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Functional calculi, regularized semigroups and integrated semigroups

by

RALPH DELAUBENFELS (Athens, Ohio) and
MUSTAPHA JAZAR (Beyrouth)

Abstract. We characterize closed linear operators A, on a Banach space, for which the corresponding abstract Cauchy problem has a unique polynomially bounded solution for all initial data in the domain of A^n, for some nonnegative integer n, in terms of functional calculi, regularized semigroups, integrated semigroups and the growth of the resolvent in the right half-plane. We construct a semigroup analogue of a spectral distribution for such operators, and an extended functional calculus: When the abstract Cauchy problem has a unique $O(1+t^k)$ solution for all initial data in the domain of A^n, for some nonnegative integer n, then a closed operator $f(A)$ is defined whenever f is the Laplace transform of a derivative of any order, in the sense of distributions, of a function F such that $t \mapsto (1+t^k)F(t)$ is in $L^1([0,\infty))$. This includes fractional powers. In general, A is neither bounded nor densely defined.

0. Introduction. Stone's theorem, for possibly unbounded linear operators on a Hilbert space, states that the operator A is self-adjoint if and only if iA generates a strongly continuous group of isometries. This is surprising, in that the existence of $f_t(A)$ for the particular family of functions

$$f_t(s) \equiv e^{its} \quad (s, t \in \mathbb{R})$$

(so that $\{f_t(A)\}_{t\in\mathbb{R}}$ becomes a strongly continuous group generated by iA) guarantees, via the functional calculus produced by the spectral theorem, the existence of $f(A)$ for f any everywhere-bounded Borel measurable function. From a practical point of view, the strongly continuous group promises only existence and uniqueness of solutions of the corresponding abstract Cauchy problem, while a functional calculus may produce many explicit constructions of the solutions.

On a general Banach space, one is not guaranteed such a large functional calculus for the generator of a bounded strongly continuous group. However, equivalences similar to Stone's theorem, between generating a bounded

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strongly continuous group, or, more generally, an $O(t^k)$ k-times integrated group, and having certain functional calculi, still hold; see [Balab-E-J] and [E-J]. The spectral measure of the spectral theorem must be replaced by the more general notion of a spectral distribution.

For bounded strongly continuous *semi*groups, even on a Hilbert space, the generator may have a functional calculus defined only for appropriate holomorphic functions. For example, take A to be multiplication by $z-1$ on the Hardy space $H^2(D)$, where D is the open unit disc in the complex plane:

$$(Ag)(z) \equiv (z-1)g(z) \qquad (|z|<1,\ g \in H^2(D)).$$

Then A generates the strongly continuous semigroup of contractions

$$(T(t)g)(z) \equiv e^{t(z-1)}g(z) \qquad (|z|<1,\ g \in H^2(D)).$$

At least if $\mathcal{F}$ is a Banach algebra of complex-valued functions on the spectrum of A in which the polynomials are dense, an $\mathcal{F}$ functional calculus for A must have the form

$$(f(A))g \equiv fg \qquad (g \in H^2(D)).$$

Clearly, the operator $f(A)$ is defined only for functions f holomorphic in $D-1$.

A characterization of generators of bounded strongly continuous semigroups in terms of functional calculi appears in [d3], and is used there to give simple proofs of standard results such as the Hille–Yosida–Phillips theorem.

Generation of a bounded strongly continous semigroup corresponds to the existence and uniqueness of bounded mild solutions of the abstract Cauchy problem, for all initial data. In this paper, we consider a much larger class of operators, those operators A for which the abstract Cauchy problem has a unique polynomially bounded mild solution for all initial data in the domain of A^n, for some nonnegative integer n. When the resolvent of A is nonempty, the existence of these solutions is equivalent to A generating a polynomially bounded $(\lambda-A)^{-n}$-regularized semigroup or n-times integrated semigroup. We give characterizations of such operators in terms of functional calculi and in terms of the resolvent of A. As in [d3], our characterizations do not require that the domain of A be dense.

We give preliminary material on functional calculi, integrated semigroups, regularized semigroups and the abstract Cauchy problem in Section I. Characterizations of operators A for which the abstract Cauchy problem has a unique polynomially bounded solution, for all initial data in the domain of A^n, in terms of resolvent, functional calculi, regularized semigroups and integrated semigroups, are in Section II. Section III presents an analogue of spectral distribution, that we call *semispectral distribution*, that plays the same role for (integrated) semigroups that spectral distributions play for (integrated) groups. Section IV extends our functional calculi to a much larger family of functions. As a special case of the extended functional calculus in Section IV, we construct fractional powers in Section V.

I. Preliminaries. Throughout this paper, all operators are linear, on a Banach space X. Denote by $\mathcal{D}(A)$ the domain of the operator A, by $\operatorname{Im}(A)$ its image, by $\sigma(A)$ the spectrum of A, by $\varrho(A)$ its resolvent set. Denote by $B(X)$ the space of all bounded operators from X to itself.

See [d1], and the references therein, for basic material on regularized and integrated semigroups, including their history. See [G] or [P] for basic material on strongly continuous semigroups and their applications.

DEFINITION 1.1. Suppose $\mathcal{F}$ is a Banach algebra of complex-valued functions defined on a subset of the complex plane, not containing the constant functions, and there exists complex λ such that $g_\lambda(z) \equiv (\lambda-z)^{-1} \in \mathcal{F}$. Then an $\mathcal{F}$ *functional calculus* for A is a continuous algebra homomorphism $f \mapsto f(A)$, from $\mathcal{F}$ into $B(X)$, such that $\lambda \in \varrho(A)$ whenever $g_\lambda \in \mathcal{F}$ and

$$g_\lambda(A) = (\lambda - A)^{-1} \qquad (g_\lambda \in \mathcal{F}).$$

Note that when $\mathcal{G}_\lambda$, defined to be the algebra (without unit) spanned by g_λ, is dense in $\mathcal{F}$, then it is sufficient to show that

$$\|f(A)\| \le M\|f\|_{\mathcal{F}}, \qquad \forall f \in \mathcal{G}_\lambda,$$

where

$$f(A) \equiv \sum_{k=0}^{m} \alpha_k(\lambda-A)^{-k} \quad \text{ when } f \equiv \sum_{k=0}^{m} \alpha_k (g_\lambda)^k;$$

the unique bounded extension of $f \mapsto f(A)$, from $\mathcal{G}_\lambda$ to $\mathcal{F}$, will automatically be an algebra homomorphism.

Strongly continuous semigroups correspond to the *abstract Cauchy problem*

$$\text{(ACP)} \qquad \frac{d}{dt}u(t,x) = A(u(t,x)) \quad (t \ge 0), \qquad u(0,x) = x,$$

being well-posed. By a *mild solution* of (ACP) we will mean $u \in C([0,\infty), X)$ such that, for all $t \ge 0$, $t \mapsto \int_0^t u(s,x)\,ds \in \mathcal{D}(A)$, with

$$u(t,x) = x + A\Big(\int_0^t u(s,x)\,ds\Big) \qquad (t \ge 0).$$

For dealing with ill-posed abstract Cauchy problems, two generalizations of strongly continuous semigroups have recently appeared.

DEFINITION 1.2. Suppose $n \in \mathbb{N}$. An *n-times integrated semigroup* is a strongly continuous family $\{S(t)\}_{t\geq 0} \subseteq B(X)$ such that $S(0)=0$ and

$$S(t_1)S(t_2)x = \frac{1}{(n-1)!}\Big[\int_{t_1}^{t_1+t_2} (t_1+t_2-r)^{n-1}S(r)x\,dr - \int_0^{t_2} (t_1+t_2-r)^{n-1}S(r)x\,dr\Big],$$

for all $x \in X$ and $t_1, t_2 \geq 0$. The algebraic properties are precisely the properties of the real-valued function $t \mapsto J^n(e^{st})$, where $Jf(s) \equiv \int_0^s f(r)\,dr$.

The *generator* is defined by

$$\mathcal{D}(A) = \Big\{x \Bigm| \exists y \text{ such that } S(t)x = \frac{t^n}{n!}x + \int_0^t S(r)y\,dr\ \forall t\geq 0\Big\}, \quad \text{with } Ax = y.$$

If ω is a real number, then the strongly continuous $O(e^{\omega t})$ family $\{S(t)\}_{t\geq 0} \subseteq B(X)$ is an n-times integrated semigroup generated by A if and only if $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > \omega\} \subseteq \varrho(A)$ with

$$(z-A)^{-1}x = z^n \int_0^\infty e^{-zt}S(t)x\,dt \quad \text{whenever } \mathrm{Re}(z) > \omega,\ x \in X.$$

DEFINITION 1.3. If C is injective, then the strongly continuous family $\{W(t)\}_{t\geq 0} \subseteq B(X)$ is a *C-regularized semigroup generated by A* if $W(0)=C$, $W(s)W(t) = CW(t+s)$ for all $s,t\geq 0$, and

$$Ax = C^{-1}\Big(\frac{d}{dt}W(t)x\Big|_{t=0}\Big),$$

with maximal domain.

It is convenient to think of $W(t)$ as $e^{tA}C$, and $t \mapsto u(t,x)$, the mild solution of (ACP), as $e^{tA}x$.

The generator of even a bounded C-regularized semigroup may have empty resolvent; what is called the *C-resolvent* plays a role analogous to the resolvent. The complex number λ is in $\varrho_C(A)$, the C-resolvent of A, if $\lambda - A$ is injective and $(\lambda-A)^{-1}C \in B(X)$.

If $\{W(t)\}_{t\geq 0}$ is $O(e^{\omega t})$, for some real ω, then $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > \omega\} \subseteq \varrho_C(A)$ with

$$(z-A)^{-1}Cx = \int_0^\infty e^{-zt}W(t)x\,dt \quad \text{whenever } \mathrm{Re}(z) > \omega,\ x \in X.$$

The converse is true when $\varrho(A)$ is nonempty; that is, if $\varrho(A)$ is nonempty and ω is a real number, then the strongly continuous $O(e^{\omega t})$ family $\{W(t)\}_{t\geq 0} \subseteq B(X)$ is a C-regularized semigroup generated by A if and only if the Laplace transformation above holds.

When A generates a C-regularized semigroup, then (ACP) has a unique mild solution for all $x \in \mathrm{Im}(C)$. When $\varrho(A)$ is nonempty, the converse is true. An n-times integrated semigroup corresponds to C chosen to be $(\lambda - A)^{-n}$; this is what we will focus on in this paper.

PROPOSITION 1.4. *The following are equivalent, if $\lambda \in \varrho(A)$ and n is a nonnegative integer.*

(a) *A generates an n-times integrated semigroup $\{S(t)\}_{t\geq 0}$.*
(b) *A generates a $(\lambda-A)^{-n}$-regularized semigroup $\{W(t)\}_{t\geq 0}$.*
(c) *(ACP) has a unique mild solution for all $x \in \mathcal{D}(A^n)$.*

Then for any $x \in \mathcal{D}(A^n)$, we have

$$u(t,x) = W(t)(\lambda-A)^n x \quad \text{and} \quad J^n(u(t,x)) = S(t)x \quad (t\geq 0).$$

In fact, under the equivalent conditions of Proposition 1.4, *any* mild solution of (ACP) then has the form $u(t,x) = (\lambda-A)^n W(t)x$.

It is clear from the representation of the solutions that (ACP) has a unique bounded mild solution for all $x \in \mathcal{D}(A^n)$ if and only if A generates a bounded $(\lambda-A)^{-n}$-regularized semigroup. It is not clear what type of integrated semigroup would correspond to the existence of such solutions.

One may combine regularizing and integrating (see [L], [L-Sha] and [W]). A once-integrated regularized semigroup turns out to be natural when the domain of the generator is not dense.

DEFINITION 1.5. If $C \in B(X)$ is injective, then the strongly continuous family $\{W(t)\}_{t\geq 0} \subseteq B(X)$ is a *once-integrated C-regularized semigroup generated by A* if $W(0)=0$,

$$W(t_1)W(t_2)x = \int_0^{t_2} [W(r+t_1) - W(r)]Cx\,dr \quad \forall x \in X,$$

$$\mathcal{D}(A) = \Big\{x \Bigm| \exists y \text{ such that } W(t)x = tCx + \int_0^t W(r)y\,dr\ \forall t\geq 0\Big\}, \quad \text{with } Ax = y.$$

If $\{W(t)\}_{t\geq 0}$ is $O(e^{\omega t})$, for some real ω, then $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > \omega\} \subseteq \varrho_C(A)$ and

$$(z-A)^{-1}Cx = z\int_0^\infty e^{-zt}W(t)x\,dt \quad \text{whenever } \mathrm{Re}(z) > \omega,\ x \in X.$$

The following is an immediate consequence of [L-Sha, Corollary 7.6].

PROPOSITION 1.6. *If $\lambda \in \varrho(A)$, n is a nonnegative integer and A generates a once-integrated $(\lambda - A)^{-n}$-regularized semigroup $\{W(t)\}_{t\geq 0}$, then* (ACP) *has a unique mild solution, for all $x \in \mathcal{D}(A^{n+1})$, given by*

$$u(t,x) = W(t)(\lambda - A)^n Ax + x.$$

The proof of the following is in [Q-Liu, Lemma 6.1].

LEMMA 1.7. *If $\varrho(A)$ is nonempty, then for any $\lambda \in \varrho(A)$ and $n \in \mathbb{N}$,*

$$\varrho(A) = \varrho_{(\lambda - A)^{-n}}(A).$$

II. Functional calculi and polynomially bounded regularized semigroups. In this section, we characterize generators of polynomially bounded $(\lambda - A)^{-n}$-regularized semigroups (or Lipschitz-continuous once-integrated $(\lambda - A)^{-n}$-regularized semigroups, if $\mathcal{D}(A)$ is not dense) in terms of functional calculi and resolvent. The connection with the abstract Cauchy problem is in Propositions 1.4 and 1.6. The main results of this section are Theorems 2.4, 2.6, 2.7 and 2.9.

We think of a C-regularized semigroup generated by A as $\{e^{tA}C\}_{t\geq 0}$. We wish to construct $f(-A)$ for functions f of the form

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad (s \geq 0). \tag{2.1}$$

F is the *determining function* for its Laplace transform f.

Replacing s with $-A$ and regularizing both sides of (2.1) with $C \equiv (1-A)^{-n}$ gives us, informally,

$$\left(s \mapsto \frac{f(s)}{(1+s)^n}\right)(-A) = f(-A)(1-A)^{-n} = \int_0^\infty (e^{tA}(1-A)^{-n}) F(t)\, dt.$$

Thus if A generates a bounded $(1-A)^{-n}$-regularized semigroup, we have a functional calculus defined for functions in the set

$$\left\{\left(s \mapsto \frac{f(s)}{(1+s)^n}\right) \,\middle|\, F \in L^1([0,\infty))\right\};$$

this equals the set of all Laplace transforms of functions in $W_0^{1,n}((0,\infty))$ (see Lemma 2.3(2)).

Thus we are led to the following Banach algebras.

DEFINITION 2.2. Define $W^{1,0}([0,\infty)) \equiv L^1([0,\infty))$, and for $n \in \mathbb{N}$,

$$W^{1,n}([0,\infty)) \equiv \{F \in C^{n-1}([0,\infty)) \mid F^{(j)} \in L^1([0,\infty)) \text{ for } j = 0,1,\dots,n\}.$$

For n a nonnegative integer, we will denote by $\mathcal{A}_n$ the Banach algebra of

Laplace transforms of functions in $W^{1,n}([0,\infty))$. We topologize $\mathcal{A}_n$ by

$$\|f\|_{\mathcal{A}_n} \equiv \sum_{j=0}^n \frac{1}{j!} \|F^{(j)}\|_{L^1([0,\infty))}$$

for f as in (2.1).

LEMMA 2.3. *For any nonnegative integer n,*

(1) *$\mathcal{P} \equiv \{t \mapsto p(t)e^{-t} \mid p$ is a polynomial$\}$ is dense in $W^{1,n}([0,\infty))$,*

(2) *there exist complex $\{\alpha_{j,k}\}_{j,k=0}^{n-1}$ so that for any $F \in W^{1,n}([0,\infty))$ and $s > 0$,*

$$\int_0^\infty e^{-st}F(t)\,dt = \sum_{k=0}^{n-1}\Big[\sum_{j=0}^{n-1} \alpha_{j,k} F^{(j)}(0)\Big](1+s)^{-k} + (1+s)^{-n}\int_0^\infty e^{-st}\left[\left(1+\frac{d}{dt}\right)^n F\right](t)\,dt.$$

Proof. (1) Let D be the generator of left-translation on $L^1([0,\infty))$. Then $(1-D)^n$ is a homeomorphism of $W^{1,n}([0,\infty))$ onto $L^1([0,\infty))$. Since $1-D$ maps $\mathcal{P}$ onto itself, it is sufficient to prove (1) when $n = 0$. So suppose $h \in L^\infty([0,\infty))$ annihilates $\mathcal{P}$; that is,

$$0 = \int_0^\infty t^k e^{-t} h(t)\,dt \qquad \text{for } k = 0,1,2,\dots$$

By [Sho-T, Theorem 1.10], $h = 0$ a.e., as desired.

Assertion (2) follows from integration by parts. ■

THEOREM 2.4. *The following are equivalent, if $\varrho(A)$ is nonempty and n is a nonnegative integer.*

(a) *$(0,\infty) \subseteq \varrho(A)$ and there exists a constant M such that*

$$\|\lambda^j(\lambda - A)^{-j}(1-A)^{-n}\| \leq M, \qquad \forall \lambda > 0,\ j \in \mathbb{N}.$$

(b) *$(0,\infty) \subseteq \varrho(A)$ and for all $x \in X$ and $x^* \in X^*$, there exists $T_{x,x^*} \in L^\infty([0,\infty))$ such that*

$$\langle(\lambda - A)^{-1}(1-A)^{-n}x, x^*\rangle = \int_0^\infty e^{-\lambda t} T_{x,x^*}(t)\,dt \qquad (\lambda > 0).$$

(c) *$-A$ has an $\mathcal{A}_n$ functional calculus.*

(d) *A generates a Lipschitz-continuous once-integrated $(1-A)^{-n}$-regularized semigroup.*

We then have

$$(*)\qquad \langle f(-A)x,x^*\rangle=\sum_{k=0}^{n-1}\Big[\sum_{j=0}^{n-1}\alpha_{j,k}F^{(j)}(0)\Big]\langle(1-A)^{-k}x,x^*\rangle+\int_0^\infty\Big[\Big(1+\frac{d}{dt}\Big)^nF\Big](t)T_{x,x^*}(t)\,dt,$$

for $\{\alpha_{j,k}\}_{j,k=0}^{n-1}$ *as in Lemma* 2.3(2), $f\in\mathcal{A}_n$ *as in* (2.1), $x\in X$ *and* $x^*\in X^*$.

If $\mathcal{D}(A)$ *is dense, then* (a)–(d) *are equivalent to the following.*

(e) A *generates a bounded* $(1-A)^{-n}$*-regularized semigroup* $\{W(t)\}_{t\ge0}$.

(f) (ACP) *has a unique bounded mild solution for all* $x\in\mathcal{D}(A^n)$.

Then

$$\langle W(t)x,x^*\rangle=T_{x,x^*}(t),\qquad\forall t\ge0,\ x\in X,\ x^*\in X^*.$$

Proof. The equivalence of (e) and (f) is Proposition 1.4. The equivalence of (a) and (d), and their equivalence with (e) for $\mathcal{D}(A)$ dense, is [d-H-W-W, Theorem 4.6] and Lemma 1.7.

(a)⇒(b) follows immediately from Widder's theorem.

(b)⇒(c). Let $\mathcal{G}$ be the algebra generated by $s\mapsto(1+s)^{-1}$. For $f\in\mathcal{G}$, $x\in X$ and $x^*\in X^*$, we claim that $(*)$ holds true. To prove this, it is sufficient to consider $f_m(s)\equiv(1+s)^{-m}$ for $m\in\mathbb{N}$, so that the determining function is

$$F_m(t)=\frac{t^{m-1}}{(m-1)!}e^{-t}.$$

If $m\le n$, then

$$\Big[\Big(1+\frac{d}{dt}\Big)^nF_m\Big](t)=0$$

for all $t\ge0$, thus the right-hand side of $(*)$ reduces to

$$\sum_{k=0}^{n-1}\Big[\sum_{j=0}^{n-1}\alpha_{j,k}F_m^{(j)}(0)\Big]\langle(1-A)^{-k}x,x^*\rangle,$$

where, by Lemma 2.3(2),

$$\sum_{k=0}^{n-1}\Big[\sum_{j=0}^{n-1}\alpha_{j,k}F_m^{(j)}(0)\Big](1+s)^{-k}=(1+s)^{-m},\qquad\forall s>0.$$

By the linear independence of $\{(1+s)^{-j}\mid j=1,\dots,n\}$,

$$\Big[\sum_{j=0}^{n-1}\alpha_{j,k}F_m^{(j)}(0)\Big]=\delta_{k,m},$$

thus the right-hand side of $(*)$ becomes $\langle(1-A)^{-m}x,x^*\rangle$, as desired.

If $m>n$, then $F_m^{(j)}(0)=0$ for $j<n$, thus the right-hand side of $(*)$ becomes

$$\begin{aligned}\int_0^\infty\Big[\Big(1+\frac{d}{dt}\Big)^nF_m\Big](t)T_{x,x^*}(t)\,dt&=\int_0^\infty F_{m-n}(t)T_{x,x^*}(t)\,dt\\&=\frac{1}{(m-n-1)!}\Big(-\frac{d}{d\lambda}\Big)^{m-n-1}\Big[\int_0^\infty e^{-\lambda t}T_{x,x^*}(t)\,dt\Big]\Big|_{\lambda=1}\\&=\frac{1}{(m-n-1)!}\Big(-\frac{d}{d\lambda}\Big)^{m-n-1}[\langle(\lambda-A)^{-1}(1-A)^{-n}x,x^*\rangle]\Big|_{\lambda=1}\\&=\langle(1-A)^{-m}x,x^*\rangle,\end{aligned}$$

as desired.

If

$$|||f|||\equiv\sum_{j=0}^{n-1}|F^{(j)}(0)|+\sum_{j=0}^{n}\|F^{(j)}\|_{L^1([0,\infty))}$$

then $|||\ |||$ is equivalent to $\|\ \|_{\mathcal{A}_n}$. Thus assertion $(*)$ implies that

$$\{|\langle f(-A)x,x^*\rangle|\mid f\in\mathcal{G},\ \|f\|_{\mathcal{A}_n}\le1\}$$

is bounded for any $x\in X$ and $x^*\in X^*$. By the uniform boundedness theorem, there exists a constant M so that

$$\|f(-A)\|\le M\|f\|_{\mathcal{A}_n},\qquad\forall f\in\mathcal{G}.$$

By Lemma 2.3(1), $\mathcal{G}$ is a dense subalgebra of $\mathcal{A}_n$. Thus the bounded algebra homomorphism $f\mapsto f(-A)$ defined on $\mathcal{G}$ extends to the desired algebra homomorphism on $\mathcal{A}_n$ (see the comments after Definition 1.1).

(c)⇒(a). Choose

$$g_{\lambda,j}(s)\equiv(\lambda+s)^{-j}(1+s)^{-n}\qquad(\lambda>0,\ j\in\mathbb{N}).$$

Using the fact that

$$g_{\lambda,j}(s)=\int_0^\infty e^{-st}\Big[\frac{t^{j-1}}{(j-1)!}e^{-\lambda t}*\frac{t^{n-1}}{(n-1)!}e^{-t}\Big]dt\qquad(s\ge0),$$

a calculation shows that $\{\lambda^j\|g_{\lambda,j}\|_{\mathcal{A}_n}\mid\lambda>0,\ j\in\mathbb{N}\}$ is bounded. Since

$$(\lambda-A)^{-j}(1-A)^{-n}=g_{\lambda,n}(-A),$$

(a) follows. ■

DEFINITION 2.5. If n and k are nonnegative integers, we define $\mathcal{A}_{n,k}$ to be the set of all Laplace transforms of F such that $t\mapsto(1+t)^kF(t)\in W^{1,n}$, with norm

$$\|f\|_{\mathcal{A}_{n,k}} \equiv \sum_{j=0}^{n} \frac{1}{j!}\|t \mapsto (1+t)^k F^{(j)}(t)\|_{L^1([0,\infty))},$$

for f as in (2.1). Note that $\mathcal{A}_{n,0} = \mathcal{A}_n$.

The same proof as that of Theorem 2.4, with [d-H-W-W, Lemma 2.2] replacing Widder's theorem, gives the following generalization.

THEOREM 2.6. *The following are equivalent, if $\varrho(A)$ is nonempty and n is a nonnegative integer.*

(a) $(0,\infty) \subseteq \varrho(A)$ *and there exists a constant M such that*

$$\|\lambda^j(\lambda - A)^{-j}(1-A)^{-n}\| \le M\left[1 + \frac{(j+k-1)!}{(j-1)!}\lambda^{-k}\right], \qquad \forall \lambda > 0,\ j \in \mathbb{N}.$$

(b) $(0,\infty) \subseteq \varrho(A)$ *and for all $x \in X$ and $x^* \in X^*$, there exists T_{x,x^*} such that*

$$t \mapsto \frac{1}{(1+t^k)} T_{x,x^*}(t) \in L^\infty([0,\infty))$$

and

$$\langle(\lambda - A)^{-1}(1-A)^{-n}x, x^*\rangle = \int_0^\infty e^{-\lambda t} T_{x,x^*}(t)\,dt \qquad (\lambda > 0).$$

(c) $-A$ *has an $\mathcal{A}_{n,k}$ functional calculus.*

(d) A *generates a once-integrated $(1-A)^{-n}$-regularized semigroup $\{W_1(t)\}_{t\ge0}$ such that*

$$\overline{\lim_{h\to0+}}\ \frac{1}{h}\|W_1(t+h) - W_1(t)\| \le M(1+t^k), \qquad \forall t \ge 0.$$

We then have $(*)$ *for $\{\alpha_{j,k}\}_{j,k=0}^{n-1}$ as in Lemma* 2.3(2), *$f \in \mathcal{A}_{n,k}$ as in* (2.1), *$x \in X$ and $x^* \in X^*$.*

If $\mathcal{D}(A)$ is dense, then (a)–(d) *are equivalent to the following.*

(e) *A generates an $O(1+t^k)$ $(1-A)^{-n}$-regularized semigroup $\{W_2(t)\}_{t\ge0}$.*

(f) (ACP) *has a unique $O(1+t^k)$ mild solution for all $x \in \mathcal{D}(A^n)$.*

Then

$$\langle W_2(t)x, x^*\rangle = T_{x,x^*}(t), \qquad \forall t \ge 0,\ x \in X,\ x^* \in X^*.$$

It would be more desirable to give spectral characterizations involving only $(z-A)^{-1}$ rather than all its powers. These do not seem possible; however, there are sufficient conditions for having an $\mathcal{A}_{n,k}$ functional calculus or generating an $O(1+t^k)$ $(1-A)^{-n}$-regularized semigroup that are close to a necessary condition. Note that A generating an $O(1+t^k)$ $(1-A)^{-n}$-regularized semigroup $\{W(t)\}_{t\ge0}$ implies that $-A$ has an $\mathcal{A}_{n,k}$ functional calculus, since

$$W_1(t)x \equiv \int_0^t W(s)x\,ds \qquad (x \in X)$$

clearly satisfies (d) of Theorem 2.6.

THEOREM 2.7. *Suppose $r \ge 0$, $s \ge 1$ and n and k are nonnegative integers.*

(1) *If $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\} \subseteq \varrho(A)$ and there exists a constant M such that*

$$\|(z-A)^{-1}\| \le M(1+|z|)^r((\mathrm{Re}(z))^{-1} + (\mathrm{Re}(z))^{-s})) \qquad (\mathrm{Re}(z) > 0),$$

then A generates an $O(1+t^s)$ $(1-A)^{-([r]+2)}$-regularized semigroup, hence $-A$ has an $\mathcal{A}_{[r]+2,k}$ functional calculus for k any integer greater than or equal to s.

(2) *If $-A$ has an $\mathcal{A}_{n,k}$ functional calculus, then $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\} \subseteq \varrho(A)$ and there exists a constant c such that*

$$\|(z-A)^{-1}\| \le c(1+|z|)^n((\mathrm{Re}(z))^{-1} + (\mathrm{Re}(z))^{-(k+1)}) \qquad (\mathrm{Re}(z) > 0).$$

Proof. (1) Let $n \equiv [r]+2$. For $t \ge 0$ and $1 > \varepsilon > 0$, define

$$W(t) \equiv \int_{i\mathbb{R}+\varepsilon} e^{-zt}(z-A)^{-1}\frac{dz}{2\pi i(1-z)^n}.$$

Note that a calculus of residues argument shows that this definition is independent of ε.

By [d1, Theorem 22.10(e) and Corollary 22.12], $\{W(t)\}_{t\ge0}$ is a norm-continuous $(1-A)^{-n}$-regularized semigroup generated by A. For any $1/2 > \varepsilon > 0$ and $t \ge 0$,

$$\|W(t)\| \le \frac{M}{\pi}2^{n-1}e^{\varepsilon t}\int_{\mathbb{R}}[\varepsilon^{-1} + \varepsilon^{-s}]\frac{(1+|y|)^r}{(1+y^2)^{n/2}}\,dy.$$

For $t > 2$, choose $\varepsilon \equiv 1/t$ and

$$K \equiv \frac{Me}{\pi}2^{n-1}\int_{\mathbb{R}}\frac{(1+|y|)^r}{(1+y^2)^{n/2}}\,dy$$

to obtain

$$\|W(t)\| \le K(t+t^s) \qquad \forall t > 2.$$

(2) Let W_1 be as in Theorem 2.6(d). Then $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\} \subseteq \varrho(A)$ and for $x \in X$, $x^* \in X^*$ and $\mathrm{Re}(z) > 0$,

$$\langle (z-A)^{-1}(1-A)^{-n}x, x^*\rangle = z\int_0^\infty e^{-zt}\langle W_1(t)x, x^*\rangle\,dt$$
$$= \int_0^\infty e^{-zt}\frac{d}{dt}\langle W_1(t)x, x^*\rangle\,dt;$$

the growth conditions on W_1 imply that

$$|\langle (z-A)^{-1}(1-A)^{-n}x, x^*\rangle| \le M\|x\|\,\|x^*\|\int_0^\infty e^{-\operatorname{Re}(z)t}(1+t^k)\,dt$$
$$= M\|x\|\,\|x^*\|\left(\frac{1}{\operatorname{Re}(z)} + \frac{k!}{(\operatorname{Re}(z))^{k+1}}\right),$$

thus

$$\|(z-A)^{-1}(1-A)^{-n}\| \le M\left(\frac{1}{\operatorname{Re}(z)} + \frac{k!}{(\operatorname{Re}(z))^{k+1}}\right) \quad (\operatorname{Re}(z) > 0).$$

A calculation using the resolvent identity n times shows that there exists a constant K so that

$$\|(z-A)^{-1}\| \le K(1+|z|)^n\|(z-A)^{-1}(1-A)^{-n}\| \quad \text{whenever } \operatorname{Re}(z) > 0. \ \blacksquare$$

REMARK 2.8. Note that, in particular, (1) of Theorem 2.7 implies that the abstract Cauchy problem (ACP) has a unique $O(1+t^s)$ mild solution, for all initial data x in $\mathcal{D}(A^{[r]+2})$. For A densely defined, a result analogous to this appears in [Nee-St, Theorem 0.3].

If we disregard the precise order of polynomial growth, we may get equivalent conditions involving only $\|(z-A)^{-1}\|$.

THEOREM 2.9. *If $\varrho(A)$ is nonempty, then the following are equivalent.*

(a) *There exist nonnegative integers n, k so that* (ACP) *has a unique $O(1+t^k)$ mild solution for all $x \in \mathcal{D}(A^n)$.*

(b) *There exist nonnegative integers n, k so that A generates an $O(1+t^k)$ $(1-A)^{-n}$-regularized semigroup.*

(c) *There exist nonnegative integers m, j so that $-A$ has an $\mathcal{A}_{m,j}$ functional calculus.*

(d) *There exist nonnegative integers n, q so that A generates an $O(1+t^q)$ n-times integrated semigroup.*

(e) *$\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\} \subseteq \varrho(A)$ and there exist $r \ge 0$, $s \ge 1$ and a constant M such that*

$$\|(z-A)^{-1}\| \le M(1+|z|)^r(\operatorname{Re}(z))^{-s} \quad (\operatorname{Re}(z) > 0).$$

Proof. (a) and (b) are equivalent by Proposition 1.4. The equivalence of (c), (b) and (e) follows from Theorem 2.7.

The equivalence of (d) and (a) will follow in turn from

$$(*) \qquad S(t)A^jx = J^n[u(t, A^jx)] = J^{n-j}\left[u(t,x) - \sum_{i=0}^{j-1}\frac{t^i}{i!}A^{j-1-i}x\right] \quad (j = 0, 1, 2, \ldots, n),$$

when $\{S(t)\}_{t\ge0}$ is an n-times integrated semigroup generated by A, $x \in \mathcal{D}(A^n)$, $(Jf)(s) \equiv \int_0^s f(r)\,dr$ and $t \mapsto u(t,x)$ is the corresponding mild solution of (ACP). First, given (d), $(*)$ implies that (ACP) has a unique $O(1+t^k)$ mild solution for all $x \in \mathcal{D}(A^n)$ if $k \equiv \sup\{q, n-1\}$. Conversely, given (a), it follows that A generates an n-times integrated semigroup $\{S(t)\}_{t\ge0}$. Then $(*)$ implies that, for any $x \in X$,

$$S(t)x = S(t)(1-A)^n(1-A)^{-n}x$$
$$= \sum_{j=0}^n \binom{n}{j}(-1)^jS(t)A^j(1-A)^{-n}x$$
$$= \sum_{j=0}^n \binom{n}{j}(-1)^jJ^{n-j}\left[u(t,(1-A)^{-n}x) - \sum_{i=0}^{j-1}\frac{t^i}{i!}A^{j-1-i}(1-A)^{-n}x\right],$$

so that $\|S(t)\|$ is $O(1+t^{n+k})$. $\blacksquare$

REMARK 2.10. For A densely defined, results analogous to the equivalence of (a), (d) and (e) in Theorem 2.9 may be found in [Nee-St].

III. Smooth semispectral distributions and integrated semigroups. A spectral distribution (see [Balab-E-J] and [E-J]) generalizes the spectral measure of the spectral theorem for self-adjoint linear operators on a Hilbert space. In this section we introduce smooth semispectral distributions and show that they play the same role with certain integrated semigroups that spectral distributions play with certain integrated groups. Our main results in this section are Theorems 3.2 and 3.6.

DEFINITION 3.1. Let $\mathcal{A}$ be the space of all Laplace transforms of functions in the Schwartz space, topologized by the seminorms

$$\|f\|_{j,k} \equiv \|t \mapsto t^jF^{(k)}(t)\|_{L^1([0,\infty))},$$

where f is as in (2.1), j and k are nonnegative integers. By a *smooth semispectral distribution* for A we mean a continuous algebra homomorphism $f \mapsto f(A)$, from $\mathcal{A}$ into $B(X)$, such that

(1) $\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) < 0\} \subseteq \varrho(A)$, with $g_\lambda(A) = (\lambda - A)^{-1}$ whenever $\operatorname{Re}(\lambda) < 0$, and

(2) $f_m(A)x \to x$ as $m \to \infty$, for all $x \in X$, $f \in \mathcal{A}$ such that $f(0) = 1$, where $f_m(s) \equiv f(s/m)$.

We think of an n-times integrated semigroup generated by A as $J^n(e^{tA})$, where $(Jg)(s) \equiv \int_0^s g(r)\,dr$. In (2.1), apply integration by parts n times, and replace s with $-A$, to obtain, informally,

$$f(-A) = (-1)^n \int_0^\infty J^n(e^{tA})F^{(n)}(t)\,dt;$$

for this formula to be valid, we need $\lim_{t\to\infty} t^j F^{(j)}(t) = 0$ for $j < n$.

Since a bounded solution of (ACP), after being integrated n times, will be $O(t^n)$, it is natural to characterize $O(t^n)$ n-times integrated semigroups rather than bounded n-times integrated semigroups; more generally, what is needed to obtain a smooth semispectral distribution is $O(t^n)$ behaviour for t near 0.

THEOREM 3.2. *If $\mathcal{D}(A)$ is dense, n and k are nonnegative integers and A generates an $O(t^n(1+t^k))$ n-times integrated semigroup, then $-A$ admits a smooth semispectral distribution.*

Proof. Let $\{S(t)\}_{t\ge0}$ be the n-times integrated semigroup generated by A. There exists a constant M so that

$$\|S(t)\| \le Mt^n(1+t^k), \qquad \forall t \ge 0.$$

Define, for $f \in \mathcal{A}$ as in (2.1),

$$f(-A)x \equiv (-1)^n \int_0^\infty [S(t)x]F^{(n)}(t)\,dt \qquad (x \in X). \tag{3.3}$$

Since $\|S(t)\|$ is polynomially bounded, $f \mapsto f(-A)$ is a continuous map from $\mathcal{A}$ into $B(X)$.

For $x \in \mathcal{D}(A^n)$, the map $t \mapsto S(t)x$ is n-times continuously differentiable, and we may rewrite (3.3) as

$$f(-A)x = \int_0^\infty [S(t)x]^{(n)}F(t)\,dt \qquad (x \in \mathcal{D}(A^n)). \tag{3.4}$$

In fact, $t \mapsto S(t)(1-A)^{-(n+1)}$ is n-times continuously differentiable in the operator norm, with

$$[S(t)(1-A)^{-(n+1)}]^{(n)}[S(r)(1-A)^{-(n+1)}]^{(n)} = [S(t+r)(1-A)^{-2(n+1)}]^{(n)}$$

for all $r, t \ge 0$.

To show that $f \mapsto f(-A)$ is an algebra homomorphism, it is sufficient, since $\mathcal{D}(A^{2(n+1)})$ is dense, to show that

$$(fg)(-A)(1-A)^{-2(n+1)} = f(-A)g(-A)(1-A)^{-2(n+1)}$$

for $f, g \in \mathcal{A}$. The calculation follows, where F is the determining function for f, and G is the determining function for g:

$$\begin{aligned}
f(-A)g(-A)(1-A)^{-2(n+1)} &= [f(-A)(1-A)^{-(n+1)}][g(-A)(1-A)^{-(n+1)}] \\
&= \int_0^\infty\int_0^\infty [S(t)(1-A)^{-(n+1)}]^{(n)} \\
&\quad\times [S(r)(1-A)^{-(n+1)}]^{(n)}F(t)G(r)\,dr\,dt \\
&= \int_0^\infty\int_0^\infty [S(r+t)(1-A)^{-2(n+1)}]^{(n)}F(t)G(r)\,dr\,dt \\
&= \int_0^\infty F(t)\int_t^\infty [S(s)(1-A)^{-2(n+1)}]^{(n)}G(s-t)\,ds\,dt \\
&= \int_0^\infty [S(s)(1-A)^{-2(n+1)}]^{(n)}\Big[\int_0^s G(s-t)F(t)\,dt\Big]ds \\
&= \int_0^\infty [S(s)(1-A)^{-2(n+1)}]^{(n)}(F*G)(s)\,ds \\
&= (fg)(-A)(1-A)^{-2(n+1)},
\end{aligned}$$

as desired.

For $\mathrm{Re}(\lambda) < 0$, note that the determining function for g_λ is $t \mapsto -e^{\lambda t}$, thus for $x \in X$,

$$\begin{aligned}
g_\lambda(-A)x &\equiv (-1)^n \int_0^\infty [S(t)x](-\lambda^n)e^{\lambda t}\,dt = -(-\lambda)^n \int_0^\infty e^{\lambda t}[S(t)x]\,dt \\
&= -(-\lambda-A)^{-1}x = (\lambda+A)^{-1}x.
\end{aligned}$$

Thus $f \mapsto f(-A)$ satisfies (1) of Definition 3.1.

For $m \in \mathbb{N}$, $x \in X$,

$$\begin{aligned}
\|(f_m)(-A)x\| &= \Big\|\int_0^\infty [S(t)x](t \mapsto mF(mt))^{(n)}\,dt\Big\| \\
&= \Big\|m^{n+1}\int_0^\infty [S(t)x]F^{(n)}(mt)\,dt\Big\| \\
&= \Big\|m^n\int_0^\infty \Big[S\Big(\frac{r}{m}\Big)x\Big]F^{(n)}(r)\,dr\Big\| \\
&\le M\|x\|\int_0^\infty r^n\Big(1+\Big(\frac{r}{m}\Big)^k\Big)|F^{(n)}(r)|\,dr
\end{aligned}$$

$$\leq M\|x\| \int_0^\infty r^n(1+r^k)|F^{(n)}(r)|\,dr.$$

Thus, to satisfy (2) of Definition 3.1, it is sufficient to satisfy (2) for all x in the dense set $\mathcal{D}(A^n)$. For $f(0)=1$, we again use (3.4):

$$\begin{aligned}(f_m)(-A)x &= \int_0^\infty [S(t)x]^{(n)} mF(mt)\,dt\\ &= \int_0^\infty [S(t)x]^{(n)}|_{t=r/m}F(r)\,dr \to [S(t)x]^{(n)}|_{t=0}\int_0^\infty F(r)\,dr\\ &= f(0)x = x,\end{aligned}$$

as $m\to\infty$. ∎

DEFINITION 3.5. Let $\mathcal{T}_n$ be the completion of $\mathcal{A}$ with respect to the norm

$$\|f\|_{\mathcal{T}_n} \equiv \|t\mapsto t^nF^{(n)}(t)\|_{L^1([0,\infty))},$$

for f as in (2.1). We say that a smooth semispectral distribution is *of degree* n if it extends continuously to a linear map from $\mathcal{T}_n$ into $B(X)$.

THEOREM 3.6. *Suppose n is a nonnegative integer and $\mathcal{D}(A)$ is dense. Then the following are equivalent.*

(a) $(0,\infty)\subseteq\varrho(A)$ *and there exists a constant* M *so that*

$$\left\|\left(\frac{d}{d\lambda}\right)^j\left(\frac{1}{\lambda^n}(\lambda-A)^{-1}\right)\right\| \leq M\left(\frac{(n+j)!}{\lambda^{n+j+1}}\right), \qquad \forall\lambda>0,\ j-1\in\mathbb{N}.$$

(b) $(0,\infty)\subseteq\varrho(A)$ *and for all* $x\in X$ *and* $x^*\in X^*$, *there exists* T_{x,x^*} *such that*

$$t\mapsto\frac{1}{t^n}T_{x,x^*}(t)\in L^\infty([0,\infty))$$

and

$$\langle(\lambda-A)^{-1}x,x^*\rangle = \lambda^n\int_0^\infty e^{-\lambda t}T_{x,x^*}(t)\,dt \qquad (\lambda>0).$$

(c) $-A$ *has a smooth semispectral distribution of degree* n.

(d) A *generates an* n*-times integrated semigroup* $\{S(t)\}_{t\geq0}$ *that is* $O(t^n)$.

We then have, for $f\in\mathcal{T}_n$ *as in* (2.1), $x\in X$ *and* $x^*\in X^*$,

$$f(-A)x = (-1)^n\int_0^\infty [S(t)x]\,F^{(n)}(t)\,dt$$

and

$$\langle S(t)x,x^*\rangle = T_{x,x^*}(t), \qquad \forall t\geq0.$$

Proof. The equivalence of (a), (b) and (d) is [d-H-W-W, Theorem 4.9 and Lemma 2.3].

(d)⇒(c). For $f\in\mathcal{T}_n$, define $f(-A)$ by (3.3). By Theorem 3.2, this defines a smooth semispectral distribution. All that remains is to show that $f\mapsto f(-A)$ is a continuous map from $\mathcal{T}_n$ into $B(X)$. There exists a constant M so that

$$\|S(t)\|\leq Mt^n, \qquad \forall t\geq0,$$

thus for $f\in\mathcal{T}_n$ as in (2.1),

$$\|f(-A)\| \leq \int_0^\infty Mt^n|F^{(n)}(t)|\,dt \equiv M\|f\|_{\mathcal{T}_n},$$

as desired.

(c)⇒(a). For any $\lambda>0$ and nonnegative integer j,

$$\begin{aligned}&\left\|\frac{\lambda^{n+j+1}}{(n+j)!}\left(\frac{d}{d\lambda}\right)^j\left(\frac{1}{\lambda^n}g_{-\lambda}\right)\right\|_{\mathcal{T}_n}\\ &\quad= \left\|t\mapsto t^n\left[\frac{\lambda^{n+j+1}}{(n+j)!}\left(\frac{d}{d\lambda}\right)^j\left(\frac{1}{\lambda^n}e^{-\lambda t}\right)\right]^{(n)}\right\|_{L^1([0,\infty))}\\ &\quad= \left\|t\mapsto\frac{\lambda^{n+j+1}}{(n+j)!}t^{n+j}e^{-\lambda t}\right\|_{L^1([0,\infty))} = 1.\end{aligned}$$

Thus

$$\begin{aligned}&\left\{\left\|\frac{\lambda^{n+j+1}}{(n+j)!}\left(\frac{d}{d\lambda}\right)^j\left(\frac{1}{\lambda^n}(\lambda-A)^{-1}\right)\right\| \;\middle|\; \lambda>0,\ j-1\in\mathbb{N}\right\}\\ &\quad= \left\{\left\|\frac{\lambda^{n+j+1}}{(n+j)!}\left(\frac{d}{d\lambda}\right)^j\left(\frac{1}{\lambda^n}g_{-\lambda}(-A)\right)\right\| \;\middle|\; \lambda>0,\ j-1\in\mathbb{N}\right\}\end{aligned}$$

is bounded, as desired. ∎

IV. An extension of the functional calculi. For operators A with an $\mathcal{A}_{n,k}$ functional calculus, for some nonnegative integers n,k (see Section II for a characterization of such operators; for a particularly simple resolvent characterization, see Theorem 2.9), we define $f(A)$ for functions f in the large algebra of complex-valued functions that we denote by $\mathcal{B}_k$ (see Definition 4.1). Our main result in this section is Theorem 4.4.

DEFINITION 4.1. For k a nonnegative integer, let

$$\mathcal{B}_k \equiv \bigcup_{m\in\mathbb{N}}\{s\mapsto(1+s)^mf(s)\mid f\in\mathcal{A}_{0,k}\}.$$

Note that this is an increasing union, thus $\mathcal{B}_k$ is an algebra.

By Lemma 2.3(2), for any nonnegative integer n,

$$\mathcal{B}_k = \bigcup_{m\in\mathbb{N}}\{s\mapsto(1+s)^mf(s)\mid f\in\mathcal{A}_{n,k}\}. \tag{4.2}$$

The algebra $\mathcal{B}_0$ is the set of all Laplace transforms of derivatives, of any order, of functions in $L^1([0,\infty))$. More generally, for any nonnegative k, $\mathcal{B}_k$ is the set of all Laplace transforms of derivatives, of any order, of functions F such that

$$t \mapsto (1+t)^k F(t) \in L^1([0,\infty)).$$

However, (4.2) will be the most convenient way to represent $\mathcal{B}_k$.

DEFINITION 4.3. Suppose n and k are nonnegative integers and A has an $\mathcal{A}_{n,k}$ functional calculus. If $g \in \mathcal{B}_k$, define

$$g(A) \equiv (\lambda + A)^m (s \mapsto (\lambda+s)^{-m} g(s))(A)$$

for any integer m and $\lambda > 0$ such that

$$(s \mapsto (\lambda+s)^{-m} g(s)) \in \mathcal{A}_{n,k}.$$

(See (4.2).) By [d2, Definition 3.4], this definition is independent of m and λ.

THEOREM 4.4. *Suppose n and k are nonnegative integers and A has an $\mathcal{A}_{n,k}$ functional calculus. Then Definition 4.3 defines a map $g \mapsto g(A)$, from $\mathcal{B}_k$ into the space of closed operators on X, extending the $\mathcal{A}_{n,k}$ functional calculus, with the following properties.*

(1) $(s \mapsto (\lambda+s)^m)(A) = (\lambda + A)^m$ *for all $\lambda > 0$ and integers m.*

(2) *If $f \in \mathcal{A}_{n,k}$ and $g \in \mathcal{B}_k$, then*

$$f(A)g(A) \subseteq g(A)f(A) = (fg)(A).$$

(3) *For all $f, g \in \mathcal{B}_k$,*

$$f(A)g(A) \subseteq (fg)(A), \quad \textit{with} \quad \mathcal{D}(f(A)g(A)) = \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A)).$$

(4) *For all $f, g \in \mathcal{B}_k$,*

$$f(A) + g(A) \subseteq (f+g)(A).$$

(5) *If f and $1/f$ are in $\mathcal{B}_k$, then $f(A)$ is injective and $(f(A))^{-1} = (1/f)(A)$.*

(6) *If $\lambda \in \mathbb{C}$, $f \in \mathcal{B}_k$ and $(s \mapsto (\lambda - f(s))^{-1}) \in \mathcal{A}_{n,k}$, then $\lambda \in \varrho(f(A))$, with*

$$(\lambda - f(A))^{-1} = (s \mapsto (\lambda - f(s))^{-1})(A).$$

(7) *If j is a nonnegative integer less than n and $f \in \mathcal{A}_{j,k}$, then $\mathcal{D}(A^{n-j}) \subseteq \mathcal{D}(f(A))$, with*

$$(s \mapsto f(s)(1+s)^{j-n})(A)(1+A)^{n-j} \subseteq f(A) = (1+A)^{n-j}(s \mapsto f(s)(1+s)^{j-n})(A).$$

Proof. Assertion (1) follows immediately from Definition 4.3. Assertions (2)–(6) are a special case of [d2, Theorem 3.7]. Assertion (7) follows from Definition 4.3 and the fact that $(s \mapsto f(s)(1+s)^{j-n}) \in \mathcal{A}_{n,k}$. ∎

Note that, as special cases of (1), $f_0(A) = I$ ($f_0(s) \equiv 1$), and

$$(s \mapsto (\lambda + s))(A) = (\lambda + A) \quad \text{for } \lambda > 0.$$

REMARK 4.5. It is interesting that the amount of regularizing required in the regularized semigroup generated by $-A$ (the n in $\mathcal{A}_{n,k}$; see Theorem 2.6) has no effect on $\mathcal{B}_k$; we still have $f(A)$ defined for any $f \in \mathcal{B}_k$. The effect of n is on how "unbounded" $f(A)$ is (see (7) of Theorem 4.4).

It is only the rate of growth of the regularized semigroup that limits our extended functional calculus, by dictating the integer k in $\mathcal{B}_k$; see Theorem 2.6.

REMARK 4.6. For $n = 0 = k$ and $\mathcal{D}(A)$ dense (that is, when $-A$ generates a bounded strongly continuous semigroup), a similar extended functional calculus is constructed in [Balak] and [Nel], by different methods.

V. Fractional powers. As an easy application of Theorem 4.4, we will now use our functional calculi to construct fractional powers, and prove analogues of the usual desired properties for a large class of operators, those with an $\mathcal{A}_{n,k}$ functional calculus for some natural numbers n, k, as in Theorems 2.7 and 2.9.

Since $(s \mapsto (\varepsilon + s)^r) \in \mathcal{A}_{n,k}$ for $\varepsilon > 0$ and $r < -n$, the $\mathcal{A}_{n,k}$ functional calculus will produce bounded fractional powers $(\varepsilon + A)^r$ for $\varepsilon > 0$ and $r < -n$. The extension of the $\mathcal{A}_{n,k}$ functional calculus, in Theorem 4.4, will define (possibly unbounded) closed fractional powers $(\varepsilon + A)^r$ for arbitrary real $r, \varepsilon > 0$.

More conveniently, we will assume that $A - \varepsilon$ has an $\mathcal{A}_{n,k}$ functional calculus for some positive ε, so that, for r real, we may define

$$A^r \equiv (s \mapsto (s+\varepsilon)^r)(A - \varepsilon).$$

See [St] and [d-Y-W] for other constructions of fractional powers of densely defined operators whose resolvent satisfies a polynomial growth condition in a sector. Our construction in this section does not require that the operators be densely defined.

It is interesting that A^r may be unbounded, even when $r < 0$ and $A - 1$ has an $\mathcal{A}_1$ functional calculus, so that $0 \in \varrho(A)$.

EXAMPLE 5.1. Suppose $2 - G$ generates a bounded holomorphic strongly continuous semigroup $\{e^{t(2-G)}\}_{t \ge 0}$ on X, and G is unbounded. Define A, on $X \times X$, by

$$A \equiv \begin{bmatrix} G & G^2 \\ 0 & G \end{bmatrix},$$

with domain

$$\mathcal{D}(A) \equiv \{(x_1, x_2) \mid x_2 \in \mathcal{D}(G),\ G^{-1}x_1 + x_2 \in \mathcal{D}(G^2)\};$$

in other words, we write A as

$$G^2\begin{bmatrix} G^{-1} & 1 \\ 0 & G^{-1}\end{bmatrix}.$$

Note that, since $2-G$ generates a bounded strongly continuous semigroup, $0\in\varrho(G)$. Then $-(A-1)$ generates a bounded A^{-1}-regularized semigroup, given by

$$W(t)\equiv\begin{bmatrix} e^{-t(G-1)} & -tG^2e^{-t(G-1)} \\ 0 & e^{-t(G-1)}\end{bmatrix}\begin{bmatrix} G^{-1} & -1 \\ 0 & G^{-1}\end{bmatrix}\qquad (t\ge 0),$$

so that $A-1$ has an $\mathcal{A}_1$ functional calculus. For any real r, the fractional power $A^r\equiv(s\mapsto(1+s)^r)(A-1)$, as defined by Theorem 4.4, may be shown to be

$$A^r=\begin{bmatrix} G^r & rG^{1+r} \\ 0 & G^r\end{bmatrix}.$$

Thus $A^r\in B(X)$ only when $r\le -1$.

In general, we expect A^r to be bounded, when A (or, to be more precise, $A-\varepsilon$, $\varepsilon>0$) has an $\mathcal{A}_{n,k}$ functional calculus, only when $r<-n$, since $s\mapsto(\varepsilon+s)^r$ is in $\mathcal{A}_{n,k}$, for $\varepsilon>0$, if and only if $r<-n$.

In the following, note that we are not assuming that A is densely defined.

THEOREM 5.2. *Suppose n and k are nonnegative integers and $A-\varepsilon$ has an $\mathcal{A}_{n,k}$ functional calculus for some $\varepsilon>0$. Then there exists a map $r\mapsto A_r$, from the real line into the space of closed operators, such that*

(1) *$A_j=A^j$ for any integer j.*
(2) *A_r is injective and $(A_r)^{-1}=A_{-r}$ for any real r.*
(3) *For $r<-n$, $A_r\in B(X)$.*
(4) *For any real r, s, $A_rA_s\subseteq A_{r+s}$, with*

$$\mathcal{D}(A_rA_s)=\mathcal{D}(A_s)\cap\mathcal{D}(A_{r+s}).$$

(5) *If $s<-n$, then $A_rA_s=A_{r+s}$ for all real r.*
(6) *$\mathcal{D}(A_s)\subseteq\mathcal{D}(A_r)$ when $s>r+n$.*

Proof. We may use the functional calculus of Definition 4.3 for $A-\varepsilon$ to define $A_r\equiv(s\mapsto(s+\varepsilon)^r)(A-\varepsilon)$.

Assertions (1)–(4) follow immediately from Theorem 4.4, since, for $r<-n$, $s\mapsto(\varepsilon+s)^{-r}\in\mathcal{A}_{n,k}$, hence $(s\mapsto(\varepsilon+s)^r)\in\mathcal{B}_k$ for any real r.

Assertion (5) is a consequence of (3) and (4).

For assertion (6), note that (5) implies that

$$A_{-r}A_{r-s}=A_{-s},$$

so that, by (2),

$$\mathcal{D}(A_s)=\mathrm{Im}(A_{-s})=\mathrm{Im}(A_{-r}A_{r-s})\subseteq\mathrm{Im}(A_{-r})=\mathcal{D}(A_r).\ \blacksquare$$

Note that, for $r,s\ge n$, $A^{-r}A^{-s}=A^{-(r+s)}\in B(X)$.

REMARK 5.3. When A has an $\mathcal{A}_{n,k}$ functional calculus for nonnegative integers n and k, one may also define A^α, for $0<\alpha<1$ as follows. For $\alpha|\arg(z)|<\pi/2$, it may be shown that the map

$$z\mapsto(s\mapsto e^{-zs^\alpha})$$

is a holomorphic map into $\mathcal{A}_{n,k}$, thus

$$e^{-zA^\alpha}\equiv(s\mapsto e^{-zs^\alpha})(A)\qquad(|\arg(z)|<\pi/(2\alpha))$$

defines a holomorphic semigroup; we may define $-A^\alpha$ as the generator, in some sense.

Unless $n=0$, this semigroup is not strongly continuous at zero, in general. To make the domain of A^α more precise, $-A^\alpha$ may be defined as the generator of the $(1+A)^{-n}$-regularized semigroup

$$W(t)\equiv(s\mapsto e^{-ts^\alpha}(1+s)^{-n})(A)\qquad(t\ge 0).$$

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Scientia Research Institute
P.O. Box 988
Athens, Ohio 45701
U.S.A.
E-mail: 72260.2403@compuserve.com

Département de Mathématiques
Université Libanaise
Hadeth Liban B.P. 11-4015
Beyrouth, Liban
E-mail: lusc1@cnrs.edu.lb

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A quasi-nilpotent operator with reflexive commutant, II

by

V. MÜLLER (Praha) and M. ZAJAC (Bratislava)

Abstract. A new example of a non-zero quasi-nilpotent operator T with reflexive commutant is presented. The norms $\|T^n\|$ converge to zero arbitrarily fast.

Let H be a complex separable Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all continuous linear operators on H. If $T \in \mathcal{B}(H)$ then $\{T\}' = \{A \in \mathcal{B}(H) : AT = TA\}$ is called the *commutant* of T. By a *subspace* we always mean a closed linear subspace. If $\mathcal{A} \subset \mathcal{B}(H)$ then $\operatorname{Alg}\mathcal{A}$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing the identity I and $\mathcal{A}$, and $\operatorname{Lat}\mathcal{A}$ denotes the set of all subspaces invariant for each $A \in \mathcal{A}$. If $\mathcal{L}$ is a set of subspaces of H, then $\operatorname{Alg}\mathcal{L} = \{T \in \mathcal{B}(H) : \mathcal{L} \subset \operatorname{Lat}\{T\}\}$. T is said to be *hyperreflexive* if $\{T\}' = \operatorname{Alg}\operatorname{Lat}\{T\}'$, i.e., if the algebra $\{T\}'$ is reflexive.

It can be shown (see [1]) that if T is a nilpotent hyperreflexive operator on a separable Hilbert space then $T = 0$. This is not true for quasinilpotent operators. An example of a non-zero quasinilpotent hyperreflexive operator was given in [5] using a modification of an idea of Wogen [4]. The powers in the example converged to zero slowly; more precisely, the following inequality was true for all positive integers:

$$\|T^n\|^{1/n} \geq 1/\log n.$$

In [6] it was shown that the convergence of the powers of T to zero can be faster, namely for each $p > 0$ there exists a non-zero hyperreflexive operator T for which

$$\|T^n\|^{1/n} \leq 1/n^p.$$

The aim of this note is to show that the convergence $\|T^n\|^{1/n} \to 0$ can be arbitrarily fast:

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