On oscillatory integral operators with folding canonical relations

by

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Abstract. Sharp $L^p$ estimates are proven for oscillatory integrals with phase functions $\Phi(x,y), (x,y) \in X \times Y$, under the assumption that the canonical relation $C_\Phi$ projects to $T^*X$ and $T^*Y$ with fold singularities.

1. Introduction. Let $X$ and $Y$ be open sets in $\mathbb{R}^d$ and let $\Omega \subset X \times Y$ be a bounded open set whose closure is contained in $X \times Y$. We consider oscillatory integral operators $T_{\lambda, \gamma}$ given by

$$T_{\lambda, \gamma}[a, f](x) = \int e^{i\lambda \Phi(x, y, \gamma)} a_{\lambda, \gamma}(x, y) f(y) dy.$$  

Here $\lambda \geq 1$ and $\gamma$ is a parameter in a manifold $T$. We assume that $a_{\lambda, \gamma} \in C^\infty_0(\Omega)$ and that $\Phi(\cdot, \cdot, \gamma) \in C^\infty(X \times Y)$ is a real-valued phase function; moreover, $a_{\lambda, \gamma}, \Phi(\cdot, \cdot, \gamma)$ and their derivatives depend continuously on $\gamma$.

It may sometimes be convenient to admit some growth in $\lambda$ for the $(x, y)$ derivatives of the amplitude. Let $0 \leq \delta \leq 1$. We shall say that the family of amplitudes $a = \{a_{\lambda, \gamma}\}$ belongs to the class $\mathcal{G}_\delta(\Omega)$ if $\text{supp} a_{\lambda, \gamma} \subset \Omega$ and if

$$\sup_{(x, y) \in \Omega} |\partial_x^\alpha \partial_y^\beta a_{\lambda, \gamma}(x, y)| \leq C_{\alpha, \beta} \lambda^{\delta(|\alpha| + |\beta|)}.$$  

This definition is made in analogy to the standard symbol classes $S_{\varrho, \delta}$, although there is no parameter $\varrho$ since we do not impose differentiability conditions with respect to the parameter $\lambda$. If $\tilde{C}_{\alpha, \beta}$ denotes the best constants in (1.2) then we define

$$\|a\|_{\mathcal{G}_\delta} = \sup \{\tilde{C}_{\alpha, \beta} : |\alpha|, |\beta| \leq N\}$$

for some large $N$; the choice $N = 10d$ is admissible and we shall make this

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choice. We shall often suppress the dependence on $\alpha$ and $\gamma$ and then denote by $T_\lambda$ the operator $f \rightarrow T_{\lambda, \gamma}[a, f]$.

Clearly, $T_\lambda$ is bounded on all $L^p(\mathbb{R}^d)$ but the interesting issue is the dependence on $\lambda$ of the operator norm. As is well known the $L^2$ bounds depend on the geometry of the canonical relation

$$C_\Phi = \{(x, \Phi_x^+ y, -\Phi_y^+) : (x, y) \in X \times Y\}.$$

(1.3)

In particular, if $C_\Phi$ is locally a canonical graph (this being equivalent to $\text{det } \Phi_x^+ \neq 0$) and $a \in \mathcal{E}_0$, then the $L^2$ operator norm of $T_\lambda$ is $O(\lambda^{-d/2})$; see Hörmander [11]. Hence the operator norm on $L^p$ is $O(\lambda^{-d/2} \lambda^{4(1/p-1/2)})$, and if $p \leq 2$ this is also a bound for the $L^p \rightarrow L^p$ operator norm. These boundedness properties can be extended to operators with amplitudes in $\mathcal{S}_{1,2}$, by combining Hörmander's proof with arguments in the proof of the Calderón-Vaillancourt theorem for pseudo-differential operators [2]; see also [9] for related statements on Fourier integral operators associated with canonical graphs.

In this paper we consider the case where the projections $\pi_L : C_\Phi \rightarrow T^* X$, $\pi_R : C_\Phi \rightarrow T^* Y$ may have at most two-sided fold singularities; that is, we assume that $\pi_L$ and $\pi_R$ are Whitney folds where they are singular. $C_\Phi$ is then called a folding canonical relation. $L^2$ estimates for operators with folding canonical relations are well known. The singularities cause a loss of $\lambda^{1/6}$ for the operator norms; i.e. $\|T_\lambda\|_{L^2 \rightarrow L^2} = O(\lambda^{-d/2-1/6})$. This was shown by Pan and Sogge [15] relying on the fundamental work of Melrose and Taylor [12] on normal forms for folding canonical relations. Different arguments and improvements have been given in a number of papers: see Phong and Stein [18], [16], Seeger [21] for averaging operators in the plane, Smith and Sogge [22] for Fourier integral operators and Cuccagna [6] for general oscillatory integral operators.

Here we prove sharp $L^p \rightarrow L^q$ inequalities except for the exponents $(p, q) = (3/2, 3/2)$ or $(p, q) = (3, 3)$. We also prove a sharpened version of the known $L^p$ inequality (cf. Theorem 2.1), which leads to sharp $L^p$ inequalities in one dimension, including endpoint estimates.

**Theorem 1.1.** Let $(\Phi(x, y), \gamma)$ be a family of phase functions defined for $(x, y)$ near a point $(x_0, y_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and near $\gamma = \gamma_0$. Let $C_\Phi$ be the canonical relation for $\Phi(x, \gamma_0)$ and assume that $C_\Phi$ is a folding canonical relation. Then there is a neighborhood $U$ of $(x_0, y_0)$ and a neighborhood $V$ of $\gamma_0$ so that for all $\gamma \in V$ and all $a \in \mathcal{S}_{1,2}(U)$,

$$\|T_{\lambda, \gamma}[a, f]\|_q \leq C \lambda^{-\alpha(p, q)} \|a\|_{\mathcal{E}_{1/3}} \|f\|_p;$$

here

(1.4) \begin{align*}
\alpha(p, q) &= \begin{cases} 
\frac{d/p'}{d/p'} & \text{if } 1 \leq p < 3/2 \text{ and } p \leq q \leq p'/2, \\
\frac{(d-2/3)/p'}{(d-2/3)/p'} + (3q)^{-1} & \text{if } 1 \leq p < 3/2 \text{ and } p'/2 \leq q \leq p', \\
\frac{d-2/3}{d-2/3} + (3q)^{-1} & \text{if } p = 3/2 \text{ and } 3/2 < q \leq p', \\
\frac{d-2/3}{d-2/3} + (3q)^{-1} & \text{if } 3/2 < p \leq 2 \text{ and } p \leq q, \\
\frac{d-1/3}{d-1/3} & \text{if } 1 \leq p \leq 2 \text{ and } p \leq q.
\end{cases}
\end{align*}

Moreover, if $2 \leq p \leq q$ then $a(p, q) = \alpha(q', p')$.

We conjecture that the missing endpoint $L^{3/2}$ inequality holds but we can prove this endpoint inequality only in the case $d = 1$. In higher dimensions we only prove a restricted weak type inequality, which still suffices to deduce the other inequalities stated in Theorem 1.1. In the following theorem $L^p$ denotes the familiar Lorentz space (see [25, Ch. VI]).

**Theorem 1.2.** With the same assumptions as in Theorem 1.1 the following holds:

(i) The operator $T_{\lambda, \gamma}[a, \cdot]$ maps $L^{3/2, 1}$ boundedly to $L^{3/2, \infty}$ and $L^{3, 1}$ boundedly to $L^{3, \infty}$, with operator norm $O(\lambda^{-d/3}[a]\|\mathcal{E}_{1/3})$.

(ii) If $d = 1$ then $T_{\lambda, \gamma}[a, \cdot]$ maps $L^{3/2}$ boundedly to $L^{3/2}$ and $L^3$ boundedly to $L^3$, with operator norm $O(\lambda^{-1/3}[a]\|\mathcal{E}_{1/3})$.

**Remarks.** (i) Theorem 1.1 for $1 \leq p \leq 2$ follows by interpolation from the cases $(p, q) = (1, 1), (1, \infty), (2, 2)$ and the restricted weak type $(3/2, 3/2)$ inequality of Theorem 1.2. The first two cases are trivial, and the $L^2$ inequality is known at least for $C_\Phi$ amplitudes ([15]). In view of the symmetry of our assumption the appropriate estimates for $p \geq 2$ follow by applying the estimates for $p \leq 2$ to the adjoint operator $T_\lambda^*$. (ii) The estimates are sharp as one can see by considering the model $\Phi(x, y) = (x', y') + (x - y_0)^2$. In fact, our proof shows that the endpoint inequality $\|T_{\lambda}[L]_L \|_{L^2 \rightarrow L^2} = O(\lambda^{-d/3})$ is true for this example. (iii) If additional curvature assumptions are imposed on the projections of the fold surface to the fibers $T^*_X, T^*_Y$ then the $L^p \rightarrow L^2$ and $L^2 \rightarrow L^p$ estimates can be improved (see Theorem 2.2 in [7]). (iv) The $L^p$ estimates should be compared with analogous results on Fourier integral operators $F$ of order $\alpha$, associated with folding canonical relations (here $d \geq 2$). Namely, $F$ is bounded on $L^p$ for $\alpha \leq -(d-1) x/(d-1) + (1/p-1/2)$ if $3/2 < p < \infty$ or $1 < p < 3/2$ and for $\alpha < -1/(6-(d-2)) |1/p-1/2|$ if $3/2 < p < 3$ (here equality is established if $d = 2$). This was proved by Smith and Sogge [22]; see also the related results for Radon transforms in [18], [21] and [19]. The analogy breaks down for the critical exponents $3/2$ and $3$, since the $L^{3/2}$ or $L^3$ boundedness of $F$ may fail to hold for operators of order $-(d-1)/6$; cf. the translation invariant counterexample by M. Christ [4].
(v) We have stressed uniformity with respect to parameters since the $L^2$ version of the theorem is applied in [8] to a family of operators with folding canonical relations in order to prove estimates for Fourier integral operators with one-sided simple cusp singularities.

(vi) It would be interesting to obtain sharp $L^p \to L^p$ results for oscilatory integral operators with one-sided fold singularities (cf. the sharp $L^2$ estimate in [7] and $L^p$ estimates for $p > 3$ or $p < 3/2$ in [21] for Radon transforms in the plane).

2. Preliminary reductions. After affine linear changes of the coordinates in $X$ and $Y$ we may impose some normalizing assumptions at a reference point $P_0 = (x_0, y_0, \gamma_0)$ and we may assume that $x_0 = y_0 = 0$.

In fact, if a canonical relation is of the form $(u, \phi'_u, \nu, -\phi'_\nu)$ one can argue as in [7] and replace $\phi(u, v)$ by $\Phi(x, y) = \phi(x_0 + B_1 x, y_0 + B_2 y)$ where $B_1$ and $B_2$ are suitable invertible linear transformation. Specifically, if $\{e_j\}$ is the standard basis in $\mathbb{R}^d$ and if $u \in \text{Ker } \Phi_{\nu y}(x_0, y_0, \gamma_0)$, $b \in \text{Coker } \Phi'_\mu(x_0, y_0, \gamma_0)$ are nonzero vectors one can arrange that $B_1 e_d = a$, $B_2 e_d = 0$ and that for $j = 1, \ldots, d - 1$ the vectors $B_1 e_j$ are orthogonal to $(a, \phi'_u)_{u/v}$, and the vectors $B_1 e_j$ are orthogonal to $(a, \phi'_\nu)_{u/v}$. This yields that with $x = (x', x_d)$, $y = (y', y_d)$ we have $\Phi'_{\nu y}(x, y_0, \gamma_0) = 0$, $\Phi'_{\mu u}(x_0, y_0, \gamma_0) = 0$ and $\Phi'_{\mu u}(x_0, y_0, \gamma_0) = 0$. Consequently, given small $\varepsilon > 0$ we may assume that the symbol is supported in a neighborhood $\Omega \times V$ of $(x_0, y_0, \gamma_0)$ so that

\begin{align}
|\phi'_{\nu y}| & \leq \varepsilon, \\
|\phi'_{\mu u}| & \leq \varepsilon, \\
|\phi'_{\mu u}| & \leq \varepsilon, \\
|\phi'_{\mu u}| & \leq \varepsilon,
\end{align}

for $(x, y) \in \Omega$ and $\gamma \in V$.

The two-sided folding assumption implies that for suitable choice of $\Omega$, $V$ we have

\begin{align}
\det \Phi'_{\nu y} & \neq 0, \\
\Phi_{\mu u} & \neq 0, \\
\Phi_{\mu u} & \neq 0.
\end{align}

We may assume that the lower bounds for $|\det \Phi'_{\nu y}|$, $|\Phi'_{\mu u}|$, $|\Phi_{\mu u}|$ are large compared to $\varepsilon$; more specifically, the $C^4$ norm of $\Phi$ in $\Omega \times V$ is assumed to be small compared to these lower bounds.

In what follows we shall use the formula

\begin{align}
\det \begin{pmatrix} A & b \\ c^t & d \end{pmatrix} = (d - c^t A^{-1} b) \det A.
\end{align}

In a neighborhood of $(x_0, y_0, \gamma_0)$ we have $\det \Phi'_{\nu y} \neq 0$ and in view of (2.5)–(2.8) we can parametrize the variety given by $\det \Phi_{\mu u} = 0$ either as a graph $y_d = u(x', x_d, y', \gamma)$ or as a graph $x_0 = u(y', y_d, x', \gamma)$, for $\gamma$ close to $\gamma_0$. Let

\begin{align}
\sigma_0 & = \Phi'_{\nu y} - \Phi'_{\mu u} \langle \nu, \mu \rangle^{-1} \Phi_{\mu u}^{-1} \Phi'_{\nu y},
\end{align}

then by (2.8),

\begin{align}
\sigma_0(x, y) = 0 \iff y_d = u(x', x_d, y', \gamma) \iff x_0 = u(y', y_d, x', \gamma),
\end{align}

for $\gamma$ close to $\gamma_0$. Moreover, in view of (2.6)–(2.7),

\begin{align}
|\sigma_0(x, y, \gamma)| & \approx |\det \Phi_{\mu u}(x, y, \gamma)| \approx |y_d - u(x', x_d, y', \gamma)| \approx |x_0 - u(y', y_d, x', \gamma)|.
\end{align}

We fix $\gamma$ and $\lambda$ and set $T = T_{\lambda, \gamma}[\alpha, \cdot]$. From now on it is assumed that all amplitudes $\alpha$ are supported in $\Omega$ and satisfy $|\alpha|_{\ell^2} \leq 1$. All estimates will be uniform in $\gamma$ provided that $\Omega$ and $V$ are chosen small enough.

Now, following [18], we shall make a decomposition of $T$ according to the size of $|\det \Phi_{\mu u}| \approx |\sigma_0|$.

Let $\eta \in \mathcal{C}_c^\infty((-1, 1))$ so that $\eta(x) = 1$ for $|x| \leq 1/2$ and let $\epsilon_0$ be small. Set

\begin{align}
\beta_\epsilon(x, y, \gamma) & = \eta(\epsilon) \sigma_0(x, y, \gamma) - \eta(\epsilon^{d+1}) \sigma_0(x, y, \gamma)
\end{align}

and

\begin{align}
\lambda_\epsilon & = 1 - \sum_{2^i < \ell \leq 1/\lambda} \beta_\epsilon,
\end{align}

so that $\sigma_0 \approx 2^{-l}$ in supp $\beta_\epsilon$ and $\sigma_0 \approx C_0 \epsilon^{-1} \lambda^{-1/3}$ in supp $\lambda_\epsilon$.

Define operators $S_\lambda$ and $T_\lambda$ by

\begin{align}
S_{\lambda, \gamma} f(x) & = \int e^{i \lambda \Phi(x, y, \gamma)} \lambda_\epsilon(x, y, \gamma) a_{\lambda, \gamma}(x, y) f(y) dy
\end{align}

and

\begin{align}
T_{\lambda} f(x) & = \int e^{i \lambda \Phi(x, y, \gamma)} \beta_\epsilon(x, y, \gamma) a_{\lambda, \gamma}(x, y) f(y) dy.
\end{align}

Our main result is

**Theorem 2.1.** For $f \in L^2(\mathbb{R}^d)$,

\begin{align}
\| \sum_{2^i < \ell < 1/\lambda} \alpha_i T_{\lambda} f \|_2 \leq C_\rho \lambda^{-d/2} \sup_i \| \alpha_i \|_2 \| f \|_2
\end{align}

and

\begin{align}
\| S_{\lambda, \gamma} f \|_2 \leq C \lambda^{-(d-1)/2} \lambda^{-1/3} \| f \|_2.
\end{align}

Since the operators $T_{\lambda}$, $S_{\lambda}$ are bounded on both $L^1$ and $L^\infty$, with operator norms $O(2^{-l})$, $O(\lambda^{-1/3})$, respectively, we can easily deduce by interpolation...
Let $\chi \in C^0(\mathbb{R}^{d-1})$ supported in $(-1, 1)^{d-1}$ so that $\sum_{n \in \mathbb{Z}^{d-1}} \chi(s' - n) = 1$ for all $s' \in \mathbb{R}^{d-1}$. For $(\mu, \nu) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1}$ let

$$T_{\mu, \nu}^i f(x) = \chi(\lambda^{1/3} x' - \mu) e^{i \lambda^0 (\phi(x, y) - \nu)} a \chi(x, y) f(y) dy.$$

We shall use the orthogonality lemma by Cotlar and Stein [24, pp. 279-281] to deduce (2.13) from the following two propositions.

**Proposition 3.1.** (a) $T_{\mu, \nu}^i * T_{\mu', \nu'}^m = 0$ if $|\mu - \mu'| \geq 2$ for some $i \in \{1, \ldots, d - 1\}$.

(b) $T_{\mu, \nu}^i (T_{\mu', \nu'}^m)^* = 0$ if $|\nu - \nu'| \geq 2$ for some $i \in \{1, \ldots, d - 1\}$.

(c) Let $m \leq l$, $2^l \leq \lambda^0 \lambda^{1/3}$. There is a constant $A$, independent of $l$, $m$, $\lambda$ and $\gamma$, such that for $|\nu - \nu'| \geq A \lambda^{1/2} \lambda^{1/3}$,

$$\| (T_{\mu, \nu}^i)^* T_{\mu', \nu'}^m \|_{L^2 \rightarrow L^2} + \| (T_{\mu, \nu}^i)^* T_{\mu', \nu'}^m \|_{L^2 \rightarrow L^2} \leq C_2 N 2^{-l(m)/2} \lambda^{-(d-1)/3} |\mu - \mu'|^{-N}.$$

and such that for $|\mu - \mu'| \geq A \lambda^{1/2} \lambda^{1/3}$,

$$\| T_{\mu, \nu}^i (T_{\mu', \nu'}^m)^* \|_{L^2 \rightarrow L^2} + \| T_{\mu, \nu}^i (T_{\mu', \nu'}^m)^* \|_{L^2 \rightarrow L^2} \leq C_2 N 2^{-l(m)/2} \lambda^{-(d-1)/3} |\mu - \mu'|^{-N}.$$

(d) There is a constant $b$, independent of $l$, $m$, $\lambda$ and $\gamma$, such that for $m < l - b$, $2^l \leq \lambda^0 \lambda^{1/3}$,

$$\| (T_{\mu, \nu}^i)^* T_{\mu', \nu'}^m \|_{L^2 \rightarrow L^2} + \| (T_{\mu, \nu}^i)^* T_{\mu', \nu'}^m \|_{L^2 \rightarrow L^2} \leq C_2 N 2^{l(m)/2} \lambda^{-d/2} \lambda^{-d/2} \lambda^{-(d-1)/3} 2^{N - 2d - 1}.$$

**Proposition 3.2.** The estimate

$$\| T_{\mu, \nu}^i \|_{L^2 \rightarrow L^2} \leq C 2^{l/2} \lambda^{-d/2}$$

holds uniformly in $l$, $\mu$, $\nu$ and $\gamma$.

We now apply the Cotlar–Stein lemma in the following form: Let $\{S_j\}$ be a family of operators on a Hilbert space, indexed by $j = (\mu, \nu, l) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1} \times \mathbb{Z}$, only finitely many being $\neq 0$. Then

$$\| \sum_j S_j \| \leq C \sum_{r \in \mathbb{Z}^{d-1}} \sup_{j, j' \in \mathbb{Z}^{d-1}} ||| S_j S_j^* \|^{1/2} + ||| S_j^* S_{j'} \|^{1/2}.$$

In order to apply this one checks that Propositions 3.1 and 3.2 with $N = 10d$ imply the weaker estimate.
\[ \|T_{\mu \nu}^m(T_{\mu \nu}^m)^*\|_{L^2 \to L^2} + \|(T_{\mu \nu}^m)^*T_{\mu \nu}^m\|_{L^2 \to L^2} \leq C_2 |l-m|^{2d} |1 + |\nu - \nu'| - 2d| \] 
and now the Cotlar–Stein lemma clearly yields (2.13).

Proof of Proposition 3.1. Parts (a) and (b) follow immediately from the definitions. Notice that \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) is the adjoint of \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) and \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) is the adjoint of \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\)^*. So it suffices to show the required bounds for \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) and \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) if \(m \leq l\). In fact, we shall only give the proof for the boundedness of \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) and in view of the symmetry of our assumptions the corresponding estimates for \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) follow by the same arguments, or by realizing that the adjoint of \((T_{\mu \nu}^m)^*\) is essentially \((T_{\mu \nu}^m)^*\). We now have to estimate the kernel \(K_{\mu \nu}^{im}(y, z)\) of \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\). Here

\[ K_{\mu \nu}^{im}(y, z) = \chi(\lambda^{1/3} y' - \nu)\chi(\lambda^{1/3} z' - \nu')K_{\mu \nu}^{im}(y, z) \]

where

\[ K_{\mu \nu}^{im}(y, z) = \int \int e^{i \lambda(\Phi(y, z) - \Phi(z, x))} \delta_{\mu \nu}(y, x, z) dx \]

and

\[ \delta_{\mu \nu}(x, y, z) = a_{\lambda, \gamma}(x, y)\delta_{\lambda}(x, y)\chi(\lambda^{1/3} z' - \mu)\chi(\lambda^{1/3} z' - \mu'). \]

We shall use Schur's lemma, by which the \(L^2\) norm of an integral operator \(T_{\mu \nu}^m(y, z)\) satisfies

\[ \|T_{\mu \nu}^m(y, z)\|_{L^2 \to L^2} \leq \left( \sup_y \int |K(y, z)| dy \right)^{1/2} \left( \sup_z \int |K(y, z)| dy \right)^{1/2}. \]

The estimate in part (c) of Proposition 3.1 for \((T_{\mu \nu}^m)^*T_{\mu \nu}^m\) follows from

Lemma 3.3. Suppose \(m \leq l\) and \(l \leq 2^l \leq C_2|l - m|^{2d} - m\). There is a constant \(A\) such that for \(|\nu - \nu'| \leq A\lambda^{1/3}2^{-m}\),

\[ \sup_y \int |K(y, z)| dy \leq C_2^{-1} - \frac{A}{\lambda^{1/3}2^{-m}}, \]

\[ \sup_y \int |K(y, z)| dy \leq C_2^{-2m} - \frac{A}{\lambda^{1/3}2^{-m}}, \]

Part (d) of Proposition 3.1 follows from

Lemma 4. There is a positive constant \(b \geq 1\) such that for \(m < l - b\),

\[ \sup_y \int |K(y, z)| dy \leq C_2^{-2m} - \frac{A}{\lambda^{1/3}2^{-m}}, \]

so that by (2.1),

\[ \frac{\partial \eta}{\partial y_{\alpha}} = 0 \varepsilon. \]

Furthermore with

\[ Q_i(z, y) := \left( \Phi_{\nu'\nu''} \frac{\partial \eta}{\partial y_{\alpha}} \frac{\partial \eta}{\partial y_{\beta}} \right) \]
and $Q = (Q_1, \ldots, Q_{d-1})$ we have

$$\partial_y \partial_y \phi'^2 + Q + 2 \phi'' \partial_y + \phi'' = 0.$$  

We expand

$$\phi'^2 (\partial_y)^2 + Q + 2 \phi'' \partial_y + \phi'' = 0.$$  

This follows from (2.7), (2.2), (2.3) and (3.6) that, near $(x_0, y_0)$,

$$|\sigma_1(x, y)| \leq C_0 > 0.$$  

Next observe for later application that $z' - \eta(y, x, z', z_0) = O(\varepsilon |y| - z_0)$

by (3.6) and therefore

$$|z' - \eta(y, x, z', z_0) = z' - \eta(y, x, z', z_0)| = O(\varepsilon |y| - z_0).$$  

From this we also see that

$$u(x, y) - u(x, z') = O(\varepsilon |y| - z_0),$$  

and

$$u(x, y') - u(x, z') = O(\varepsilon |y| - z_0).$$  

Finally, observe that

$$\frac{\partial y}{\partial y} = (\partial_y)^{-1}(x, y, y) \partial_y (x, y) = I + O(\varepsilon).$$  

Proof of Lemma 3.1. Suppose that $m < l - b$ and that $(x, y, z) \in \text{supp } \phi''$, so $C_0^{-1} \leq |y| - u(x, y') \leq C_0^{-1}$ and $C_0^{-1} \leq |z| - u(x, z') \leq C_0^{-1}$, and we may assume that $2^{-b} \leq C_0$. Then by (3.7),

$$|\phi''(x, y) - \phi''(z, x)| \leq |y' - \eta(y, x, z)|,$$

$$|\phi''(x, y) - \phi''(z, x)| \geq C_0|y| - z_0|^{2 - |y' - \eta(y, x, z)|}$$

and, for our choice of $y, x$

$$(3.11) |y| - z_0| \geq C_0|y' - \eta(y, x, z)|$$

(cf. (3.8)). Therefore

$$|\phi''(x, y) - \phi''(z, x)| \geq C_0|2^{-2m}| + |y' - \eta(y, x, z)|.$$
and integration by parts yields
\[
|H_{\sigma',\tau'}(y,z)| \leq C \int_{|y\cdot z| \leq 2} \int_{|z| \leq 2} \int_{|y| \leq \lambda^{1/3}} \frac{\lambda^{N/3}}{(y\cdot z - z\cdot y)^{d-1} - \lambda^{1/3}|y - z|^{d-1}} \, dx. 
\]
This is a favorable estimate for the range $|y\cdot z - z\cdot y| \geq \varepsilon^2 \lambda^{-2/3}$, in particular, always when $|\sigma - \sigma'| \geq 2$.

Indeed, if $E_{\sigma,\tau}(y,z) = \{z' : |y - \sigma\cdot y| \leq 2^{-1-n}|y\cdot z - z\cdot y|\} \leq (2^{-1-n}|y\cdot z - z\cdot y|)^{-2}$ and therefore, for $|\sigma - \sigma'| \geq 2$,
\[
\int |H_{\sigma',\tau'}(y,z)| \, dx \leq C \sum_{n>0} (2^{-1/3}\lambda^{N/3}) (2^{-1/3}\lambda^{-2/3} |\tau - \sigma'|)^{-N} (\lambda^{-2/3} |\tau - \sigma'|)^{-d-1} - \lambda^{1/3}|y - z|^{d-1}.
\]
(10)
(<\big> x integration yields a factor of $\lambda^{-d/3}$ and the $z$ integration yields a factor of $\lambda^{-d/3})$. As we chose $N = 10d$ we may certainly sum in $n$ and the result is that
\[
\int |H_{\sigma',\tau'}(y,z)| \, dx \leq C \int_{2^{-1/3}\lambda^{-2/3} |\tau - \sigma'|}^{2} \int_{2^{-1/3}\lambda^{-2/3} |\tau - \sigma'|}^{2} \int_{2^{-1/3}\lambda^{-2/3} |\tau - \sigma'|}^{2} \frac{\lambda^{N/3}}{(y\cdot z - z\cdot y)^{d-1} - \lambda^{1/3}|y - z|^{d-1} - \lambda^{1/3}|y - z|^{d-1}} \, dx. 
\]

Proof of Lemma 3.6. We shall estimate the kernel $H_{\sigma',\tau'} := H_{\sigma,\tau}(y,z)$ of $R_{\sigma,\tau} \sigma,\tau$. Now if we argued as in the proof of Lemma 3.5 we would not be able to get the favorable estimate $|y\cdot z - z\cdot y| \geq \varepsilon^2 \lambda^{-2/3}$. Instead we have to make a linear change of coordinates taking into account the geometry of the fold surface.

Let
\[
X = X_{\mu} = (\sigma \lambda^{-1/3}, \nu \lambda^{-1/3}), \quad Y = Y_{\mu} = (\nu \lambda^{-1/3}, \nu (X_{\mu}, \nu \lambda^{-1/3})).
\]
Let $A = \varphi_{\mu}(X_{\mu}, Y_{\mu})$, then rank $A = d - 1$ and we may choose a unit vector $U$ in the kernel of $A$. Let $H_{U1}$ be the projection to the hyperplane orthogonal to $U$.

Now for the relevant $(x,z)$ we have $|(x,z) - (X_{\mu}, Y_{\mu})| \leq C 2^{-1}$ and therefore
\[
\Phi''_{xU}(x,z) U = O(2^{-1})
\]
(improving the previous estimate $O(\varepsilon)$ in (2.2)). Also note that $|H_{U1} \cdot e_1| = O(\varepsilon)$.

We argue as in the previous proofs (taking the better estimate (3.17) into account) and obtain
\[
|\Phi'_x(x,y) - \Phi'_x(x,z)| \geq C_1 |H_{U1} (y,z) - C_2| \geq C_1 |y\cdot z - z\cdot y| - C_2 |y - z|^2,
\]
\[
|\Phi_x(x,y) - \Phi_x(x,z)| \geq C_0 2^{-1} |y\cdot z - z\cdot y| - C_1 |y - z|^2.
\]
and consequently
\[ |\Phi_{x}^{s}(x, y) - \Phi_{x}^{s}(x, z)| \geq c_{1}2^{-1}|y_{d} - z_{d}| + c_{2}||I_{U_{-}}(y - z)|. \]

Integration by parts yields
\[ |H_{\sigma, \tau}(y, z)| \leq C\lambda^{-d/3} \frac{\lambda^{N/3}}{\lambda^{N}(2^{-1}|y_{d} - z_{d}| + ||I_{U_{-}}(y - z)||)}N. \]

We use this if either \(|y_{d} - z_{d}| \geq 2\lambda^{-2/3} / ||I_{U_{-}}(y - z)|| \geq \lambda^{-2/3}.

If both \(|y_{d} - z_{d}| \leq 2\lambda^{-2/3} / ||I_{U_{-}}(y - z)|| \leq \lambda^{-2/3} \lambda^{-d/3} \text{ then we just use the trivial estimate } |H_{\sigma, \tau}(y, z)| \leq C\lambda^{-d/3}. \text{ Combining the estimates and integrating we obtain}\]
\[ \sup_{y} \left| H_{\sigma, \tau}(y, z) \right| dz + \sup_{z} \left| H_{\sigma, \tau}(y, z) \right| dy \leq C2^{d}\lambda^{-d}. \]

References


