).

In $\mathbb{R}^2$ we take coordinates ξ,η and we observe that

$$(3.3)\quad (g *_{\mathbb{R}^2} \gamma_{x,u})(\xi,\eta)$$
$$= \int_{I_k(u)} g\Big(\xi-\phi_1(v+u)+\phi_1(v),\eta-\phi_2(v+u)+\phi_2(v) - \frac{(x+v)(\phi_1(v+u)-\phi_1(v))-u\phi_1(v)}{2}\Big)\,dv$$
$$= \int_{I_k(u)} g\Big(u\Big(\frac{\xi}{u}-\frac{\phi_1(v+u)-\phi_1(v)}{u}\Big),u\Big(\frac{\eta}{u}-\frac{\phi_2(v+u)-\phi_2(v)}{u} - \frac{(x+v)(\phi_1(v+u)-\phi_1(v))-u\phi_1(v)}{2u}\Big)\Big)\,dv$$
$$= D_{1/u}(D_ug *_{\mathbb{R}^2} \widetilde{\gamma}_{x,u})(\xi,\eta)$$

where for functions g on $\mathbb{R}^2$ and $\varepsilon\in\mathbb{R}$, we put

$$D_\varepsilon g(\xi,\eta) = g(\varepsilon\xi,\varepsilon\eta)$$

and $\widetilde{\gamma}_{x,u}$ is the curve

$$(3.4)\quad \widetilde{\gamma}_{x,u}(v) = \Big(\frac{\phi_1(v+u)-\phi_1(v)}{u},\frac{\phi_2(v+u)-\phi_2(v)}{u} + \frac{(x+v)(\phi_1(v+u)-\phi_1(v))-u\phi_1(v)}{2u}\Big),$$

$v\in I_k(u)$. Therefore we are reduced to proving the estimate

$$(3.5)\quad \|g *_{\mathbb{R}^2} \widetilde{\gamma}_{x,u}\|_{L^3(\mathbb{R}^2)} \le C\|g\|_{L^{3/2}(\mathbb{R}^2)}$$

where C is a positive constant which is independent of x and u.

In fact, once (3.5) is established, (3.3) implies (3.2) by a scaling argument. In order to prove (3.5), we conjugate the convolution operator by the linear mapping

$$(\xi,\eta)\to\Big(\xi,\eta-\frac{x}{2}\xi\Big).$$

This allows us to replace the curve $\widetilde{\gamma}_{x,u}(v)$ defined in (3.4) with the curve

$$(3.6)\quad \overline{\gamma}_u(v) = \Big(\frac{\phi_1(v+u)-\phi_1(v)}{u},\frac{\phi_2(v+u)-\phi_2(v)}{u} + \frac{v(\phi_1(v+u)-\phi_1(v))-u\phi_1(v)}{2u}\Big),\quad v\in I_k(u),$$

which is independent of x. The $L^{3/2}(\mathbb{R}^2)$-$L^3(\mathbb{R}^2)$ norm of the convolution operator by $\widetilde{\gamma}_{x,u}(v)$ is equal to the $L^{3/2}(\mathbb{R}^2)$-$L^3(\mathbb{R}^2)$ norm of the convolution operator by $\overline{\gamma}_u$, therefore to get (3.5) we have to prove the estimate

$$(3.7)\quad \|g *_{\mathbb{R}^2} \overline{\gamma}_u\|_{L^3(\mathbb{R}^2)} \le C\|g\|_{L^{3/2}(\mathbb{R}^2)}$$

where C is a positive constant which is independent of u.

We apply Lemma 2.1 to the curve $\overline{\gamma}_u(v)$, $v\in I_k(u)$. By the mean-value theorem there exist $\tau_i\in(v,v+u)$, $i=1,\dots,5$, such that

$$(3.8)\quad \begin{aligned}\overline{\gamma}_u'(v) &= \Big(\phi_1''(\tau_1),\phi_2''(\tau_2)+\frac{\phi_1'(\tau_3)-\phi_1'(v)}{2}+\frac{v\phi_1''(\tau_1)}{2}\Big),\\ \overline{\gamma}_u''(v) &= \Big(\phi_1'''(\tau_4),\phi_2'''(\tau_5)+\phi_1''(\tau_1)-\frac{\phi_1''(v)}{2}+\frac{v\phi_1'''(\tau_4)}{2}\Big).\end{aligned}$$

Condition (1.4) can be rewritten as

$$(3.9)\quad \Big|\phi_1''(s)\phi_2'''(s)-\phi_1'''(s)\phi_2''(s)+\frac{(\phi_1''(s))^2}{2}\Big|\ge C_1$$

for all $s\in I$, where C_1 is a positive constant. In order to apply Remark 2.2 we want to prove that (3.9) implies

$$(3.10)\quad \Big|\phi_1''(\tau_1)\phi_2'''(\tau_5)+(\phi_1''(\tau_1))^2-\frac{\phi_1''(\tau_1)\phi_1''(v)}{2} -\phi_1'''(\tau_4)\phi_2''(\tau_2)-\frac{\phi_1'''(\tau_4)\phi_1'(\tau_3)}{2}+\frac{\phi_1'''(\tau_4)\phi_1'(v)}{2}\Big|\ge\frac{C_1}{2}$$

for $v\in I_k(u)$, $\tau_i\in(v,v+u)$, $i=1,\dots,5$.

The left-hand side of (3.10) is a continuous function of $v, \tau_1, \ldots, \tau_5$, which we denote as $J(v, \tau_1, \ldots, \tau_5)$.

Assuming now $|u| \le b-a$, we consider the interval $I(u)$ defined as

$$I(u) = [a,b] \cap [a-u, b-u] = \begin{cases} [a, b-u], & 0 \le u \le b-a, \\ [a-u, b], & a-b \le u < 0. \end{cases}$$

and we suppose $v \in I(u)$. Set

$$E = \{(u, v, \tau_1, \ldots, \tau_5) : |u| \le b-a,\ v, \tau_1, \ldots, \tau_5 \in I(u)\}.$$

Since (3.9) holds, we see that condition (3.10) is satisfied on the diagonal

$$R = \{(u, v, \tau_1, \ldots, \tau_5) \in E : \tau_i = v,\ i = 1, \ldots, 5\}.$$

Given a point $P \in R$, there exists an open neighborhood B in E centered at P of radius r such that condition (3.10) is satisfied by every point of B. Since R is compact, we may cover it by a finite number of open neighborhoods B_i centered at $P_i \in R$ of radius r_i in which condition (3.10) is satisfied. Taking $r = \min_i\{r_i\}$, we see that condition (3.10) holds for every v and $\tau_1, \ldots, \tau_5$ belonging to a neighborhood of v of radius $r/2$. Therefore, since $\tau_i \in (v, v+u)$, $i = 1, \ldots, 5$, if we take $\delta \le r/2$ then the assumption (3.9) implies condition (3.10) for $v \in I_k(u)$ and $\tau_i \in (v, v+u)$, $i = 1, \ldots, 5$.

Next we notice that it is possible to split every interval $I_k(u)$ into a finite number N of disjoint subintervals where the curve (3.6) is a C^2-graph and this finite number is independent of u. To see this we put

$$\overline{\gamma}_u(v) = (\theta_u(v), \zeta_u(v))$$

where

$$\theta_u(v) = \frac{\phi_1(v+u) - \phi_1(v)}{u},$$

$$\zeta_u(v) = \frac{\phi_2(v+u) - \phi_2(v)}{u} + \frac{v(\phi_1(v+u) - \phi_1(v))}{2u} - \frac{\phi_1(v)}{2},$$

$v \in I_k(u)$, $u \in J = [-\delta, \delta]$. Set

$$H_k = \{(v, u) : u \in J,\ v \in I_k(u)\}.$$

We take a point $(v_0, u_0) \in H_k$ where $\theta'_{u_0}(v_0) \ne 0$ or $\zeta'_{u_0}(v_0) \ne 0$. Such a point exists because condition (3.10) implies that $\overline{\gamma}'_u(v)$ and $\overline{\gamma}''_u(v)$ are linearly independent in $I_k(u)$. Since θ and ζ are continuous functions of v and u, there exists a neighborhood V of (v_0, u_0) in H_k such that $\theta'_u(v) \ne 0$ or $\zeta'_u(v) \ne 0$ for all $(v, u) \in V$. Since H_k is compact, we may cover it by a finite number N of such neighborhoods V. Relating to this covering we get a subdivision of $I_k(u)$ and J into a finite number N of subintervals $I_{k,i}(u)$ and J_i, respectively, $i = 1, \ldots, N$, where the curve $\overline{\gamma}_u(v)$ is a graph and N is independent of u. So

$$g *_{\mathbb{R}^2} \overline{\gamma}_u(v) = \sum_{i=1}^{N} g *_{\mathbb{R}^2} \overline{\gamma}_{u,i} \tag{3.11}$$

where

$$\overline{\gamma}_{u,i} = (\theta_u(v), \zeta_u(v)), \qquad v \in I_{k,i}(u).$$

Let u be fixed in J_i and let $v \in I_{k,i}(u)$. We want to apply Remark 2.2 to estimate the $L^3(\mathbb{R}^2)$ norm of $g *_{\mathbb{R}^2} \overline{\gamma}_{u,i}$, checking that the constants which appear in this estimate do not depend on u.

Suppose that for all $v \in I_{k,i}(u)$ we have $\theta'_u(v) \ne 0$ (but we can proceed in the same way if we suppose $\zeta'_u(v) \ne 0$). Then, since $\theta_u(v)$ is a continuous function of u and u varies in the compact set J_i, there exist positive constants C_2 and M which are independent of u and such that

$$C_2 \le |\theta'_u(v)| \le M \tag{3.12}$$

for all $v \in I_{k,i}(u)$. Since (3.10) and (3.12) hold for all $v \in I_{k,i}(u)$ we are in the hypothesis of Remark 2.2. Moreover, we can notice that since $\theta''_u(v)$ is a continuous function of u and u varies in a compact set, we may find a non-negative constant C_3 which is independent of u and such that

$$|\theta''_u(v)| \le C_3 \tag{3.13}$$

for all $v \in I_{k,i}(u)$. Therefore, by (2.12), using (3.10), (3.12), (3.13) and the fact that $|I_{k,i}(u)| \le |I|$ we get

$$\|g *_{\mathbb{R}^2} \overline{\gamma}_{u,i}\|_{L^3(\mathbb{R}^2)} \le c\left(\frac{C_1 C_2}{M^3}\right)^{-1/3} \left(\frac{1}{C_2} + \frac{C_3}{C_2^3}|I_{k,i}(u)|\right)^{2/3} \|g\|_{L^{3/2}(\mathbb{R}^2)} \le C\|g\|_{L^{3/2}(\mathbb{R}^2)} \tag{3.14}$$

where C is a positive constant which is independent of u. Hence, taking into account (3.11) and (3.14) we have

$$\|g *_{\mathbb{R}^2} \overline{\gamma}_u\|_{L^3(\mathbb{R}^2)} \le CN\|g\|_{L^{3/2}(\mathbb{R}^2)},$$

which gives (3.7). This concludes the proof of the first part of Theorem 3.1.

(ii) Arguing as in part (i), after decomposing the operator T as a finite sum $T = \sum_{k=1}^{n} T_k$, we can prove that $T_k T_k^*$ is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$ for all k. This implies that the operator T^* is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^2(\mathbb{H}_1)$, which is equivalent to saying that T is bounded from $L^2(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$. ■

4. A geometric interpretation of conditions (1.4) and (1.5). Let us consider a basis $\mathcal{B}_R$ of the Lie algebra of $\mathbb{H}_1$ consisting of the rightinvariant vector fields

$$X_R = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t}, \quad Y_R = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t}, \quad T_R = \frac{\partial}{\partial t}.$$

Given the curve $\gamma(s)$ defined in (1.1), we express the vector $\gamma'(s) = (1, \phi_1'(s), \phi_2'(s))$ with respect to the basis $\mathcal{B}_R$ (or equivalently, using right translations we transport this vector to the tangent space $T_0\mathbb{H}_1$). Hence we obtain the vector

$$\gamma_R'(s) = \left(1, \phi_1'(s), -\frac{\phi_1(s)}{2} + \frac{s}{2}\phi_1'(s) + \phi_2'(s)\right).$$

Now, since the tangent vectors $\gamma_R'(s)$, $s \in I$, are applied at the same point, we can calculate the vectors

$$\gamma_R''(s) = \left(0, \phi_1''(s), \frac{s}{2}\phi_1''(s) + \phi_2''(s)\right),$$

$$\gamma_R'''(s) = \left(0, \phi_1'''(s), \frac{\phi_1''(s)}{2} + \frac{s}{2}\phi_1'''(s) + \phi_2'''(s)\right).$$

The condition of linear independence of $\gamma_R'(s), \gamma_R''(s), \gamma_R'''(s)$ at a point $s \in I$ then states that

$$\begin{vmatrix} \phi_1''(s) & \phi_2''(s) \\ \phi_1'''(s) & \phi_2'''(s) \end{vmatrix} \neq -\frac{(\phi_1''(s))^2}{2} \quad \text{for all } s \in I,$$

which is condition (1.4).

In an analogous way we can consider a basis $\mathcal{B}_L$ of the Lie algebra of $\mathbb{H}_1$ consisting of the left-invariant vector fields

$$X_L = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial t}, \quad Y_L = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial t}, \quad T_L = \frac{\partial}{\partial t}$$

and we may express the vector $\gamma'(s) = (1, \phi_1'(s), \phi_2'(s))$ in terms of the basis $\mathcal{B}_L$ (or equivalently, using left translations we transport this vector to the tangent space $T_0\mathbb{H}_1$). Therefore we get the vectors

$$\gamma_L'(s) = \left(1, \phi_1'(s), \frac{\phi_1(s)}{2} - \frac{s}{2}\phi_1'(s) + \phi_2'(s)\right),$$

$$\gamma_L''(s) = \left(0, \phi_1''(s), -\frac{s}{2}\phi_1''(s) + \phi_2''(s)\right),$$

$$\gamma_L'''(s) = \left(0, \phi_1'''(s), -\frac{\phi_1''(s)}{2} - \frac{s}{2}\phi_1'''(s) + \phi_2'''(s)\right),$$

and the condition of linear independence of $\gamma_L'(s), \gamma_L''(s), \gamma_L'''(s)$ at a point $s \in I$ then states that

$$\begin{vmatrix} \phi_1''(s) & \phi_2''(s) \\ \phi_1'''(s) & \phi_2'''(s) \end{vmatrix} \neq \frac{(\phi_1''(s))^2}{2} \quad \text{for all } s \in I,$$

which is condition (1.5).

Therefore we can restate Theorem 3.1 as follows.

THEOREM 4.1. *Let T be the operator defined in* (1.3).

(i) *If $\gamma_R'(s), \gamma_R''(s), \gamma_R'''(s)$ are linearly independent for all $s \in I$, then T is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^2(\mathbb{H}_1)$.*

(ii) *If $\gamma_L'(s), \gamma_L''(s), \gamma_L'''(s)$ are linearly independent for all $s \in I$, then T is bounded from $L^2(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$.*

5. The twisted cubic. In this section we study the type set $\mathcal{T}_\alpha$ of the operator T_α defined in (1.7).

LEMMA 5.1. *For any α, $\mathcal{T}_\alpha$ is contained in the closed trapezoid with vertices $A = (0,0)$, $B = (1,1)$, $C = (2/3, 1/2)$, $D = (1/2, 1/3)$.*

Proof. We notice that T_α is a convolution operator so it is certainly necessary that $p \le q$ if $(1/p, 1/q) \in \mathcal{T}_\alpha$. Two further tests allow us to obtain the other limitations on $\mathcal{T}_\alpha$.

(1) Let f_ε be the characteristic function of a small Euclidean ball of radius $\varepsilon > 0$ centered at the origin. We apply T_α to f_ε:

$$T_\alpha f_\varepsilon(x,y,t) = \int_0^1 f_\varepsilon((x,y,t)\cdot(s,s^2,\alpha s^3)^{-1})\,ds$$
$$= \int_0^1 f_\varepsilon\big(x-s, y-s^2, t-\alpha s^3 + \tfrac{1}{2}(-xs^2+ys)\big)\,ds.$$

Let S_ε be the set defined by

$$S_\varepsilon = \{(x,y,t) \in \mathbb{H}_1 : \exists s_0 \in [0,1] : |x-s_0| < c_1\varepsilon,\ |y-s_0^2| < c_1\varepsilon,\ |t-\alpha s_0^3| < c_1\varepsilon\}$$

where c_1 is a small positive constant. For $(x,y,t) \in S_\varepsilon$, let s_0 be a point in $[0,1]$ such that the conditions $|x-s_0| < c_1\varepsilon$, $|y-s_0^2| < c_1\varepsilon$, $|t-\alpha s_0^3| < c_1\varepsilon$ are satisfied. If we put

$$V = \{s \in [0,1] : |s-s_0| < c\varepsilon\}$$

where c is a positive constant, then since

$$|x-s| \le |x-s_0| + |s_0-s|, \qquad |y-s^2| \le |y-s_0^2| + |s_0^2-s^2|,$$
$$|t-\alpha s^3 + \tfrac{1}{2}ys - \tfrac{1}{2}xs^2| \le |t-\alpha s_0^3| + |\alpha s_0^3 - \alpha s^3| + \tfrac{1}{2}(|y-s^2| + |s^2-xs|),$$

it is easy to check that we can choose c so small that the inequalities

$$|x-s| < \varepsilon, \qquad |y-s^2| < \varepsilon, \qquad |t-\alpha s^3 + \tfrac{1}{2}ys - \tfrac{1}{2}xs^2| < \varepsilon$$

hold simultaneously for $s \in V$ and $(x,y,t) \in S_\varepsilon$. Then

$$T_\alpha f_\varepsilon(x,y,t) > \int_V ds > c\varepsilon$$

on S_ε.

Since the codimension of the curve (1.6) is 2, the Lebesgue measure of S_ε is greater than a constant times ε^2, so that

$$\|T_\alpha f_\varepsilon\|_{L^q(\mathbb{H}_1)} = \Big(\int_{\mathbb{H}_1} |T_\alpha f_\varepsilon(x,y,t)|^q\,dx\,dy\,dt\Big)^{1/q} > c\varepsilon(m(S_\varepsilon))^{1/q} > c\varepsilon^{1+2/q}.$$

Moreover,

$$\|f_\varepsilon\|_{L^p(\mathbb{H}_1)} = C\varepsilon^{3/p}.$$

Imposing the condition that T_α is bounded from $L^p(\mathbb{H}_1)$ to $L^q(\mathbb{H}_1)$ we must have the inequality

$$\varepsilon^{1+2/q} \le C\varepsilon^{3/p}$$

for $\varepsilon < 1$. This implies that

$$\frac{3}{p} - \frac{2}{q} \le 1.$$

By duality it is also necessary that

$$\frac{3}{q} \ge \frac{2}{p}.$$

Therefore $\mathcal{T}_\alpha$ is contained in the region determined by these two conditions and by the condition $p \le q$, i.e. the triangle with vertices $A = (0,0)$, $B = (1,1)$, $Q = (3/5, 2/5)$.

(2) The operator T_α is homogeneous with respect to the family of group automorphisms

$$\delta\cdot(x,y,t) = (\delta x, \delta^2 y, \delta^3 t), \quad \delta > 0, \tag{5.1}$$

and the homogeneous dimension of $\mathbb{H}_1$ with respect to the dilations (5.1) is 6.

Let f_ε be the characteristic function of the set

$$\{(x,y,t)\in\mathbb{H}_1 : |x|<\varepsilon,\ |y|<\varepsilon^2,\ |t|<\varepsilon^3\}$$

for a small positive ε. Given (x,y,t) in the set

$$S_\varepsilon = \{(x,y,t)\in\mathbb{H}_1 : |x|<c_1\varepsilon,\ |y|<c_1\varepsilon^2,\ |t|<c_1\varepsilon^3\}$$

where c_1 is a small positive constant, if we put $0 < s < c\varepsilon$ for a sufficiently small constant $c > 0$, term by term majorization yields

$$|x-s| < \varepsilon, \quad |y-s^2| < \varepsilon^2, \quad \Big|t - \alpha s^3 - \frac{xs^2}{2} + \frac{ys}{2}\Big| < \varepsilon^3.$$

Hence

$$T_\alpha f_\varepsilon(x,y,t) > c\varepsilon$$

on S_ε. Therefore, since the Lebesgue measure of S_ε is greater than a positive constant times ε^6, we have

$$\|T_\alpha f_\varepsilon\|_{L^q(\mathbb{H}_1)} = \Big(\int_{\mathbb{H}_1} |T_\alpha f_\varepsilon(x,y,t)|^q\,dx\,dy\,dt\Big)^{1/q} > c\varepsilon(m(S_\varepsilon))^{1/q} = c\varepsilon^{1+6/q},$$

while

$$\|f_\varepsilon\|_{L^p(\mathbb{H}_1)} = C\varepsilon^{6/p}.$$

Imposing the condition that T_α is bounded from $L^p(\mathbb{H}_1)$ to $L^q(\mathbb{H}_1)$ we get

$$\varepsilon^{1+6/q} < C\varepsilon^{6/p}.$$

Letting $\varepsilon \to 0$ we have

$$\frac{1}{p} - \frac{1}{q} \le \frac{1}{6}.$$

So $\mathcal{T}_\alpha$ is contained in the subset of the triangle ABQ which lies on or above the line $1/p - 1/q = 1/6$, i.e. in the closed trapezoid with vertices $A = (0,0)$, $B = (1,1)$, $C = (2/3, 1/2)$, $D = (1/2, 1/3)$. ∎

Combining Lemma 5.1 and Theorem 3.1 we get

THEOREM 5.2. *Let T_α be the operator defined in* (1.7). *If $\alpha \ne \pm 1/6$ then $\mathcal{T}_\alpha$ is the closed trapezoid $ABCD$.*

We now look at $\alpha = \pm 1/6$.

THEOREM 5.3. *Let T_α be the operator defined in* (1.7).

(i) *If $\alpha = -1/6$, then $\mathcal{T}_\alpha$ is the closed triangle with vertices $A = (0,0)$, $B = (1,1)$, $D = (1/2, 1/3)$.*

(ii) *If $\alpha = 1/6$, then $\mathcal{T}_\alpha$ is the closed triangle with vertices $A = (0,0)$, $B = (1,1)$, $C = (2/3, 1/2)$.*

Proof. By Lemma 5.1, $\mathcal{T}_{\pm 1/6}$ is contained in the closed trapezoid $ABCD$ and by Theorem 3.1, $\mathcal{T}_{-1/6}$ contains the closed triangle ABD and $\mathcal{T}_{1/6}$ contains the closed triangle ABC.

Let $E = (5/8, 1/2)$ be the point of intersection of the segment BD with the line $1/q = 1/2$. To prove assertion (i) it is sufficient to show that $\mathcal{T}_{-1/6}$ does not contain any point of this line on the right of E and this is equivalent to proving that the operator $T^*_{-1/6}T_{-1/6}$ cannot be bounded from $L^p(\mathbb{H}_1)$ to $L^{p'}(\mathbb{H}_1)$ if $1/p > 5/8$. Let f_ε be the characteristic function of the set

$$\{(x,y,t)\in\mathbb{H}_1 : |x|<\varepsilon,\ |y|<\varepsilon,\ |t|<\varepsilon^2\}$$

for a small positive ε. We have

$$(5.2)\quad T^*_{-1/6}T_{-1/6}f_\varepsilon(x,y,t)$$
$$= f_\varepsilon * \mu_{-1/6} * \check{\mu}_{-1/6}(x,y,t)$$
$$= \int_0^1\int_0^1 f_\varepsilon\left((x,y,t)\cdot\left(s,s^2,-\frac{s^3}{6}\right)\cdot\left(r,r^3,-\frac{r^3}{6}\right)^{-1}\right) dr\,ds$$
$$= \int_0^1\int_0^1 f_\varepsilon\left((x,y,t)\cdot\left(r-s,r^2-s^2,-\frac{1}{6}(r^3-s^3)+\frac{rs(r-s)}{2}\right)^{-1}\right) dr\,ds.$$

Making the change of variable $r-s=u$, $r+s=v$ in (5.2) we obtain

$$T^*_{-1/6}T_{-1/6}f_\varepsilon(x,y,t) > C\int_0^1\int_u^1 f_\varepsilon\left((x,y,t)\cdot\left(u,uv,-\frac{u^3}{6}\right)^{-1}\right) dv\,du.$$

For a fixed point (x,y,t) in the set

$$M_\varepsilon = \{(x,y,t)\in\mathbb{H}_1 : |x|<c_1\varepsilon,\ |y|<c_1\varepsilon,\ |t|<c_1\varepsilon^2\},$$

where c_1 is a small positive constant, we want to find a sufficiently small neighborhood V of u and v such that for $(u,v)\in V$ we have

$$(5.3)\quad |x-u|<\varepsilon,\qquad |y-uv|<\varepsilon,\qquad \left|t+\tfrac{1}{6}u^3+\tfrac{1}{2}(-xuv+yu)\right|<\varepsilon^2.$$

It is easy to see that if we consider $V=\{(v,u)\in\mathbb{R}^2 : 0<u<c\varepsilon,\ u<v<c\}$ for a positive constant c small enough, then inequalities (5.3) are satisfied for $(x,y,t)\in M_\varepsilon$ and $(v,u)\in V$. Therefore

$$T^*_{-1/6}T_{-1/6}f_\varepsilon(x,y,t) > \int_V dv\,du > C\varepsilon$$

on M_ε. Hence

$$\|T^*_{-1/6}T_{-1/6}f_\varepsilon\|_{L^{p'}(\mathbb{H}_1)} = \Big(\int_{\mathbb{H}_1} |T^*_{-1/6}T_{-1/6}f_\varepsilon(x,y,t)|^{p'}\,dx\,dy\,dt\Big)^{1/p'}$$
$$> C\varepsilon(m(M_\varepsilon))^{1/p'} = C\varepsilon^{1+4/p'},$$

while

$$\|f_\varepsilon\|_{L^p(\mathbb{H}_1)} = C\varepsilon^{4/p}.$$

If we impose the condition that $T^*_{-1/6}T_{-1/6}$ is bounded from $L^p(\mathbb{H}_1)$ to $L^{p'}(\mathbb{H}_1)$ we must have the inequality

$$\varepsilon^{1+4/p'} < C\varepsilon^{4/p}$$

for $\varepsilon<1$. This implies

$$\frac{1}{p}\le\frac{5}{8},$$

which concludes the proof of assertion (i).

Now let $F=(1/2,3/8)$ be the point of intersection of the segment AC with the line $1/p=1/2$. Arguing as in the proof of (i) we can prove that $\mathcal{T}_{1/6}$ does not contain any point of this line below F. This is equivalent to the fact that the operator $T_{1/6}T^*_{1/6}$ cannot be bounded from $L^{q'}(\mathbb{H}_1)$ to $L^q(\mathbb{H}_1)$ if $1/q<3/8$. Assertion (ii) is proved. ∎

6. Type set of U_σ. Using the results that we have obtained in the previous section, we now study the type set $\mathcal{U}_\sigma$ of the operator U_σ defined in (1.9).

THEOREM 6.1. *Let U_σ be the operator defined in* (1.9). *Then $\mathcal{U}_\sigma$ is contained in the closed quadrilateral with vertices*

$$A=(0,0),\qquad B=(1,1),\qquad P=\left(\frac{\sigma+1}{\sigma+3},\frac{\sigma}{\sigma+3}\right),\qquad D=\left(\frac{1}{2},\frac{1}{3}\right)$$

with the possible exception of the open segment DP and the point P.

Proof. First of all, adapting the arguments used in the proof of Lemma 5.1, we can prove that $\mathcal{U}_\sigma$ is contained in the closed trapezoid $ABCD$. Next we divide the proof into two steps.

(1) Assuming that U_σ is bounded from $L^p(\mathbb{H}_1)$ to $L^2(\mathbb{H}_1)$, we will prove that

$$\frac{1}{p}\le\frac{5\sigma-7}{4(2\sigma-3)},$$

modifying the argument used in the proof of Theorem 5.3.

For a small positive ε, let f_ε be the characteristic function of the set

$$\{(x,y,t)\in\mathbb{H}_1 : |x|<\varepsilon,\ |y|<\varepsilon^\alpha,\ |t|<\varepsilon^{\alpha+1}\}$$

for $1<\alpha<2$. We have

$$U^*_\sigma U_\sigma f_\varepsilon(x,y,t) = \int_0^\delta\int_0^\delta f_\varepsilon\left((x,y,t)\cdot\left(s,s^2,-\frac{s^3}{6}+s^\sigma\right)\right.$$
$$\left.\cdot\left(r,r^2,-\frac{r^3}{6}+r^\sigma\right)^{-1}\right) dr\,ds$$
$$= \int_0^\delta\int_0^\delta f_\varepsilon\left((x,y,t)\cdot\left(r-s,r^2-s^2,-\frac{r^3-s^3}{6}\right.\right.$$
$$\left.\left.+r^\sigma-s^\sigma+\frac{rs(r-s)}{2}\right)^{-1}\right) dr\,ds.$$

By the change of variable $r-s=u$, $s=v$, we can write

$$U_\sigma^* U_\sigma f_\varepsilon(x,y,t) > \int_0^{\delta/2}\int_0^{\delta/2} f_\varepsilon\Big((x,y,t)\cdot\Big(u, u(u+2v), (v+u)^\sigma - v^\sigma - \frac{u^3}{6}\Big)^{-1}\Big)\,dv\,du$$
$$= \int_0^{\delta/2}\int_0^{\delta/2} f_\varepsilon\Big(x-u, y-u(u+2v), t-(v+u)^\sigma + v^\sigma + \frac{u^3}{6} + \frac{yu - xu(u+2v)}{2}\Big)\,dv\,du.$$

Given a point (x,y,t) in the set

$$S_\varepsilon = \{(x,y,t)\in \mathbb{H}_1 : |x| < c_1\varepsilon,\ |y| < c_1\varepsilon^\alpha,\ |t| < c_1\varepsilon^{\alpha+1}\},$$

where c_1 is a small positive constant, we want to find a sufficiently small neighborhood V of u and v such that for $(u,v)\in V$ we have

$$\begin{gathered}|x-u| < \varepsilon, \quad |y - u(u+2v)| < \varepsilon^\alpha,\\ \big|t + \tfrac16 u^3 - (v+u)^\sigma + v^\sigma + \tfrac12 yu - \tfrac12 xu(u+2v)\big| < \varepsilon^{\alpha+1}.\end{gathered} \tag{6.1}$$

Suppose that

$$V = \{(u,v)\in [0,\delta/2]\times[0,\delta/2] : u < c\varepsilon,\ v < c\varepsilon^{\alpha-1}\}$$

for a positive constant c small enough. Then since $1<\alpha<2$, applying the triangular inequality we get

$$|x-u| \le |x| + u < \varepsilon, \quad |y - u(u+2v)| \le |y| + u^2 + 2uv < \varepsilon^\alpha.$$

Using the triangular inequality and the fact that

$$(v+u)^\sigma - v^\sigma = \sigma v^{\sigma-1}u + \sigma(\sigma-1)v^{\sigma-2}\frac{u^2}{2} + \sigma(\sigma-1)(\sigma-2)\tau^{\sigma-3}\frac{u^3}{6},$$

where $\tau\in(0,\delta/2)$, we get

$$\big|t + \tfrac16 u^3 - (v+u)^\sigma + v^\sigma + \tfrac12 yu - \tfrac12 xu(u+2v)\big| < \varepsilon^{\alpha+1}$$

if $\alpha \ge (\sigma-1)/(\sigma-2)$.

So if we take $\alpha = (\sigma-1)/(\sigma-2)$ then the inequalities (6.1) are satisfied on

$$S_\varepsilon = \{(x,y,t)\in\mathbb{H}_1 : |x| < c_1\varepsilon,\ |y| < c_1\varepsilon^{(\sigma-1)/(\sigma-2)},\ |t| < c_1\varepsilon^{(2\sigma-3)/(\sigma-2)}\}$$

and for $(u,v)\in V$, where

$$V = \{(u,v)\in[0,\delta/2]\times[0,\delta/2] : u < c\varepsilon,\ v < c\varepsilon^{1/(\sigma-2)}\}.$$

Therefore

$$U_\sigma^* U_\sigma f_\varepsilon(x,y,t) > \int_V du\,dv = C\varepsilon^{(\sigma-1)/(\sigma-2)}$$

when $(x,y,t)\in S_\varepsilon$. Hence

$$\begin{aligned}\|U_\sigma^* U_\sigma f_\varepsilon\|_{L^{p'}(\mathbb{H}_1)} &= \Big(\int_{\mathbb{H}_1} |U_\sigma^* U_\sigma f_\varepsilon(x,y,t)|^{p'}\,dx\,dy\,dt\Big)^{1/p'}\\ &> C\varepsilon^{(\sigma-1)/(\sigma-2)}(m(S_\varepsilon))^{1/p'}\\ &= C\varepsilon^{(\sigma-1)/(\sigma-2)+((4\sigma-6)/(\sigma-2))(1/p')},\end{aligned}$$

while

$$\|f_\varepsilon\|_{L^p(\mathbb{H}_1)} = C\varepsilon^{((4\sigma-6)/(\sigma-2))(1/p)}.$$

Requiring that $U_\sigma^* U_\sigma$ is bounded from $L^p(\mathbb{H}_1)$ to $L^{p'}(\mathbb{H}_1)$ we must have

$$\varepsilon^{(\sigma-1)/(\sigma-2)+((4\sigma-6)/(\sigma-2))(1/p')} \le C\varepsilon^{((4\sigma-6)/(\sigma-2))(1/p)}$$

for $\varepsilon < 1$. This implies

$$\frac{4\sigma-6}{\sigma-2}\cdot\frac1p \le \frac{\sigma-1}{\sigma-2} + \frac{4\sigma-6}{\sigma-2}\cdot\frac{1}{p'},$$

that is,

$$\frac1p \le \frac{5\sigma-7}{4(2\sigma-3)}.$$

(2) Since δ is sufficiently small, the curve $\Gamma_\sigma(s)$ defined in (1.8) satisfies condition (1.5) for all $s\in J$, therefore by Theorem 3.1, U_σ is bounded from $L^2(\mathbb{H}_1)$ to $L^3(\mathbb{H}_1)$.

We now make a dyadic decomposition of J into the intervals $J_j = [2^{-j-1}\delta, 2^{-j}\delta]$, $j = 0,1,\ldots$, and we define the measure

$$\langle \nu_{\sigma,j}, f\rangle = \int_{2^{-j-1}\delta}^{2^{-j}\delta} f\Big(s, s^2, -\frac{s^3}{6} + s^\sigma\Big)\,ds.$$

The corresponding right convolution operator $U_{\sigma,j}$ is

$$\begin{aligned}U_{\sigma,j}f(x,y,t) &= f * \nu_{\sigma,j}(x,y,t)\\ &= \int_{2^{-j-1}\delta}^{2^{-j}\delta} f\Big((x,y,t)\cdot\Big(s, s^2, -\frac{s^3}{6} + s^\sigma\Big)^{-1}\Big)\,ds,\end{aligned} \tag{6.2}$$

hence

$$U_\sigma = \sum_{j=0}^\infty U_{\sigma,j}.$$

Every operator $U_{\sigma,j}$ can be obtained from an operator $U_{\sigma,0}^{(j)}$ by dilation and with a multiplicative factor. In fact, by changing variable in (6.2) we get

$$U_{\sigma,j}f(x,y,t) = 2^{-j}D_{(2^j,2^{2j},2^{3j})}(U_{\sigma,0}^{(j)}D_{(2^{-j},2^{-2j},2^{-3j})}f)(x,y,t)$$

where

$$(6.3)\qquad U_{\sigma,0}^{(j)}f(x,y,t)=\int_{\delta/2}^{\delta} f\Big((x,y,t)\cdot\Big(s,s^2,-\frac{s^3}{6}+2^{(3-\sigma)j}s^\sigma\Big)^{-1}\Big)\,ds.$$

Hence

$$(6.4)\qquad \|U_\sigma f\|_{L^q(\mathbb{H}_1)} \le \sum_{j=0}^{\infty} 2^{-j}\|D_{(2^j,2^{2j},2^{3j})}(U_{\sigma,0}^{(j)}D_{(2^{-j},2^{-2j},2^{-3j})}f)\|_{L^q(\mathbb{H}_1)}$$
$$= \sum_{j=0}^{\infty} 2^{-(1+6/q)j}\|U_{\sigma,0}^{(j)}D_{(2^{-j},2^{-2j},2^{-3j})}f\|_{L^q(\mathbb{H}_1)}$$
$$\le \sum_{j=0}^{\infty} 2^{-(1+6/q-6/p)j}\|U_{\sigma,0}^{(j)}\|_{L^p(\mathbb{H}_1),L^q(\mathbb{H}_1)}\|f\|_{L^p(\mathbb{H}_1)}.$$

Since the curve $(s,s^2,-s^3/6+2^{(3-\sigma)j}s^\sigma)$ satisfies condition (1.4) for all $s\in J_0=[\delta/2,\delta]$, applying Theorem 3.1 we know that the operator $U_{\sigma,0}^{(j)}$ is bounded from $L^{3/2}(\mathbb{H}_1)$ to $L^2(\mathbb{H}_1)$, but we want to analyse how the $L^{3/2}(\mathbb{H}_1)$-$L^2(\mathbb{H}_1)$ norm of $U_{\sigma,0}^{(j)}$ depends on j. Therefore, following the proof of part (i) in Theorem 3.1, we consider the operator

$$(6.5)\qquad (U_{\sigma,0}^{(j)})^*U_{\sigma,0}^{(j)}f(x,y,t)=\int_{\delta/2}^{\delta}\int_{\delta/2}^{\delta} f\Big((x,y,t)\cdot\Big(s,s^2,-\frac{s^3}{6}+2^{(3-\sigma)j}s^\sigma\Big)\cdot\Big(r,r^2,-\frac{r^3}{6}+2^{(3-\sigma)j}r^\sigma\Big)^{-1}\Big)\,dr\,ds$$
$$=\int_{\delta/2}^{\delta}\int_{\delta/2}^{\delta} f\Big((x,y,t)\cdot\Big(r-s,r^2-s^2,-\frac{r^3-s^3}{6}+2^{(3-\sigma)j}(r^\sigma-s^\sigma)+\frac{rs(r-s)}{2}\Big)^{-1}\Big)\,dr\,ds.$$

By making in (6.5) the change of variables $r-s=u$, $s=v$ we get

$$(U_{\sigma,0}^{(j)})^*U_{\sigma,0}^{(j)}f(x,y,t)=\int_{-\delta/2}^{\delta/2}(f_{x-u}*_{\mathbb{R}^2}\Gamma_{\sigma,x,u,j})\Big(y,t+\frac12uy\Big)\,du$$

where

$$f_{x-u}(y,t)=f(x-u,y,t),$$
$$\Gamma_{\sigma,x,u,j}(v)=\Big(u(u+2v),2^{(3-\sigma)j}((v+u)^\sigma-v^\sigma)-\frac{u^3}{6}+\frac{xu(u+2v)}{2}\Big),\qquad v\in J_0(u),$$

$$J_0(u)=[\delta/2,\delta]\cap[-u+\delta/2,\delta-u]=\begin{cases}[\delta/2,\delta-u], & 0\le u\le\delta/2,\\ [-u+\delta/2,\delta], & -\delta/2\le u<0.\end{cases}$$

We need to prove that there exists a positive constant $C(j)$ which depends only on j and not on x and u such that

$$(6.6)\qquad \|g*_{\mathbb{R}^2}\Gamma_{\sigma,x,u,j}\|_{L^3(\mathbb{R}^2)}\le C(j)|u|^{-2/3}\|g\|_{L^{3/2}(\mathbb{R}^2)}$$

where g is a continuous compactly supported function on $\mathbb{R}^2$.

Adapting the argument used in the proof of part (i) of Theorem 3.1 we reduce ourselves to estimating the $L^3(\mathbb{R}^2)$ norm of $g*_{\mathbb{R}^2}\overline{\Gamma}_{\sigma,u,j}$ where $\overline{\Gamma}_{\sigma,u,j}$ is the curve

$$\overline{\Gamma}_{\sigma,u,j}(v)=(2v,2^{(3-\sigma)j}u((v+u)^\sigma-v^\sigma)),\qquad v\in J_0(u).$$

Since by the mean value theorem we have

$$\overline{\Gamma}''_{\sigma,u,j}(v)=\Big(0,2^{(3-\sigma)j}\sigma(\sigma-1)\frac{(v+u)^{\sigma-2}-v^{\sigma-2}}{u}\Big)$$
$$=(0,2^{(3-\sigma)j}\sigma(\sigma-1)(\sigma-2)\tau^{\sigma-3})$$

for $\tau\in[\delta/2,\delta]$, it follows that $|\overline{\Gamma}''_{\sigma,u,j}(v)|\ge c2^{(3-\sigma)j}$ for all $v\in J_0(u)$, and c is a positive constant which is independent of u and j. Therefore, applying Lemma 2.1 (here we have $\eta\equiv1$) we get

$$(6.7)\qquad \|g*_{\mathbb{R}^2}\overline{\Gamma}_{\sigma,x,u,j}\|_{L^3(\mathbb{R}^2)}\le C2^{(\sigma-3)j/3}\|g\|_{L^{3/2}(\mathbb{R}^2)}$$

where C is a positive constant which is independent of u,j, and (6.7) implies (6.6) with

$$(6.8)\qquad C(j)=C2^{(\sigma-3)j/3}.$$

So from (6.6) and (6.8) and from the $L^{3/2}(\mathbb{R})$-$L^3(\mathbb{R})$ boundedness of the Riesz potential of order 1/3 we have

$$(6.9)\qquad \|(U_{\sigma,0}^{(j)})^*U_{\sigma,0}^{(j)}f\|_{L^3(\mathbb{H}_1)}$$
$$=\Big\|\Big\|\int_{-\delta/2}^{\delta/2}(f_{x-u}*_{\mathbb{R}^2}\Gamma_{\sigma,x,u,j})\Big(y,t+\frac{uy}{2}\Big)\,du\Big\|_{L^3(\mathbb{R}^2)}\Big\|_{L^3(\mathbb{R})}$$
$$\le\Big\|\int_{-\delta/2}^{\delta/2}\Big\|(f_{x-u}*_{\mathbb{R}^2}\Gamma_{\sigma,x,u,j})\Big(y,t+\frac{uy}{2}\Big)\Big\|_{L^3(\mathbb{R}^2)}du\Big\|_{L^3(\mathbb{R})}$$
$$\le C2^{(\sigma-3)j/3}\Big\|\int_{-\delta/2}^{\delta/2}|u|^{-2/3}\|f_{x-u}\|_{L^{3/2}(\mathbb{R}^2)}\,du\Big\|_{L^3(\mathbb{R})}\le C2^{(\sigma-3)j/3}\|f\|_{L^{3/2}(\mathbb{H}_1)}.$$

From (6.9) we then obtain

$$\|U_{\sigma,0}^{(j)}f\|^2_{L^2(\mathbb{H}_1)}\le\|(U_{\sigma,0}^{(j)})^*U_{\sigma,0}^{(j)}f\|_{L^3(\mathbb{H}_1)}\|f\|_{L^{3/2}(\mathbb{H}_1)}\le C2^{(\sigma-3)j/3}\|f\|^2_{L^{3/2}(\mathbb{H}_1)},$$

that is,

$$\|U_{\sigma,0}^{(j)} f\|_{L^2(\mathbb{H}_1)} \le C 2^{(\sigma-3)j/6} \|f\|_{L^{3/2}(\mathbb{H}_1)}. \tag{6.10}$$

Now let $(1/p, 3/(2p)-1/2)$, $1/p \in (2/3, 1]$, be a point on the segment BC. We estimate the $L^p(\mathbb{H}_1)$-$L^{2p/(3-p)}(\mathbb{H}_1)$ norm of the operator $U_{\sigma,0}^{(j)}$ by interpolating between the estimates $L^{3/2}(\mathbb{H}_1)$-$L^2(\mathbb{H}_1)$, $L^1(\mathbb{H}_1)$-$L^1(\mathbb{H}_1)$. Let $t \in (0,1)$ be a value such that

$$\frac{1}{p} = \frac{1-t}{3/2} + \frac{t}{1},$$

that is,

$$t = \frac{3-2p}{p}.$$

Since the operator $U_{\sigma,0}^{(j)}$ is uniformly bounded on the diagonal AB and since (6.10) holds, applying the Riesz–Thorin interpolation theorem we get

$$\begin{aligned} &\|U_{\sigma,0}^{(j)}\|_{L^p(\mathbb{H}_1), L^{2p/(3-p)}(\mathbb{H}_1)} \\ &\quad \le \|U_{\sigma,0}^{(j)}\|^{1-t}_{L^{3/2}(\mathbb{H}_1), L^2(\mathbb{H}_1)} \|U_{\sigma,0}^{(j)}\|^{t}_{L^1(\mathbb{H}_1), L^1(\mathbb{H}_1)} \le C 2^{(\sigma-3)(p-1)j/(2p)}. \end{aligned} \tag{6.11}$$

Then by (6.4) and (6.11) we have

$$\|U_\sigma f\|_{L^{2p/(3-p)}(\mathbb{H}_1)} \le C \sum_{j=0}^{\infty} 2^{((2p-3)/p + (\sigma-3)(p-1)/(2p))j} \|f\|_{L^p(\mathbb{H}_1)}$$

and this series converges if

$$\frac{2p-3}{p} + \frac{(\sigma-3)(p-1)}{2p} < 0,$$

that is,

$$\frac{1}{p} > \frac{\sigma+1}{\sigma+3}.$$

On the segment BC consider the point

$$P = \left(\frac{\sigma+1}{\sigma+3}, \frac{\sigma}{\sigma+3}\right).$$

It is easy to see that P is just the intersection of the segment BC and the line joining the point $D = (1/2, 1/3)$ to the point

$$R = \left(\frac{5\sigma-7}{4(2\sigma-3)}, \frac{1}{2}\right).$$

Finally, considering the triangles PRC and DRC, notice that U_σ cannot be bounded from $L^{p_0}(\mathbb{H}_1)$ to $L^{q_0}(\mathbb{H}_1)$ if $(1/p_0, 1/q_0)$ is a point inside the triangle PRC (respectively if $(1/p_0, 1/q_0)$ is a point inside the triangle DRC) since otherwise, interpolating between D (respectively an appropriate point between B and P) and $(1/p_0, 1/q_0)$ we see that U_σ is bounded at some point $(1/p, 1/2)$ with

$$\frac{5\sigma-7}{4(2\sigma-3)} < \frac{1}{p} \le \frac{2}{3},$$

contrary to what we have proved in part (1). This concludes the proof of Theorem 6.1. ∎

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