A quasi-nilpotent operator with reflexive commutant, II

by

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Abstract. A new example of a non-zero quasi-nilpotent operator $T$ with reflexive commutant is presented. The norms $\|T^n\|$ converge to zero arbitrarily fast.

Let $H$ be a complex separable Hilbert space and let $B(H)$ denote the algebra of all continuous linear operators on $H$. If $T \in B(H)$ then $\{T\}' = \{A \in B(H) : AT = TA\}$ is called the commutant of $T$. By a subspace we always mean a closed linear subspace. If $A \subset B(H)$ then Alg $A$ denotes the smallest weakly closed subalgebra of $B(H)$ containing the identity $I$ and $A$, and Lat $A$ denotes the set of all subspaces invariant for each $A \subset A$. If $\mathcal{L}$ is a set of subspaces of $H$, then Alg $\mathcal{L} = \{T \in B(H) : \mathcal{L} \subset \text{Lat}(T)\}$. $T$ is said to be hyperreflexive if $\{T\}' = \text{Alg Lat}(T)'$, i.e., if the algebra $\{T\}'$ is reflexive.

It can be shown (see [1]) that if $T$ is a nilpotent hyperreflexive operator on a separable Hilbert space then $T = 0$. This is not true for quasi-nilpotent operators. An example of a non-zero quasi-nilpotent hyperreflexive operator was given in [5] using a modification of an idea of Wogen [4]. The powers in the example converged to zero slowly; more precisely, the following inequality was true for all positive integers:

$$\|T^n\|^{1/n} \geq 1/\log n.$$

In [6] it was shown that the convergence of the powers of $T$ to zero can be faster, namely for each $p > 0$ there exists a non-zero hyperreflexive operator $T$ for which

$$\|T^n\|^{1/n} \leq 1/n^p.$$

The aim of this note is to show that the convergence $\|T^n\|^{1/n} \to 0$ can be arbitrarily fast:

1991 Mathematics Subject Classification: 47A10, 47A15.

Key words and phrases: quasi-nilpotent operator, commutant, reflexivity.

The first author was supported by the grant No. 201/96/0411 of GA CR. The second author was supported by grant 1/4997/97 of the Grant Agency of Slovak Republic.
Theorem 1. Let \((\beta_n)_{n \geq 1}\) be a sequence of positive numbers. Then there exists a non-zero hyperreflexive operator \(T\) on a separable Hilbert space \(H\) such that \(\|T^n\|^{1/n} \leq \beta_n\) for all \(n \geq 1\).

Proof. The set of all non-negative integers will be denoted by \(\mathbb{N}\). Set formally \(\beta_0 = 1\). Without loss of generality we can assume that \(1 = \beta_0 \geq \beta_1 \geq \beta_2 \geq \ldots\) (if necessary, we can replace \(\beta_n\) by \(\min\{\beta_j : 0 \leq j \leq n\}\)).

For \(k = 0, 1, \ldots\) set \(m_k = 3k(k+1)+1\). For \(n \in \mathbb{N}\) let \(f(n) = \min\{k : m_k > n\}\). Thus \(f(n) = k\) if and only if \(m_{k-1} \leq n < m_k\).

Finally, set \(s_0 = 1\) and, for \(k, j \in \mathbb{N}\) with \(j^2 < k \leq (j+1)^2\), set

\[ s_k = \min \left\{ \frac{1}{f(n)} \frac{\beta_n^{f(n)} - \beta_n^{f(n)+1}}{\beta_n^2} : 0 \leq n \leq m_{(j+1)^2} \right\}. \]

Clearly, \(1 = s_0 \geq s_1 \geq s_2 \geq \ldots\). Further, \(s_{j+1} = s_{j+2} = \ldots = s_{(j+1)^2}\) so that the sequence \((s_n)\) contains constant subsequences of arbitrary length.

If \(n \in \mathbb{N}\), \(f(n) = k\) and \(j^2 < k \leq (j+1)^2\) then \(m_{k-1} \leq n < m_k \leq m_{(j+1)^2}\) so that

\[ s_f(n) \leq \frac{1}{f(n)} \frac{\beta_n^{f(n)} - \beta_n^{f(n)+1}}{\beta_n^2} \quad (n \in \mathbb{N}). \]

Now let \(R\) be a complex Hilbert space with \(\dim R = 2\). Let \(\{a, b\}\) be its orthonormal basis and let \(c = \frac{1}{\sqrt{2}}(a + b), d = \frac{1}{\sqrt{2}}(a - b)\). Note that \(\{c, d\}\) is also an orthonormal basis of \(R\).

For \(x \in R\), \(x \neq 0\), we denote by \(P_a\) the orthogonal projection in \(B(R)\) onto the one-dimensional space spanned by \(\{x\}\). For any integer \(n \geq 0\) write

\[
A_n = (I - P_a) + s_0 s_1 \ldots s_n P_a = P_b + s_0 s_1 \ldots s_n P_a,
B_n = (I - P_a) + s_0 s_1 \ldots s_n P_a = P_b + s_0 s_1 \ldots s_n P_b,
C_n = (I - P_a) + s_0 s_1 \ldots s_n P_a = P_d + s_0 s_1 \ldots s_n P_c.
\]

Note that \(A_0 = B_0 = C_0 = I\). Define the sequence \((R_n)_{n \geq 0}\) of operators in \(B(R)\) as follows:

\[
I, A_1, I, A_1, I, A_1, A_2, A_1, I, B_1, B_2, B_1, I, C_1, C_2, C_1,
I, A_1, A_2, A_3, A_2, \ldots
\]

More precisely, if \(i, k \in \mathbb{N}\) then

\[
R_n = \begin{cases} 
A_i & \text{if } n = m_k + i, \ 0 \leq i \leq k + 1, \\
B_i & \text{if } n = m_k + 2(k+1) - i, \ 1 \leq i \leq k, \\
B_i & \text{if } n = m_k + 2(k+1) + i, \ 0 \leq i \leq k + 1, \\
C_i & \text{if } n = m_k + 4(k+1) - i, \ 1 \leq i \leq k, \\
C_i & \text{if } n = m_{k+1} - i, \ 1 \leq i \leq k.
\end{cases}
\]

For \(n \in \mathbb{N}\) set \(g(n) = i\) if and only if \(R_n \in \{A_i, B_i, C_i\}\). By the definition of \(f(n)\) we have \(g(n) \leq f(n)\) for all \(n \geq 0\).

Note that \(R_n\) is invertible, \(\|R_n\| = 1\) and

\[
\|R_n + \frac{1}{s_f(n)} R_n^{-1}\| = \max \left\{ 1, \frac{s_0 s_2 \ldots s_{g(n) + 1}}{s_0 s_2 \ldots s_{g(n)}} \right\}
\]

where \(|g(n + 1) - g(n)| = 1\). If \(g(n + 1) > g(n)\) then \(\|R_{n+1} R_n^{-1}\| \leq 1\). If \(g(n + 1) < g(n)\) then \(\|R_{n+1} R_n^{-1}\| \leq 1/s_f(n)\leq 1/s_f(n)\). Thus \(\|R_{n+1} R_n^{-1}\| \leq 1/s_f(n)\) \((n \in \mathbb{N})\). For \(0 \leq i < j\) we have

\[
\|R_j R_i^{-1}\| \leq \|R_j R_i^{-1}\| \|R_{j-1} R_i^{-1}\| \|R_{i+1} R_i^{-1}\| \ldots \|R_{j+1} R_i^{-1}\| \leq 1/\frac{s_f(j-1) s_f(j-2) \ldots s_f(i)}{s_f(i+1) s_f(i+2) \ldots s_f(j+1)}.
\]

Let \(H\) be the orthogonal sum of infinitely many copies of \(R\):

\[
H = R \oplus R \oplus \ldots
\]

For \(n \geq 0\) set

\[
\alpha_n = s_f(n) \beta_n^{n+1}/\beta_n^n \quad \text{and} \quad T_n = \alpha_n R_{n+1} R_n^{-1}.
\]

Let \(T \in B(H)\) be the weighted shift with weights \(T_n\),

\[
T(x_0 \oplus x_1 \oplus \ldots) = 0 \oplus T_0 x_0 \oplus T_1 x_1 \oplus \ldots
\]

We show that \(T\) satisfies the required conditions.

Let \(n \geq 1\). Then

\[
T^n \left( \bigoplus_{i=0}^\infty x_i \right) = \bigoplus_{i=0}^\infty x_i \oplus \bigoplus_{i=0}^\infty \alpha_i \alpha_{i+1} \ldots \alpha_{i+n-1} R_{n+i} R_i^{-1} x_i.
\]

Thus

\[
\|T^n\| = \sup_i \|\alpha_i \alpha_{i+1} \ldots \alpha_{i+n-1} R_{n+i} R_i^{-1}\|^{1/n} \leq \sup_i \frac{s_f(i+1) s_f(i+2) \ldots s_f(i+n)}{s_f(i+1) s_f(i+2) \ldots s_f(i)} \frac{\beta_{i+n}}{\beta_i} \frac{\beta_{i+n}}{\beta_{i+n-1}} \leq \sup_i \frac{\beta_{i+n}}{\beta_i} \leq \sup_i \frac{\beta_{i+n}}{\beta_{i+n}} \leq \beta_n.
\]

Hence

\[
\|T^n\|^{1/n} \leq \beta_n \quad (n \geq 1).
\]

The above-defined operator-weighted shift \(T\) is reflexive since it has injective weights of dimension 2 [2, Corollary 3.5]. We shall show that \(\{T\}' = \text{Alg} T\) and then \(T\) is also hyperreflexive. Similarly to [5, p. 281]...
let \((U_{ij})_{i,j \geq 0}\) be the matrix of an operator \(U \in \{T^i\}^\prime\) in the decomposition (1). Then

\[
0 = TU - UT = \begin{pmatrix}
-U_{01}T_0 & -U_{02}T_1 & -U_{03}T_2 & \cdots \\
T_0U_{00} - U_{11}T_0 & T_0U_{01} - U_{12}T_1 & T_0U_{02} - U_{13}T_2 & \cdots \\
T_1U_{10} - U_{21}T_0 & T_1U_{11} - U_{22}T_1 & T_1U_{12} - U_{23}T_2 & \cdots \\
T_2U_{20} - U_{31}T_0 & T_2U_{21} - U_{32}T_1 & T_2U_{22} - U_{33}T_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Since \(T_n\)'s are invertible we obtain from the first row \(U_{0i} = 0\) for all \(i \geq 1\). Similarly we obtain by induction \(U_{ij} = 0\) if \(i < j\), i.e., the matrix \(U\) is lower triangular.

Further, for \(i \geq j \geq 1\), we have \(T_{i-1}U_{i-1,j-1} - U_{ij}T_{j-1} = 0\) so that

\[
U_{ij} = T_{i-1}U_{i-1,j-1}T_{j-1}^{-1}.
\]

Thus for \(i, n \geq 0\) we have by induction

\[
U_{n+i,n} = T_{n+i-1}T_{n+i-2} \cdots T_0U_{0i}T_0^{-1} \cdots T_{i-1}^{-1} = (T_{n+i-1}T_{n+i-2} \cdots T_0)S_i(T_{n-1}T_{n-2} \cdots T_0)^{-1}
\]

\[
= \alpha_n\alpha_{n-1} \cdots \alpha_{n+i-1}R_{n+i+1}R_n^{-1},
\]

where \(S_i = (T_{i-1}T_{i-2} \cdots T_0)^{-1}U_{00}\).

We are now going to show that each \(S_i\) is a scalar multiple of identity. Fix \(i \geq 0\). Suppose that \(s_i = \lambda_i a + \mu_i b\). To show that \(\mu_i = 0\) find \(k \in \mathbb{N}, k > i\), such that \(s_k = s_{k-1} = \ldots = s_{k-i}\). Let \(n = m_{k-1} + k\). Then \(R_n = A_k, R_{n+i} = A_{k+i}, f(n) = f(n+1) = \ldots = f(n+i) = k\) and we have

\[
\|U\| \geq \|U_{n+i,n}^0\| \geq \|U_{n+i,n}a\| = \alpha_n\alpha_{n-1} \cdots \alpha_{n+i-1}\|R_{n+i}S_iR_n^{-1}a\|
\]

\[
= \alpha_n\alpha_{n-1} \cdots \alpha_{n+i-1}\|A_{k+i}(\lambda_i a + \mu_i b)\|
\]

\[
= \alpha_n\alpha_{n-1} \cdots \alpha_{n+i-1}\|s_0s_1 \cdots s_{k-i}\lambda_i a + \mu_i b\|
\]

\[
\geq \|\mu_i\| \alpha_n\alpha_{n+1} \cdots \alpha_{n+i-1}\|s_0s_1 \cdots s_k\lambda_i a + \mu_i b\|
\]

\[
\geq \|\mu_i\| \frac{s_0s_1 \cdots s_k}{s_0s_1 \cdots s_k} \beta_n^{n+i} = \|\mu_i\| \frac{1}{s_0s_1 \cdots s_k} \beta_n^{n+i} \geq \|\mu_i\| k.
\]

Since \(k\) could have been chosen arbitrarily large, we conclude that \(\mu_i = 0\). Thus \(S_i a = \lambda_i a\). Similarly (for \(n = m_{k-1} + 3k\) and \(n = m_{k-1} + 5k\), respectively) we can prove that \(S_i b = \lambda_i^2 b\) and that \(S_i c = \lambda_i^2 c\) for some complex numbers \(\lambda_i, \lambda_i^2\). Thus

\[
\frac{1}{\sqrt{2}} \lambda_i^2(a + b) = \lambda_i^2 c = S_i c = S_i \left(\frac{1}{\sqrt{2}} (a + b)\right) = \frac{1}{\sqrt{2}} \lambda_i a + \frac{1}{\sqrt{2}} \lambda_i^2 b.
\]

Thus \(\lambda_i = \lambda_i^2\), i.e., \(S_i = \lambda_i^2 I\). Hence \(U_{n+i,n} = \lambda_i T_{n+i-1}T_{n+i-2} \cdots T_n\) for all \(i, n \geq 0\).

Observe that the only non-zero entries of the matrix of the operator \(T^i\) are \(\lambda_i T_{n+i,n} = T_{n+i-1}T_{n+i-2} \cdots T_n\) for \(n = 0, 1, 2, \ldots\) and so formally \(U = \sum \lambda_i T^i\).

The rest of the proof is exactly the same as that of Lemma 2.3 in [3]. The operator \(U\) can be written as a formal power series \(\sum \lambda_i T^i\). The series need not converge but its Cesàro means converge to \(U\) strongly. So the commutant of \(T\) coincides with \(\text{Alg} T\) and therefore it is reflexive. This finishes the proof of Theorem 1.

References


