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On oscillatory integral operators with folding canonical relations

by

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Abstract. Sharp L^p estimates are proven for oscillatory integrals with phase functions $\Phi(x,y)$, $(x,y)\in X\times Y$, under the assumption that the canonical relation C_Φ projects to T^*X and T^*Y with fold singularities.

1. Introduction. Let X and Y be open sets in $\mathbb{R}^d$ and let $\Omega\subset X\times Y$ be a bounded open set whose closure is contained in $X\times Y$. We consider oscillatory integral operators $T_{\lambda,\gamma}$ given by

$$T_{\lambda,\gamma}[a,f](x)=\int e^{i\lambda\Phi(x,y,\gamma)}a_{\lambda,\gamma}(x,y)f(y)\,dy. \tag{1.1}$$

Here $\lambda\ge 1$ and γ is a parameter in a manifold Γ. We assume that $a_{\lambda,\gamma}\in C_0^\infty(\Omega)$ and that $\Phi(\cdot,\cdot,\gamma)\in C^\infty(X\times Y)$ is a real-valued phase function; moreover, $a_{\lambda,\gamma}$, $\Phi(\cdot,\cdot,\gamma)$ and their derivatives depend continuously on γ.

It may sometimes be convenient to admit some growth in λ for the (x,y) derivatives of the amplitude. Let $0\le\delta\le 1$. We shall say that the family of amplitudes $a=\{a_{\lambda,\gamma}\}$ belongs to the class $\mathfrak{S}_\delta(\Omega)$ if $\operatorname{supp} a_{\lambda,\gamma}\subset\Omega$ and if

$$\sup_{(x,y)\in\Omega}|\partial_x^\alpha\partial_y^\beta a_{\lambda,\gamma}(x,y)|\le C_{\alpha,\beta}\lambda^{\delta(|\alpha|+|\beta|)}. \tag{1.2}$$

This definition is made in analogy to the standard symbol classes $S_{\varrho,\delta}$, although there is no parameter ϱ since we do not impose differentiability conditions with respect to the parameter λ. If $\widetilde{C}_{\alpha,\beta}$ denote the best constants in (1.2) then we define

$$\|a\|_{\mathfrak{S}_\delta}=\sup\{\widetilde{C}_{\alpha,\beta}:|\alpha|,|\beta|\le N\}$$

for some large N; the choice $N=10d$ is admissible and we shall make this

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choice. We shall often suppress the dependence on a and γ and then denote by T_λ the operator $f \mapsto T_{\lambda,\gamma}[a,f]$.

Clearly, T_λ is bounded on all $L^p(\mathbb{R}^d)$ but the interesting issue is the dependence on λ of the operator norm. As is well known the L^2 bounds depend on the geometry of the canonical relation

$$(1.3)\qquad \mathcal{C}_\Phi = \{(x,\Phi'_x,y,-\Phi'_y) : (x,y) \in X \times Y\}.$$

In particular, if $\mathcal{C}_\Phi$ is locally a canonical graph (this being equivalent to $\det \Phi''_{xy} \neq 0$) and $a \in \mathfrak{S}_0$, then the L^2 operator norm of T_λ is $O(\lambda^{-d/2})$; see Hörmander [11]. Hence the operator norm on L^p is $O(\lambda^{-d/2}\lambda^{d|1/p-1/2|})$, and if $p \le 2$ this is also a bound for the $L^p \to L^{p'}$ operator norm. These boundedness properties can be extended to operators with amplitudes in $\mathfrak{S}_{1/2}$, by combining Hörmander's proof with arguments in the proof of the Calderón–Vaillancourt theorem for pseudo-differential operators [2]; see also [9] for related statements on Fourier integral operators associated with canonical graphs.

In this paper we consider the case where the projections $\pi_L : \mathcal{C}_\Phi \to T^*X$, $\pi_R : \mathcal{C}_\Phi \to T^*Y$ may have at most two-sided fold singularities; that is, we assume that π_L and π_R are Whitney folds where they are singular. $\mathcal{C}_\Phi$ is then called a *folding canonical relation.* L^2 estimates for operators with folding canonical relations are well known. The singularities cause a loss of $\lambda^{1/6}$ for the operator norms; i.e. $\|T_\lambda\|_{L^2\to L^2} = O(\lambda^{-d/2+1/6})$. This was shown by Pan and Sogge [15] relying on the fundamental work of Melrose and Taylor [12] on normal forms for folding canonical relations. Different arguments and improvements have been given in a number of papers: see Phong and Stein [18], [16], Seeger [21] for averaging operators in the plane, Smith and Sogge [22] for Fourier integral operators and Cuccagna [6] for general oscillatory integral operators.

Here we prove sharp $L^p \to L^q$ inequalities except for the exponents $(p,q) = (3/2,3/2)$ or $(p,q) = (3,3)$. We also prove a sharpened version of the known L^2 inequality (cf. Theorem 2.1), which leads to sharp L^p inequalities in one dimension, including endpoint estimates.

THEOREM 1.1. *Let $\{\Phi(\cdot,\cdot,\gamma)\}$ be a family of phase functions defined for (x,y) near a point $(x_0,y_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and near $\gamma = \gamma_0$. Let $\mathcal{C}_0$ be the canonical relation for $\Phi(\cdot,\cdot,\gamma_0)$ and assume that $\mathcal{C}_0$ is a folding canonical relation. Then there is a neighborhood Ω of (x_0,y_0) and a neighborhood V of γ_0 so that for all $\gamma \in V$ and all $a \in \mathfrak{S}_{1/3}(\Omega)$,*

$$\|T_{\lambda,\gamma}[a,f]\|_q \le C\lambda^{-\alpha(p,q)}\|a\|_{\mathfrak{S}_{1/3}}\|f\|_p;$$

here

$$(1.4)\qquad \alpha(p,q) = \begin{cases} d/p' & \text{if } 1 \le p < 3/2 \text{ and } p \le q \le p'/2,\\ (d-2/3)/p' + (3q)^{-1} & \text{if } 1 \le p < 3/2 \text{ and } p'/2 \le q \le p',\\ (d-2/3)/3 + (3q)^{-1} & \text{if } p = 3/2 \text{ and } 3/2 < q \le p',\\ (d-2/3)/p' + (3q)^{-1} & \text{if } 3/2 < p \le 2 \text{ and } p \le q \le p',\\ (d-1/3)/q & \text{if } 1 \le p \le 2 \text{ and } p' \le q. \end{cases}$$

Moreover, if $2 \le p \le q$ then $\alpha(p,q) = \alpha(q',p')$.

We conjecture that the missing endpoint $L^{3/2}$ inequality holds but we can prove this endpoint inequality only in the case $d = 1$. In higher dimensions we only prove a restricted weak type inequality, which still suffices to deduce the other inequalities stated in Theorem 1.1. In the following theorem L^{pq} denotes the familiar Lorentz space (see [25, Ch. V]).

THEOREM 1.2. *With the same assumptions as in Theorem 1.1 the following holds.*

(i) *The operator $T_{\lambda,\gamma}[a,\cdot]$ maps $L^{3/2,1}$ boundedly to $L^{3/2,\infty}$ and $L^{3,1}$ boundedly to $L^{3,\infty}$, with operator norm $O(\lambda^{-d/3}\|a\|_{\mathfrak{S}_{1/3}})$.*

(ii) *If $d = 1$ then $T_{\lambda,\gamma}[a,\cdot]$ maps $L^{3/2}$ boundedly to $L^{3/2}$ and L^3 boundedly to L^3, with operator norm $O(\lambda^{-1/3}\|a\|_{\mathfrak{S}_{1/3}})$.*

REMARKS. (i) Theorem 1.1 for $1 \le p \le 2$ follows by interpolation from the cases $(p,q) = (1,1), (1,\infty), (2,2)$ and the restricted weak type $(3/2,3/2)$ inequality of Theorem 1.2. The first two cases are trivial, and the L^2 inequality is known at least for $\mathfrak{S}_0$ amplitudes ([15]). In view of the symmetry of our assumption the appropriate estimates for $p \ge 2$ follow by applying the estimates for $p \le 2$ to the adjoint operator T^*_λ.

(ii) The estimates are sharp as one can see by considering the model $\Phi(x,y) = \langle x',y'\rangle + (x_d - y_d)^3$. In fact, our proof shows that the endpoint inequality $\|T_\lambda\|_{L^3\to L^3} = O(\lambda^{-d/3})$ is true for this example.

(iii) If additional curvature assumptions are imposed on the projections of the fold surface to the fibers T^*_xX, T^*_yY then the $L^p \to L^2$ and $L^2 \to L^{p'}$ estimates can be improved (see Theorem 2.2 in [7]).

(iv) The L^p estimates should be compared with analogous results on Fourier integral operators $\mathcal{F}$ of order α, associated with folding canonical relations (here $d \ge 2$). Namely, $\mathcal{F}$ is bounded on L^p for $\alpha \le -(d-1)\times|1/p-1/2|$ if $3 < p < \infty$ or $1 < p < 3/2$ and for $\alpha < -1/6-(d-2)|1/p-1/2|$ if $3/2 < p < 3$ (here equality is established if $d = 2$). This was proved by Smith and Sogge [22]; see also the related results for Radon transforms in [18], [21] and [19]. The analogy breaks down for the critical exponents 3/2 and 3, since the $L^{3/2}$ or L^3 boundedness of $\mathcal{F}$ may fail to hold for operators of order $-(d-1)/6$; cf. the translation invariant counterexample by M. Christ [4].

(v) We have stressed uniformity with respect to parameters since the L^2 version of the theorem is applied in [8] to a family of operators with folding canonical relations in order to prove estimates for Fourier integral operators with one-sided simple cusp singularities.

(vi) It would be interesting to obtain sharp $L^p \to L^p$ results for oscillatory integral operators with one-sided fold singularities (cf. the sharp L^2 estimate in [7] and L^p estimates for $p > 3$ or $p < 3/2$ in [21] for Radon transforms in the plane).

2. Preliminary reductions. After affine linear changes of the coordinates in X and Y we may impose some normalizing assumptions at a reference point $P_0 = (x_0, y_0, \gamma_0)$ and we may assume that $x_0 = y_0 = 0$.

In fact, if a canonical relation is of the form $\{u, \phi'_u, v, -\phi'_v\}$ one can argue as in [7] and replace $\phi(u,v)$ by $\Phi(x,y) = \phi(x_0 + B_1 x, y_0 + B_2 y)$ where B_1 and B_2 are suitable invertible linear transformation. Specifically, if $\{e_j\}$ is the standard basis in $\mathbb{R}^d$ and if $a \in \operatorname{Ker} \Phi''_{xy}(x_0, y_0, \gamma_0)$, $b \in \operatorname{Coker} \Phi''_{xy}(x_0, y_0, \gamma_0)$ are nonzero vectors one can arrange that $B_1 e_d = a$, $B_2 e_d = a$ and that for $j = 1, \ldots, d-1$ the vectors $B_2 e_j$ are orthogonal to $\langle a, \Phi'_u\rangle''_{vv} b$ and the vectors $B_1 e_j$ are orthogonal to $\langle a, \Phi'_v\rangle''_{uu} b$. This yields that with $x = (x', x_d)$, $y = (y', y_d)$ we have $\Phi''_{x'y_d}(x_0, y_0\gamma_0) = 0$, $\Phi''_{x_d y'}(x_0, y_0, \gamma_0) = 0$, $\Phi'''_{x_d y' y_d}(x_0, y_0, \gamma_0) = 0$ and $\Phi'''_{x' x_d y_d}(x_0, y_0, \gamma_0) = 0$. Consequently, given small $\varepsilon > 0$ we may assume that the symbol is supported in a neighborhood $\Omega \times V$ of $((x_0, y_0), \gamma_0)$ so that

$$|\Phi''_{xy_d}| \le \varepsilon, \tag{2.1}$$

$$|\Phi''_{x_d y}| \le \varepsilon, \tag{2.2}$$

$$|\Phi'''_{x_d y' y_d}| \le \varepsilon, \tag{2.3}$$

$$|\Phi'''_{x' x_d y_d}| \le \varepsilon \tag{2.4}$$

for $(x,y) \in \Omega$ and $\gamma \in V$.

The two-sided folding assumption implies that for suitable choice of Ω, V we have

$$\det \Phi''_{x'y'} \ne 0, \tag{2.5}$$

$$\Phi'''_{x_d y_d y_d} \ne 0, \tag{2.6}$$

$$\Phi'''_{x_d x_d y_d} \ne 0. \tag{2.7}$$

We may assume that the lower bounds for $|\det \Phi''_{x'y'}|$, $|\Phi'''_{x_d y_d y_d}|$, $|\Phi'''_{x_d x_d y_d}|$ are large compared to ε; more specifically, the C^4 norm of $\varepsilon\Phi$ in $\Omega \times V$ is assumed to be small compared to these lower bounds.

In what follows we shall use the formula

$$\det \begin{pmatrix} A & b \\ c^t & d \end{pmatrix} = (d - c^t A^{-1} b) \det A. \tag{2.8}$$

In a neighborhood of (x_0, y_0, γ_0) we have $\det \Phi''_{x'y'} \ne 0$ and in view of (2.5)–(2.8) we can parametrize the variety given by $\det \Phi''_{xy} = 0$ either as a graph $y_d = u(x', x_d, y', \gamma)$ or as a graph $x_d = v(y', y_d, x', \gamma)$, for γ close to γ_0. Let

$$\sigma_0 = \Phi''_{x_d y_d} - \Phi''_{x_d y'} (\Phi''_{x'y'})^{-1} \Phi''_{x' y_d}; \tag{2.9}$$

then by (2.8),

$$\sigma_0(x,y) = 0 \Leftrightarrow y_d = u(x', x_d, y', \gamma) \Leftrightarrow x_d = v(y', y_d, x', \gamma),$$

for γ close to γ_0. Moreover, in view of (2.6)–(2.7),

$$\begin{aligned} |\sigma_0(x,y,\gamma)| &\approx |\det \Phi''_{xy}(x,y,\gamma)| \approx |y_d - u(x', x_d, y', \gamma)| \\ &\approx |x_d - v(y', y_d, x', \gamma)|. \end{aligned} \tag{2.10}$$

We fix γ and λ and set $T = T_{\lambda,\gamma}[a, \cdot]$. From now on it is assumed that all amplitudes a are supported in Ω and satisfy $\|a\|_{\mathfrak{S}_{1/3}} \le 1$. All estimates will be uniform in γ provided that Ω and V are chosen small enough.

Now, following [18], we shall make a decomposition of T according to the size of $|\det \Phi''_{xy}| \approx |\sigma_0|$.

Let $\eta \in C_0^\infty(-1,1)$ so that $\eta(s) = 1$ for $|s| \le 1/2$ and let ε_0 be small. Set

$$\beta_l(x,y,\gamma) = \eta(2^l \sigma_0(x,y,\gamma)) - \eta(2^{l+1}\sigma_0(x,y,\gamma))$$

and

$$\zeta_\lambda = 1 - \sum_{2^l < \varepsilon_0 \lambda^{1/3}} \beta_l,$$

so that $\sigma_0 \approx 2^{-l}$ in $\operatorname{supp} \beta_l$ and $\sigma_0 \le C\varepsilon_0^{-1}\lambda^{-1/3}$ in $\operatorname{supp} \zeta_l$.

Define operators S_λ and T^l by

$$S_{\lambda,\gamma} f(x) = \int e^{i\lambda\Phi(x,y,\gamma)} \zeta_\lambda(x,y,\gamma) a_{\lambda,\gamma}(x,y) f(y)\, dy \tag{2.11}$$

and

$$T^l_\gamma f(x) = \int e^{i\lambda\Phi(x,y,\gamma)} \beta_l(x,y,\gamma) a_{\lambda,\gamma}(x,y) f(y)\, dy. \tag{2.12}$$

Our main result is

THEOREM 2.1. *For $f \in L^2(\mathbb{R}^d)$,*

$$\Big\| \sum_{2^l \le \varepsilon_0 \lambda^{1/3}} \alpha_l T^l_\gamma f \Big\|_2 \le C_p \lambda^{-d/2} \sup_l [2^{l/2} |\alpha_l|]\, \|f\|_2 \tag{2.13}$$

and

$$\|S_{\lambda,\gamma} f\|_2 \le C \lambda^{-(d-1)/2 - 1/3} \|f\|_2. \tag{2.14}$$

Since the operators T^l, S_λ are bounded on both L^1 and L^∞, with operator norms $O(2^{-l})$, $O(\lambda^{-1/3})$, respectively, we can easily deduce by interpolation

COROLLARY 2.2. *Let $1 \le p \le 2$. Then*

$$\|T^l_\gamma f\|_p \le C\lambda^{-d/p'} 2^{l(2-3/p)} \|f\|_p \tag{2.15}$$

and

$$\|S_{\lambda,\gamma} f\|_p \le C\lambda^{-(d-1)/p'-1/3}\|f\|_p. \tag{2.16}$$

Moreover,

$$\Big\| \sum_{2^l \le \varepsilon_0 \lambda^{1/3}} \alpha_l T^l_\gamma f\Big\|_{L^{3/2,\infty}} \le C_p \lambda^{-d/3} \sup_l |\alpha_l| \, \|f\|_{L^{3/2,1}}. \tag{2.17}$$

Here (2.17) follows by an argument of Bourgain [1] (see also the appendix in [3] for a more general version). Now all estimates in Theorem 1.1 follow by interpolation of (2.17) with trivial $L^1 \to L^1$ and $L^p \to L^\infty$ estimates.

The stronger version in one dimension follows from (the proof of) a theorem by Pan ([13], [14]), using a modification of Hardy space theory (cf. also [17], [20] for related earlier results). To describe this let for every bounded interval Q with center x_Q the function e_Q be defined by $e_Q(y) = e^{i\lambda\Phi(x_Q,y)}$. Denote by E the family of functions $\{e_Q\}$. An *E-atom* associated with Q is a bounded function supported in Q such that $\|a\|_\infty \le |Q|^{-1}$, and $\int a e_Q \, dy = 0$. A function $f \in L^1$ belongs to H^1_E if $f = \sum \lambda_Q a_Q$ where $\sum_Q |\lambda_Q| < \infty$ and where the a_Q are E-atoms. The norm in H^1_E is $\inf \sum_Q |\lambda_Q|$ where the infimum is taken over all possible representations of f in the form $\sum \lambda_Q a_Q$. As pointed out in [23], the proof of the standard interpolation theorem for the pair (H^1, L^p) carries over to H^1_E.

If $d = 1$ the argument of Pan yields

$$\Big\| \sum_{2^l \le \varepsilon_0 \lambda^{1/3}} \alpha_l T^l_\gamma f\Big\|_1 \le C_p \sup_l 2^l |\alpha_l| \, \|f\|_{H^1_E}. \tag{2.18}$$

Now if $1 < p < 2$ we can deduce the inequality

$$\Big\| \sum_{2^l \le \varepsilon_0 \lambda^{1/3}} \alpha_l T^l_\gamma f\Big\|_p \le C_p \lambda^{-1/p'} \sup_l [2^{l(2-3/p)} |\alpha_l|] \, \|f\|_p \tag{2.19}$$

from (2.18) and Theorem 2.1 by analytic interpolation. The proof of Theorem 2.1 will be given in §3.

3. L^2 estimates. We shall only prove the inequality (2.13). The proof of (2.14) is similar and in fact somewhat easier. As in [18], [21], [6] we need finer decompositions motivated in part by the geometry of the situation and in part by the proof of the Calderón–Vaillancourt theorem [2]. For the sake of notational simplicity we shall omit the parameter γ, but all our estimates will be uniform in γ if chosen in a sufficiently small neighborhood of γ_0.

Let $\chi \in C^\infty(\mathbb{R}^{d-1})$ supported in $(-1,1)^{d-1}$ so that $\sum_{n\in\mathbb{Z}^{d-1}} \chi(s'-n) = 1$ for all $s' \in \mathbb{R}^{d-1}$. For $(\mu,\nu) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1}$ let

$$T^l_{\mu\nu} f(x) = \chi(\lambda^{1/3}x' - \mu) \int e^{i\lambda\Phi(x,y)} \beta_l(x,y) \chi(\lambda^{1/3}y' - \nu) a_\lambda(x,y) f(y)\, dy.$$

We shall use the orthogonality lemma by Cotlar and Stein [24, pp. 279–281] to deduce (2.13) from the following two propositions.

PROPOSITION 3.1. (a) *$(T^l_{\mu\nu})^* T^m_{\mu'\nu'} = 0$ if $|\mu_i - \mu'_i| \ge 2$ for some $i \in \{1,\dots,d-1\}$.*

(b) *$T^l_{\mu\nu}(T^m_{\mu'\nu'})^* = 0$ if $|\nu_i - \nu'_i| \ge 2$ for some $i \in \{1,\dots,d-1\}$.*

(c) *Let $m \le l$, $2^l \le \varepsilon_0\lambda^{1/3}$. There is a constant A, independent of l, m, λ and γ, such that for $|\nu - \nu'| \ge A\lambda^{1/3}2^{-m}$,*

$$\|(T^l_{\mu\nu})^* T^m_{\mu'\nu'}\|_{L^2\to L^2} + \|(T^m_{\mu\nu})^* T^l_{\mu'\nu'}\|_{L^2\to L^2} \le C_N 2^{-(l+m)/2} \lambda^{-(d-1+N)/3} |\nu - \nu'|^{-N}$$

and such that for $|\mu - \mu'| \ge A\lambda^{1/3}2^{-m}$,

$$\|T^l_{\mu\nu} (T^m_{\mu'\nu'})^*\|_{L^2\to L^2} + \|T^m_{\mu\nu} (T^l_{\mu'\nu'})^*\|_{L^2\to L^2} \le C_N 2^{-(l+m)/2} \lambda^{-(d-1+N)/3} |\mu - \mu'|^{-N}.$$

(d) *There is a constant b, independent of l, m, λ and γ, such that for $m < l - b$, $2^l \le \varepsilon_0\lambda^{1/3}$,*

$$\|(T^l_{\mu\nu})^* T^m_{\mu'\nu'}\|_{L^2\to L^2} + \|(T^m_{\mu\nu})^* T^l_{\mu'\nu'}\|_{L^2\to L^2} \le C_N 2^{(l+m)/2} \lambda^{-d} 2^{m-l} (2^m\lambda^{-1/3})^{2N-2d-1},$$

$$\|T^l_{\mu\nu} (T^m_{\mu'\nu'})^*\|_{L^2\to L^2} + \|T^m_{\mu\nu} (T^l_{\mu'\nu'})^*\|_{L^2\to L^2} \le C_N 2^{(l+m)/2} \lambda^{-d} 2^{m-l} (2^m\lambda^{-1/3})^{2N-2d-1}.$$

PROPOSITION 3.2. *The estimate*

$$\|T^l_{\mu\nu}\|_{L^2\to L^2} \le C 2^{l/2} \lambda^{-d/2}$$

holds uniformly in l, μ, ν and γ.

We now apply the Cotlar–Stein lemma in the following form: Let $\{S_j\}$ be a family of operators on a Hilbert space, indexed by $j = (\mu,\nu;l) \in \mathbb{Z}^{d-1} \times \mathbb{Z}^{d-1} \times \mathbb{Z}$, only finitely many being $\ne 0$. Then

$$\Big\| \sum_j S_j \Big\| \le C \sum_{r\in\mathbb{Z}^{2d-1}} \sup_{j-j'=r} [\|S_j S^*_{j'}\|^{1/2} + \|S^*_j S_{j'}\|^{1/2}].$$

In order to apply this one checks that Propositions 3.1 and 3.2 with $N = 10d$ imply the weaker estimate

$$\|T^l_{\mu\nu}(T^m_{\mu'\nu'})^*\|_{L^2\to L^2}+\|(T^l_{\mu\nu})^*T^m_{\mu'\nu'}\|_{L^2\to L^2}\le C2^{(l+m)/2}\lambda^{-d}2^{-|l-m|}(1+|\nu-\nu|')^{-2d}(1+|\mu-\mu|')^{-2d}$$

and now the Cotlar–Stein lemma clearly yields (2.13).

Proof of Proposition 3.1. Parts (a) and (b) follow immediately from the definitions. Now notice that $(T^m_{\mu\nu})^*T^l_{\mu'\nu'}$ is the adjoint of $(T^l_{\mu\nu})^*T^m_{\mu'\nu'}$ and $T^m_{\mu\nu}(T^l_{\mu'\nu'})^*$ is the adjoint of $T^l_{\mu\nu}(T^m_{\mu'\nu'})^*$. So it suffices to show the required bounds for $(T^m_{\mu\nu})^*T^l_{\mu'\nu'}$ and $T^m_{\mu\nu}(T^l_{\mu'\nu'})^*$ if $m\le l$. In fact, we shall only give the proof for the boundedness of $(T^l_{\mu\nu})^*T^m_{\mu'\nu'}$ and in view of the symmetry of our assumptions the corresponding estimates for $T^l_{\mu\nu}(T^m_{\mu'\nu'})^*$ follow by the same arguments, or by realizing that the adjoint of $T^l_{\mu\nu}$ is essentially $(T^*)^l_{\nu\mu}$.

We now have to estimate the kernel $K^{lm}_{\mu\mu',\nu\nu'}$ of $(T^m_{\mu\nu})^*T^l_{\mu'\nu'}$. Here

$$K^{lm}_{\mu\mu',\nu\nu'}(y,z)=\overline{\chi}(\lambda^{1/3}y'-\nu)\chi(\lambda^{1/3}z'-\nu')K^{lm}_{\mu\mu'}(y,z)$$

where

$$K^{lm}_{\mu\mu'}(y,z)=\int e^{i\lambda(\Phi(x,y)-\Phi(x,z))}\varrho^{lm}_{\mu\mu'}(x,y,z)\,dx$$

and

$$\varrho^{lm}_{\mu\mu'}(x,y,z)=\overline{a_{\lambda,\gamma}(x,y)\beta_m(x,y)\chi(\lambda^{1/3}x'-\mu)}a_{\lambda,\gamma}(x,z)\beta_l(x,z)\chi(\lambda^{1/3}x'-\mu').$$

We shall use Schur's lemma, by which the L^2 norm of an integral operator T_K with kernel $K(x,y)$ satisfies

$$\|T_K\|_{L^2\to L^2}\le\Big(\sup_x\int|K(x,y)|\,dy\Big)^{1/2}\Big(\sup_y\int|K(x,y)|\,dx\Big)^{1/2}.$$

The estimate in part (c) of Proposition 3.1 for $(T^m_{\mu\nu})^*T^l_{\mu'\nu'}$ follows from

LEMMA 3.3. *Suppose $m\le l$ and $2^l\le\varepsilon_0\lambda^{1/3}$. There is a constant A such that for $|\nu-\nu'|\ge A\lambda^{1/3}2^{-m}$,*

$$\sup_y\int|K^{lm}_{\mu\mu',\nu\nu'}(y,z)|\,dz\le C_N2^{-l}\lambda^{-(d-1+N)/3}|\nu-\nu'|^{-N},\tag{3.1}$$

$$\sup_z\int|K^{lm}_{\mu\mu',\nu\nu'}(y,z)|\,dy\le C_N2^{-m}\lambda^{-(d-1+N)/3}|\nu-\nu'|^{-N}.\tag{3.2}$$

Part (d) of Proposition 3.1 follows from

LEMMA 3.4. *There is a positive constant $b\ge1$ such that for $m<l-b$, $2^l\le\varepsilon_0\lambda^{1/3}$,*

$$\sup_z\int|K^{lm}_{\mu\mu',\nu\nu'}(y,z)|\,dy\le C_N2^m\lambda^{-d}(2^m\lambda^{-1/3})^{2N-2d-1},\tag{3.3}$$

$$\sup_y\int|K^{lm}_{\mu\mu',\nu\nu'}(y,z)|\,dz\le C_N2^{2m-l}\lambda^{-d}(2^m\lambda^{-1/3})^{2N-2d-1}.\ \blacksquare\tag{3.4}$$

Proof of Lemma 3.3. We integrate by parts with respect to the x' variables. Note that $|y'-z'|\approx\lambda^{-1/3}|\nu-\nu'|\ge A2^{-m}$ for the relevant y',z' (for which $\overline{\chi}(\lambda^{1/3}y'-\nu)\chi(\lambda^{1/3}z'-\nu')\ne0$) and that

$$y_d-z_d=u(x,y')-u(x,z')+O(2^{-m})=O(2^{-m}+|y'-z'|).\tag{3.5}$$

Now $\Phi''_{x'y_d}=O(\varepsilon)$ by (2.1) and therefore

$$\Phi'_{x'}(x,y)-\Phi'_{x'}(x,z)=\Phi''_{x'y'}(x,z)(y'-z')+O(\varepsilon|y'-z'|+\varepsilon2^{-m}+|y'-z'|^2)$$

so that with our assumption on ν, ν',

$$|\Phi'_{x'}(x,y)-\Phi'_{x'}(x,z)|\ge c\lambda^{-1/3}|\nu-\nu'|.$$

For any smooth F we see from (3.5) that

$$|F(x,y)-F(x,z)|\le C[2^{-m}+|y'-z'|]\le C'\lambda^{-1/3}|\nu-\nu'|.$$

Since $a\in\mathfrak{S}_{1/3}$ and since $\varrho^{lm}_{\mu\mu'}\in\mathfrak{S}_{1/3}$ uniformly in l,m,μ,μ' (for $2^l,2^m\le\lambda^{1/3}$) we obtain by integration by parts

$$\int|K^{lm}_{\mu\mu',\nu\nu'}(y,z)|\,dz\le C\iiiint\limits_{\substack{|z'-\lambda^{-1/3}\nu'|\le2\lambda^{-1/3}\\|x'-\lambda^{-1/3}\mu|\le2\lambda^{-1/3}\\|z_d-u(x,z')|\le C2^{-l}}}\lambda^{N/3}(\lambda^{2/3}|\nu-\nu'|)^{-N}\times\chi_\Omega(x,z)\,dz_d\,dx'\,dx_d\,dz'\le C2^{-l}\lambda^{-2(d-1)/3}(\lambda^{1/3}|\nu-\nu'|)^{-N},$$

which is (3.1). Similarly we prove (3.2). ■

In order to prove Lemma 3.4 and Proposition 3.2 we need to examine the kernel of $K^{lm}_{\mu\mu',\nu\nu'}$ for $|\nu-\nu'|\le A\lambda^{1/3}2^{-m}$; here we have to use more refined integration by parts arguments. In the process we have to examine equations of the form $\Phi'_x(x,y)=\Phi'_x(x,z)$ for fixed (x,z) or $\Phi'_y(x,y)=\Phi'_y(w,y)$ for fixed (w,y). In view of (2.5) we may solve in y' in the first equation and define a function $y'=\mathfrak{y}$ by

$$\Phi'_{x'}(x,\mathfrak{y}(y_d,x,z),y_d)=\Phi'_{x'}(x,z).$$

Implicit differentiation yields

$$\Phi''_{x'y'}\frac{\partial\mathfrak{y}}{\partial y_d}+\Phi''_{x'y_d}\Big|_{(x,\mathfrak{y},y_d)}=0$$

so that by (2.1),

$$\frac{\partial\mathfrak{y}}{dy_d}=O(\varepsilon).\tag{3.6}$$

Furthermore with

$$Q_i(x,y):=\Big\langle\Phi'''_{x_iy'y'}\frac{\partial\mathfrak{y}}{\partial y_d},\frac{\partial\mathfrak{y}}{\partial y_d}\Big\rangle$$

and $Q = (Q_1, \dots, Q_{d-1})$ we have

$$\Phi''_{x'y'} \frac{\partial^2 \mathfrak{y}}{(\partial y_d)^2} + Q + 2\Phi'''_{x'y_dy'} \frac{\partial \mathfrak{y}}{\partial y_d} + \Phi'''_{x'y_dy_d} = 0.$$

We expand

(3.7) $$\Phi'_{x_d}(x, \mathfrak{y}, y_d) - \Phi'_{x_d}(x, z', z_d) = \sigma_0(x,z)(y_d - z_d) + \sigma_1(x,z)(y_d - z_d)^2 + \sigma_2(x,y,z)(y_d - z_d)^3$$

where σ_0 is as in (2.9) and

$$2\sigma_1(x,z) = \left\langle \Phi'''_{x_dy'y'} \frac{\partial \mathfrak{y}}{\partial y_d}, \frac{\partial \mathfrak{y}}{\partial y_d} \right\rangle + 2\Phi'''_{x_dy_dy'} \frac{\partial \mathfrak{y}}{\partial y_d} + \Phi''_{x_dy'} \frac{\partial^2 \mathfrak{y}}{(\partial y_d)^2} + \Phi'''_{x_dy_dy_d} \Big|_{(x,z)};$$

it follows from (2.7), (2.2), (2.3) and (3.6) that, near (x_0, y_0),

(3.8) $$|\sigma_1(x,y)| \ge c_0 > 0.$$

Next observe for later application that $z' - \mathfrak{y}(y_d, x, z', z_d) = O(\varepsilon|y_d - z_d|)$ by (3.6) and therefore

$$y' - \mathfrak{y}(y_d, x, z', z_d) = y' - z' + O(\varepsilon|y_d - z_d|).$$

From this we also see that

$$u(x, \mathfrak{y}(y_d, x, z', z_d)) - u(x, z') = O(\varepsilon|y_d - z_d|)$$

and

(3.9) $$u(x, y') - u(x, z') = O(|y' - \mathfrak{y}(y_d, x, z)| + \varepsilon|y_d - z_d|).$$

Finally, observe that

(3.10) $$\frac{\partial \mathfrak{y}}{\partial z'} = (\Phi''_{xy})^{-1}(x, \mathfrak{y}, y_d)\Phi''_{xy}(x,z) = \mathrm{Id} + O(\varepsilon).$$

Proof of Lemma 3.4. Suppose that $m < l - b$ and that $(x,y,z) \in \operatorname{supp} \varrho^{lm}_{\mu\mu'}$, so $C_0^{-1}2^{-m} \le |y_d - u(x,y')| \le C_0 2^{-m}$ and $C_0^{-1}2^{-l} \le |z_d - u(x,z')| \le C_0 2^{-m}$, and we may assume that $2^{-b} \ll C_0^{-1}$. Then by (3.7), (3.8),

$$|\Phi'_{x'}(x,y) - \Phi'_{x'}(x,z)| \approx |y' - \mathfrak{y}(y_d, x, z)|,$$
$$|\Phi'_{x_d}(x,y) - \Phi'_{x_d}(x,z)| \ge c_1[|y_d - z_d|^2 - |y' - \mathfrak{y}(y_d, x, z)|]$$

and, for our choice of y, z,

(3.11) $$|y_d - z_d| \ge c2^{-m} - C|y' - \mathfrak{y}(y_d, x, z)|$$

(cf. (3.8)). Therefore

$$|\Phi'_x(x,y) - \Phi'_x(x,z)| \ge c_2[2^{-2m} + |y' - \mathfrak{y}(y_d, x, z)|].$$

Since $\mathfrak{y}(z_d, x, z) = z'$ we see that for any smooth function

$$|F(x,y) - F(x,z)| \le C[|y_d - z_d| + |y' - \mathfrak{y}(y_d, x, z)|],$$

which is used for F being a higher order derivative of Φ.

Now straightforward integration by parts yields

$$|K^{lm}_{\mu\mu'}(y,z)| \le C_N \int_{D_{l\mu}(z)\cap D_{m\mu}(y)} \frac{\lambda^{N/3}}{(\lambda[2^{-2m} + |y' - \mathfrak{y}(y_d, x, z', z_d)|])^N}\, dx$$

with $D_{l\mu}(z) = \{x : (x,z) \in \operatorname{supp} \sigma_0\}$, and $D_{m\mu}(y)$ is similarly defined.

We now estimate $\int |K^{lm}_{\mu\mu',\nu\nu'}(y,z)|\, dz$. For fixed x', z_d let

$$\mathcal{E}_{lm\nu',n}(x', z_d) = \{(z', x_d) : |\lambda^{1/3}z' - \nu'| \le 2,\ 2^{-l-1} \le |\sigma_0(x,z')| \le 2^{-l+1},\ 2^{-2m+n-1} \le |y' - \mathfrak{y}(y_d, x, z)| \le 2^{-2m+n}\}.$$

We claim that

(3.12) $$|\mathcal{E}_{lm\nu',n}(x', z_d)| \le C2^{-l} \min\{2^{(n-2m)(d-1)}, \lambda^{-(d-1)/3}\}.$$

This is clear if $2^{n-2m} \ge \lambda^{-1/3}$ since $\mathcal{E}_{lm\nu',n}(z', z_d)$ is contained in the set of all (z', x_d) with $|z' - \lambda^{-1/3}\nu'| \le 2\lambda^{-1/3}$ and $|x_d - v(z,x')| \le C2^{-l}$. If $2^{n-2m} \le \lambda^{-1/3}$ observe that $x_d - v(z,x') = x_d - v(\lambda^{-1/3}\nu', z_d, x') + O(\lambda^{-1/3})$ so since $2^l \le \lambda^{1/3}$ we see that in this case $\mathcal{E}_{lm\nu',n}(x', z_d)$ is contained in the set of all (z', x_d) with $|x_d - v(\lambda^{-1/3}\nu', z_d, x')| \le C2^{-l}$ and $|y' - \mathfrak{y}(y_d, x, z)| \le 2^{-2m+n+1}$. Note that by (3.10) the set $\{z' : (z', x_d) \in \mathcal{E}_{lm\nu',n}\}$ has measure $O(2^{(n-2m)(d-1)})$. In either case (3.12) follows by Fubini's theorem.

We obtain

$$\int |K^{lm}_{\mu\mu',\nu\nu'}(y,z)|\, dz \le \sum_{0\le n\le 2m} \int_{z_d} \int_{|\lambda^{1/3}x' - \mu| \le 2} \lambda^{N/3}(\lambda 2^{-2m+n})^{-N} \times |\mathcal{E}_{lm\nu',n}(x', z_d)|\, dx'\, dz_d \le C'_N 2^{-l} 2^{2m(N-d+1)} \lambda^{-(2N+d-1)/3},$$

which is (3.4). The estimate (3.3) is slightly easier and follows by a similar argument. ∎

Proof of Proposition 3.2. We need a finer localization with respect to the x_d variables. Denote by $\mathcal{K}^l_{\mu\nu}$ the kernel of $T^l_{\mu\nu}$, let $\delta > \varepsilon_0$ be small (not depending on l) and $n = (n_1, n_2) \in \mathbb{Z}^2$, and let $T^{l,n}_{\mu\nu}$ be the integral operator with kernel

$$\mathcal{K}^{l,n}_{\mu\nu}(x,y) = \mathcal{K}^l_{\mu\nu}(x,y)\chi(2^l\delta^{-1}x_d - n_1)\chi(2^l\delta^{-1}y_d - n_2).$$

It is immediate that $(T^{l,n}_{\mu\nu})^* T^{l,n'}_{\mu\nu} = 0$ if $|n_1 - n'_1| > 2$ and $T^{l,n}_{\mu\nu}(T^{l,n'}_{\mu\nu})^* = 0$ if $|n_2 - n'_2| > 2$. Now if $|y_d - u(x,y')| \approx 2^{-l} \approx |z_d - u(x,z')|$, then $|x' - y'| \le \delta 2^{-l}$ forces $|y_d - z_d| \approx 2^{-l}$. This shows that $T^{l,n}_{\mu\nu}(T^{l,n'}_{\mu\nu})^* = 0$ if $|n_1 - n'_1| \ge C\delta^{-1}$, for appropriate C independent of l. Similarly, since $|y_d - u(x,y')|$

$\approx |x_d - v(y,x')|$ we see that by the same argument $T^{l,n}_{\mu\nu}(T^{l,n'}_{\mu\nu})^* = 0$ if $|n_2 - n_2'| \ge C\delta^{-1}$.

Hence

$$T^{l,n}_{\mu\nu}(T^{l,n'}_{\mu\nu})^* = 0 \quad \text{if } |n - n'| \ge C'\delta^{-1}$$

and by the Cotlar–Stein lemma it suffices to prove the uniform bound

$$\|T^{l,n}_{\mu\nu}\|_{L^2\to L^2} = O(2^l\lambda^{-d}). \tag{3.13}$$

We fix l, n, μ and ν and set $R = T^{l,n}_{\mu\nu}$. To check (3.13) we need a finer decomposition in the x_d variables, in order to handle $\mathfrak{S}_{1/3}$ amplitudes. Let $\mathcal{K}$ denote the kernel of R. Given integers σ and τ we let $R_{\sigma,\tau}$ be the integral operator with kernel

$$\mathcal{K}_{\sigma,\tau}(x,y) = \mathcal{K}(x,y)\chi(\lambda^{1/3}x_d - \sigma)\chi(\lambda^{1/3}y_d - \tau)$$

so that $R = \sum_{\sigma,\tau} R_{\sigma,\tau}$. We shall show that

$$\|R^*_{\sigma\tau}R_{\sigma'\tau'}\|_{L^2\to L^2} + \|R_{\sigma\tau}R^*_{\sigma'\tau'}\|_{L^2\to L^2} \le C2^l\lambda^{-d}(1+|\sigma-\sigma'|)^{-2}(1+|\tau-\tau'|)^{-2}.$$

Observe that we only get a nontrivial contribution if $|\sigma - \sigma'| + |\tau - \tau'| \ll \lambda^{1/3}2^{-l}$. Moreover, $R^*_{\sigma\tau}R_{\sigma'\tau'} = 0$ if $|\sigma - \sigma'| \ge 2$ and $R_{\sigma\tau}R^*_{\sigma'\tau'} = 0$ if $|\tau - \tau'| \ge 2$.

The proposition follows from almost orthogonality if we take into account the following lemmata.

LEMMA 3.5. *For* $|\tau - \tau'| \ge 2$,

$$\|R^*_{\sigma\tau}R_{\sigma'\tau'}\|_{L^2\to L^2} \le C2^l\lambda^{-d}(1+|\tau-\tau'|)^{-2d}, \tag{3.14}$$

$$\|R_{\sigma\tau}R^*_{\sigma'\tau'}\|_{L^2\to L^2} \le C2^l\lambda^{-d}(1+|\sigma-\sigma'|)^{-2d}. \tag{3.15}$$

LEMMA 3.6. *We have*

$$\|R_{\sigma\tau}\|_{L^2\to L^2} \le C2^{l/2}\lambda^{-d/2}. \ \blacksquare \tag{3.16}$$

Proof of Lemma 3.5. We shall only show (3.14), in fact a better estimate involving decay factors of the form $(2^l\lambda^{-1/3})^M$. By the symmetry of the assumptions, (3.15) follows in the same way.

As before, we shall use integration by parts to estimate the kernel $H_{\sigma\sigma',\tau\tau'}$ of $R^*_{\sigma\tau}R_{\sigma'\tau'}$; to this end we have to examine the behavior of $\Phi'_{x_d}(x,y) - \Phi'_{x_d}(x,z)$ given that $|y' - \lambda^{-1/3}\nu| \le 2\lambda^{-1/3}$, $|z' - \lambda^{-1/3}\nu| \le 2\lambda^{-1/3}$, $|x' - \lambda^{-1/3}\mu| \le 2\lambda^{-1/3}$, $|y_d - \lambda^{-1/3}\tau| \le 2\lambda^{-1/3}$, $|z_d - \lambda^{-1/3}\tau'| \le 2\lambda^{-1/3}$, $|x_d - \lambda^{-1/3}\sigma| \le 2\lambda^{-1/3}$.

In view of our previous localization we assume that $|y_d - z_d| \le \lambda^{-1/3}|\tau-\tau'| \ll 2^{-l}$ so that in the expansion (3.7) the first term is dominant and comparable to $2^{-l}|y_d - z_d|$. Arguing as in the proof of Lemma 3.4 we see that

$$|\Phi'_x(x,y) - \Phi'_x(x,z)| \ge c[2^{-l}|y_d - z_d| + |y' - \mathfrak{y}(y_d,x,z)|]$$

and integration by parts yields

$$|H_{\sigma\sigma',\tau\tau'}(y,z)| \le C \iint\limits_{\substack{|\lambda^{1/3}x'-\mu|\le 2\\ |\lambda^{1/3}x_d-\sigma|\le 2}} \frac{\lambda^{N/3}}{\lambda^N(2^{-l}|y_d - z_d| + |y' - \mathfrak{y}(y_d,x,z)|)^N}\,dx.$$

This is a favorable estimate for the range $|y_d - z_d| \ge \varepsilon 2^l\lambda^{-2/3}$; in particular, always when $|\tau - \tau'| \ge 2$.

Indeed, if $\mathcal{E}_{l,y_d,z_d,n} = \{z' : |y' - \mathfrak{y}(y_d,x,z)| \le 2^{-l+n}|y_d - z_d|\}$ then $|\mathcal{E}_{l,y_d,z_d,n}| \le C(2^{-l+n}|y_d - z_d|)^{d-1}$ and therefore, for $|\tau - \tau'| \ge 2$,

$$\int |H_{\sigma\sigma',\tau\tau'}(y,z)|\,dz \le C\sum_{n>0}\lambda^{-(d+1)/3}\lambda^{N/3}(\lambda^{2/3}2^{-l+n}|\tau-\tau'|)^{-N}(2^{-l+n}\lambda^{-1/3}|\tau-\tau'|)^{d-1}$$

(the x integration yields a factor of $\lambda^{-d/3}$ and the z_d integration yields a factor of $\lambda^{-1/3}$). As we chose $N = 10d$ we may certainly sum in n and the result is that

$$\int |H_{\sigma\sigma',\tau\tau'}(y,z)|\,dz \le C2^l\lambda^{-d}(2^l\lambda^{-1/3})^{N-d}|\tau-\tau'|^{d-1-N}. \ \blacksquare$$

Proof of Lemma 3.6. We shall estimate the kernel $H_{\sigma,\tau} := H_{\sigma\sigma,\tau\tau}$ of $R^*_{\sigma\tau}R_{\sigma\tau}$. Now if we argued as in the proof of Lemma 3.5 we would not be able to get the favorable estimate if $\lambda^{-2/3} \le |y_d - z_d| \le \varepsilon 2^l\lambda^{-2/3}$. Instead we have to make a linear change of coordinates taking into account the geometry of the fold surface.

Let

$$X = X_{\mu\sigma} = (\mu\lambda^{-1/3}, \sigma\lambda^{-1/3}), \qquad Y = Y_{\mu\nu\tau} = (\nu\lambda^{-1/3}, u(X_{\mu\sigma}, \nu\lambda^{-1/3})).$$

Let $A = \Phi''_{xy}(X_{\mu\sigma}, Y_{\mu\nu\tau})$; then $\operatorname{rank} A = d-1$ and we may choose a unit vector U in the kernel of A. Let $\Pi_{U^\perp}$ be the projection to the hyperplane orthogonal to U.

Now for the relevant (x,z) we have $|(x,z) - (X_{\sigma\tau}, Y_{\mu\nu\tau})| \le C2^{-l}$ and therefore

$$\Phi''_{xy}(x,z)U = O(2^{-l}) \tag{3.17}$$

(improving the previous estimate $O(\varepsilon)$ in (2.2)). Also note that $|\Pi_{U^\perp}e_d| = O(\varepsilon)$.

We argue as in the previous proofs (taking the better estimate (3.17) into account) and obtain

$$|\Phi'_{x'}(x,y) - \Phi'_{x'}(x,z)| \ge C_0|\Pi_{U^\perp}(y-z)| - C_1 2^{-l}|y_d - z_d| - C_2|y-z|^2,$$

$$|\Phi'_{x_d}(x,y) - \Phi'_{x_d}(x,z)| \ge C_0 2^{-l}|y_d - z_d| - C_1\varepsilon|\Pi_{U^\perp}(y-z)| - C_2|y-z|^2$$

and consequently

$$|\Phi'_x(x,y)-\Phi'_x(x,z)|\ge c_1 2^{-l}|y_d-z_d|+c_2|\Pi_{U^\perp}(y-z)|.$$

Integration by parts yields

$$|H_{\sigma,\tau}(y,z)|\le C\lambda^{-d/3}\frac{\lambda^{N/3}}{\lambda^N(2^{-l}|y_d-z_d|+|\Pi_{U^\perp}(y-z)|)^N}.$$

We use this if either $|y_d-z_d|\ge 2^l\lambda^{-2/3}$ or $|\Pi_{U^\perp}(y-z)|\ge\lambda^{-2/3}$.

If both $|y_d-z_d|\le 2^l\lambda^{-2/3}$ and $|\Pi_{U^\perp}(y-z)|\le\lambda^{-2/3}$ then we just use the trivial estimate $|H_{\sigma,\tau}(y,z)|\le C\lambda^{-d/3}$. Combining the estimates and integrating we obtain

$$\sup_y\int|H_{\sigma,\tau}(y,z)|\,dz+\sup_z\int|H_{\sigma,\tau}(y,z)|\,dy\le C2^l\lambda^{-d}. \qquad\blacksquare$$

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