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Translation-invariant operators on Lorentz spaces $L(1,q)$ with $0 < q < 1$

by

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Abstract. We study convolution operators bounded on the non-normable Lorentz spaces $L^{1,q}$ of the real line and the torus. Here $0 < q < 1$. On the real line, such an operator is given by convolution with a discrete measure, but on the torus a convolutor can also be an integrable function. We then give some necessary and some sufficient conditions for a measure or a function to be a convolutor on $L^{1,q}$. In particular, when the positions of the atoms of a discrete measure are linearly independent over the rationals, we give a necessary and sufficient condition. This condition is, however, only sufficient in the general case.

0. Introduction and results. On the real line $\mathbb{R}$ and the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0,1)$, we shall consider convolution operators bounded on the Lorentz spaces $L^{1,q}$, $0 < q < 1$. A linear operator which commutes with translations and which is bounded on $L^{1,q}$ is given by convolution with a finite measure. Denote by $C^{1,q}$ the space of such convolutors on $L^{1,q}$. In the classical case $q = 1$, one has $L^{1,1} = L^1$, and $C^{1,1}$ coincides with the space of all finite Borel measures. The spaces $C^{1,q}$ with $1 < q < \infty$ and $q = \infty$ have been studied in [SJ-2] and [SJ-1]. In this paper we shall examine the spaces $C^{1,q}$ of convolutors on the Lorentz spaces $L^{1,q}$ for $0 < q < 1$. Convolutors on the Lorentz spaces $L^{p,q}$ with $0 < p < 1$ have been studied and determined by several authors, like [CO] and [SH]. See the further references given in [SJ-1] and [SJ-2].

Given a Lebesgue measurable function f on $\mathbb{R}$ or $\mathbb{T}$, or more generally on some measure space, we denote by f^* the nonincreasing rearrangement of $|f|$ on $\mathbb{R}_+$ with Lebesgue measure. The Lorentz space $L^{1,q}$, $0 < q < \infty$,

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is then defined to consist of those f for which the quasinorm

$$\|f\|_{1,q} = \left(q \int_0^\infty (tf^*(t))^q \frac{dt}{t} \right)^{1/q}$$

is finite.

An equivalent quasinorm in $L^{1,q}$ is given in terms of the distribution function,

$$\|f\|_{1,q} \approx \left(q \int_0^\infty (s|\{x : |f(x)| > s\}|)^q \frac{ds}{s} \right)^{1/q},$$

and we also have the discrete analogs

$$\|f\|_{1,q} \approx \Big(\sum_{h=-\infty}^{\infty} (2^h f^*(2^h))^q \Big)^{1/q},$$

$$\|f\|_{1,q} \approx \Big(\sum_{k=-\infty}^{\infty} (2^k |\{x : |f(x)| > 2^k\}|)^q \Big)^{1/q}.$$

Here and below we denote the Lebesgue measure of a set E by $|E|$. We denote by $c > 0$ and $C < \infty$ various constants. By $A \approx B$ we mean that A and B are comparable in the sense that $c \le A/B \le C$. The index q will be fixed in the sequel, with $0 < q < 1$. A general reference for Lorentz spaces is [HU].

In the sequel, the setting will be $\mathbb{R}$ or $\mathbb{T}$, except where otherwise explicitly stated.

As is well known, the space $L^{1,q}$ is not normable. To see this, notice first that for characteristic functions of measurable sets one has $\|\chi_E\|_{1,q} = |E|$. If $\{E_k\}$ is a sequence of disjoint sets with $|E_k| = 2^k$ and the sequence $\{2^{-k}|\alpha_k|\}$ is decreasing, then $\|\sum_k \alpha_k 2^{-k}\chi_{E_k}\|_{1,q} \approx \{\sum_k |\alpha_k|^q\}^{1/q}$. It is clear that the last expression can be much larger than $\sum_k |\alpha_k| = \sum_k \|\alpha_k 2^{-k}\chi_{E_k}\|_{1,q}$. Thus $\|\cdot\|_{1,q}$ is not equivalent with a norm. In some sense, the more different values a function takes, the more the $L^{1,q}$-quasinorm differs from the L^1-norm.

The lack of subadditivity of the quasinorm of $L^{1,q}$ is described by means of the following notions.

The *galb* of $L^{1,q}$ is defined as the set of all numerical sequences $\{\alpha_j\}$ such that $\sum_j \alpha_j f_j$ is in $L^{1,q}$ whenever the functions $\{f_j\}$ are in $L^{1,q}$ with $\|f_j\|_{1,q} \le 1$. For such a sequence, the galb quasinorm is defined as

$$\sup \Big\{ \Big\| \sum_j \alpha_j f_j \Big\|_{1,q} : \|f_j\|_{1,q} \le 1 \Big\}.$$

The notion of galb was first introduced in [TU]. As an example, we mention that the galb of $L^{1,\infty}$ is $\ell \log \ell$. The main part of this result follows from [SNW, Lemma 2.3].

Similarly, the *equidistribution galb* of $L^{1,q}$ is the set of all sequences $\{\alpha_j\}$ such that $\sum_j \alpha_j f_j$ is in $L^{1,q}$ whenever the functions $\{f_j\}$ are equidistributed and in $L^{1,q}$. The equidistribution galb quasinorm has an obvious definition. The equidistribution galb was defined and studied for $L^{1,r}$ with $1 < r < \infty$ in [SJ-2].

It is clear that the galb is contained in the equidistribution galb, and it is easy to see that the equidistribution galb of $L^{1,q}$ is contained in the Lorentz sequence space $\ell^{1,q}$ and hence in ℓ^1. In our case these inclusions are strict, as the following theorem shows. This theorem actually holds in any nonatomic σ-finite measure space.

THEOREM 1. (a) *The galb of $L^{1,q}$ is ℓ^q, with equivalence of quasinorms.*

(b) *Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence in ℓ^1 and write $\{\alpha_j^*\}_{j=1}^\infty$ for the decreasing rearrangement of the sequence $\{|\alpha_j|\}_{j=1}^\infty$. The following three conditions are equivalent:*

(i) $\{\alpha_j\}_{j=1}^\infty$ *is in the equidistribution galb of $L^{1,q}$;*

(ii) $$\Big(\sum_{n=0}^\infty 2^{n(1-q)} \Big(\sum_{j=2^{2^n}-1}^{2^{2^{n+1}}-2} \alpha_j^* \Big)^q \Big)^{1/q} < \infty;$$

(iii) $$\Big(\sum_{n=0}^\infty 2^{n(1-q)} \Big(\sum_{2^{1-2^{n+1}} < |\alpha_j|/\|\alpha\|_1 \le 2^{1-2^n}} |\alpha_j| \Big)^q \Big)^{1/q} < \infty.$$

The expressions in (ii) *and* (iii) *are equivalent to the equidistribution galb quasinorm.*

Part (a) of this theorem means in particular that the space $L^{1,q}$ is q-normed, that is, the quasinorm $\|\cdot\|_{1,q}$ is equivalent to a quasinorm whose qth power is subadditive.

We remark that it follows from Theorem 1 and [SJ-2], pp. 399–400, that the condition

$$\sum_{n=0}^\infty \Big(2^{n|1/r-1|} \sum_{j=2^{2^n}-1}^{2^{2^{n+1}}-2} \alpha_j^* \Big)^{r\wedge r/|r-1|} < \infty$$

describes the equidistribution galb in all the spaces $L^{1,r}$, $0 < r < \infty$.

The next theorem states some general, and essentially known, properties of operators on our Lorentz spaces.

THEOREM 2. (a) *Let S be a linear operator from the space of simple functions into $L^{1,q}$, and assume that there exists a positive constant d such that for every measurable characteristic function χ_E, $\|S\chi_E\|_{1,q} \le d|E|$. Then for any measurable simple function $f = \sum_k \gamma_k \chi_{E_k}$ one has $\|Sf\|_{1,q} \le Cd\|f\|_{1,q}$. In particular, S has an extension to a continuous operator on $L^{1,q}$.*

(b) *A bounded linear operator on $L^{1,q}$ has an extension to a bounded linear operator on L^1.*

(c) *Any bounded linear operator on $L^{1,q}$ which commutes with translations is given by convolution with a finite measure.*

The space $C^{1,q}$ of convolutors on $L^{1,q}$ has the natural quasinorm

$$\|\mu\|_{C^{1,q}} = \sup\{\|\mu * f\|_{1,q} : \|f\|_{1,q} \le 1\}.$$

The results stated so far are valid not only for $\mathbb{R}$ and $\mathbb{T}$, but in much greater generality. However, now we start distinguishing between these groups.

THEOREM 3. *Every bounded linear operator S on $L^{1,q}(\mathbb{R})$ which commutes with translations is given by convolution with a finite discrete measure $\mu = \sum_j \alpha_j \delta_{x_j}$, i.e., $Sf(x) = \sum_j \alpha_j f(x - x_j)$.*

It is obvious that if $\mu = \sum_j \alpha_j \delta_{x_j}$ has coefficients $\{\alpha_j\}$ in the equidistribution galb of $L^{1,q}(\mathbb{R})$, then this measure is in $C^{1,q}(\mathbb{R})$. The next results show that in many cases, but not always, this sufficient condition is also necessary and the equidistribution galb and convolutor quasinorms are equivalent.

In the sequel, it will always be understood that in the expression $\sum_j \alpha_j \delta_{x_j}$ the points x_j are distinct.

THEOREM 4. (a) *Let the set $\{x_j\} \subset \mathbb{R}$ be linearly independent over the rationals. Then the measure $\mu = \sum_j \alpha_j \delta_{x_j}$ is a convolutor on $L^{1,q}(\mathbb{R})$ if and only if the sequence $\{\alpha_j\}$ is in the equidistribution galb of $L^{1,q}(\mathbb{R})$.*

(b) *There exists a measure $\mu = \sum_j \alpha_j \delta_{x_j}$ which is a convolutor on $L^{1,q}(\mathbb{R})$ but such that the sequence $\{\alpha_j\}$ does not belong to the equidistribution galb of $L^{1,q}(\mathbb{R})$.*

The example we shall produce to prove Theorem 4(b) is a signed measure. We do not know whether the equidistribution galb condition is necessary for all positive discrete measures, as happens for $L^{1,r}(\mathbb{R})$ with $2 \le r \le \infty$. See [SJ-1] and [SJ-2]. However, in this direction we have the following two results.

THEOREM 5. *Let $\mu = \sum_j \alpha_j \delta_{x_j}$, with $\{\alpha_j\}$ positive. Then the convolutor quasinorm on $L^{1,q}(\mathbb{R})$ of this measure is at least*

$$c\Big(\sum_j \alpha_j\Big)\left(1 + \log\left(\frac{\sum_j \alpha_j}{\sqrt{\sum_j \alpha_j^2}}\right)\right)^{1/q-1}.$$

We point out that in some cases of interest the last expression is close to the quantities characterizing the equidistribution galb in Theorem 1(b). For example, Theorems 1 and 5 imply that when the measure is a sum of N equal point masses, $\mu = \sum_{j=1}^N \delta_{x_j}$, then the equidistribution galb and the convolutor quasinorm are both of the order of $N(\log(N))^{1/q-1}$.

THEOREM 6. *Let $\mu = \sum_{j=1}^\infty \alpha_j \delta_j$ be a positive measure supported on the positive integers and with masses $\{\alpha_j\}$ that form a decreasing sequence. Then μ is a convolutor on $L^{1,q}(\mathbb{R})$ if and only if the sequence $\{\alpha_j\}$ is in the equidistribution galb of $L^{1,q}(\mathbb{R})$, and the convolutor quasinorm of μ is equivalent to the equidistribution galb quasinorm of $\{\alpha_j\}$.*

Turning now to the torus, we remark that some results for $\mathbb{R}$ are easily extended to $\mathbb{T}$. For example, Theorem 4 holds unchanged for $C^{1,q}(\mathbb{T})$. In Theorem 3 we saw that convolutors in $C^{1,q}(\mathbb{R})$ are atomic measures. However, for the torus the situation is different, because the space $C^{1,q}(\mathbb{T})$ contains, for instance, all functions in $L^p(\mathbb{T})$ with $p > 1$. Indeed, we have the inclusions

$$L^p(\mathbb{T}) * L^{1,q}(\mathbb{T}) \subseteq L^p(\mathbb{T}) * L^1(\mathbb{T}) \subseteq L^p(\mathbb{T}) \subsetneq L^{1,q}(\mathbb{T}).$$

The following results give more precise estimates of the size of functions which are convolutors on $L^{1,q}(\mathbb{T})$.

THEOREM 7. *The convolutor quasinorm on $L^{1,q}(\mathbb{T})$ of the characteristic function of a measurable set $E \subseteq \mathbb{T}$ is comparable to $|E|(1+\log(1/|E|))^{1/q-1}$.*

This result gives a sharp estimate for the convolutor quasinorm of a characteristic function. Since the space $L^{1,q}(\mathbb{T})$ is q-normed, to estimate the convolutor quasinorm of an arbitrary function, it is natural to consider an ℓ^q sum of expressions analogous to that of Theorem 7. Our next result shows that this gives an upper estimate for the convolutor quasinorm which is sharp in at least some cases.

THEOREM 8. (a) *Assume that f is a measurable function on $\mathbb{T}$ and that*

$$\Big(\sum_{m=0}^{\infty}\Big(2^{m(1/q-1)}\int_{2^{-2^{m+1}}}^{2^{-2^m}} f^*(t)\,dt\Big)^q\Big)^{1/q} < \infty.$$

Then the function f is a convolutor on $L^{1,q}(\mathbb{T})$, and its convolutor quasinorm is dominated by the above quantity.

(b) *Let us identify functions on the torus $\mathbb{T}$ with functions on the interval $[0,1)$. If the function f is positive and decreasing on $[0,1)$, then the condition in (a) is necessary for f to be a convolutor on $L^{1,q}(\mathbb{T})$ and the convolutor quasinorm of f is equivalent to the quantity in (a).*

Observe that the monotonicity of the function in part (b) of this theorem is analogous to the monotonicity of the point masses in Theorem 6. Both results say that the sufficient condition is also necessary. Moreover, the proofs of these two results are based on the same ideas.

Before the proofs of the theorems, we make some remarks.

For the torus, we have a proof that measures like the triadic Cantor measure are not convolutors on $L^{1,q}(\mathbb{T})$. However, we do not know whether $C^{1,q}(\mathbb{T})$ contains measures which are continuous but singular with respect to Lebesgue measure.

A (*Fourier*) *multiplier* on $L^p(\mathbb{R})$, or on $L^p(\mathbb{T})$, is the Fourier transform of a convolutor. In particular, a multiplier on $L^p(\mathbb{R})$ is a function on $\mathbb{R}$, while a multiplier on $L^p(\mathbb{T})$ is a sequence on $\mathbb{Z}$.

When $1 \le p \le \infty$, there exists a close relation between multipliers on $L^p(\mathbb{R})$ and $L^p(\mathbb{T})$: The restriction to the integers of a continuous function which is a multiplier on $L^p(\mathbb{R})$ is a multiplier on $L^p(\mathbb{T})$. Conversely, we can interpolate a sequence which is a multiplier on $L^p(\mathbb{T})$ and obtain a function which is a multiplier on $L^p(\mathbb{R})$. These classical results are due to K. de Leeuw [DLE]; see also the book [SGW], Section VII.3.

We do not know whether the restriction of a multiplier on $L^{1,q}(\mathbb{R})$ is a multiplier on $L^{1,q}(\mathbb{T})$. However, in some cases it is impossible to extend a multiplier on $L^{1,q}(\mathbb{T})$ to a multiplier on $L^{1,q}(\mathbb{R})$. The reason is the following. A multiplier on $L^{1,q}(\mathbb{R})$ is the Fourier transform of a discrete and finite measure on $\mathbb{R}$. Hence, the restriction to the integers of a multiplier on $L^{1,q}(\mathbb{R})$ is the Fourier transform of a discrete and finite measure on $\mathbb{T}$. But on $L^{1,q}(\mathbb{T})$ there are multipliers which are the Fourier transforms of absolutely continuous measures, and these multipliers cannot be extended to multipliers on $L^{1,q}(\mathbb{R})$.

1. Proof of Theorem 1. To prove (a), we observe that a function f is in $L^{1,q}$ if and only if $|f|^q$ is in $L^{1/q,1}$, and this last space is Banach; see also [HU, p. 258]. Thus ℓ^q is contained in the galb of $L^{1,q}$.

For the converse inclusion, let $f_j = 2^j \chi_{E_j}$ for $j = 1, 2, \ldots$, where the sets E_j are pairwise disjoint with $|E_j| = 2^{-j}$. Then $\|\sum \alpha_j f_j\|_{1,q} \ge c\|\alpha\|_q$ for any sequence $\alpha = \{\alpha_j\}_{j=1}^{\infty}$.

Part (b) will be obtained from the characterization of the equidistribution galb given in [SJ-2, Theorem 2]. Observe first that a slight modification of the simple Lemma 10 of [SJ-2] implies the equivalence of (ii) and (iii). Let, as in [SJ-2], for $n = 1, 2, \ldots$,

$$c_n = \sum_{2^{-n} < \alpha_i/\|\alpha\|_1 \le 2^{1-n}} |\alpha_i|.$$

Then (iii) is equivalent to

$$\sum_{m=0}^{\infty} 2^{(1-q)m} \Big(\sum_{n=2^m}^{2^{m+1}-1} c_n \Big)^q < \infty.$$

Theorem 2 of [SJ-2] states that α is in the equidistribution galb of $L^{1,q}$ if and only if the operator

$$T : \{x_j\}_{j=-\infty}^{\infty} \to \{y_k\}_{k=-\infty}^{\infty}, \qquad y_k = \sum_{n=1}^{\infty} \sum_{j=k}^{k+n} m_{kj}^n x_j,$$

is bounded on ℓ^q for any nonnegative numbers m_{kj}^n, defined for integer k, j, n with $n \ge 1$ and $k \le j \le k+n$, and satisfying $\sum_{k=j-n}^{j} m_{kj}^n \le c_n$ for each j and n.

To verify that (iii) implies that T is bounded, we observe that for $x_j \ge 0$,

$$\sum_k y_k^q = \sum_k \Big(\sum_{n=1}^{\infty} \sum_{j=k}^{k+n} m_{kj}^n x_j \Big)^q \le \sum_{m=0}^{\infty} \sum_k \Big(\sum_{n=2^m}^{2^{m+1}-1} \sum_{j=k}^{k+n} m_{kj}^n x_j \Big)^q.$$

We split the sum over k in the last expression into parts corresponding to $2^{m+1}\nu \le k < 2^{m+1}(\nu+1)$, with $\nu \in \mathbb{Z}$. For each fixed m and ν, we then have the sum

$$\sum_{k=2^{m+1}\nu}^{2^{m+1}(\nu+1)-1} \Big(\sum_{n=2^m}^{2^{m+1}-1} \sum_{j=k}^{k+n} m_{kj}^n x_j \Big)^q$$

which by Hölder's inequality is at most

$$\begin{aligned} C2^{(1-q)m} & \Big(\sum_{k=2^{m+1}\nu}^{2^{m+1}(\nu+1)-1} \sum_{n=2^m}^{2^{m+1}-1} \sum_{j=k}^{k+n} m_{kj}^n x_j \Big)^q \\ & \le C2^{(1-q)m} \Big(\sum_{n=2^m}^{2^{m+1}-1} \sum_{j=2^{m+1}\nu}^{2^{m+1}(\nu+2)-1} x_j \sum_{k=j-n}^{j} m_{kj}^n \Big)^q \\ & \le C2^{(1-q)m} \Big(\sum_{n=2^m}^{2^{m+1}-1} \sum_{j=2^{m+1}\nu}^{2^{m+1}(\nu+2)-1} x_j c_n \Big)^q \\ & \le C2^{(1-q)m} \Big(\sum_{n=2^m}^{2^{m+1}-1} c_n \Big)^q \sum_{j=2^{m+1}\nu}^{2^{m+1}(\nu+2)-1} x_j^q. \end{aligned}$$

Summing over ν and m, we obtain

$$\sum_k y_k^q \le C \sum_{m=0}^{\infty} 2^{(1-q)m} \Big(\sum_{n=2^m}^{2^{m+1}-1} c_n \Big)^q \sum_j x_j^q,$$

and so (iii) implies (i).

For the converse implication, we let $m_{kj}^n = 2^{1-m} c_n$ for $j = 0$, $2^m \le n < 2^{m+1}$, $-2^m \le k < -2^{m-1}$ with $m = 0, 1, \ldots$, and $m_{kj}^n = 0$ otherwise.

With $x_0=1$ and $x_j=0$ if $j\neq 0$, for $-2^m\le k<-2^{m-1}$ we get

$$y_k=\sum_{n=2^m}^{2^{m+1}-1}2^{1-m}c_n.$$

Thus

$$\sum_k y_k^q\ge\sum_{m=0}^{\infty}2^{m-1}2^{q(1-m)}\Big(\sum_{n=2^m}^{2^{m+1}-1}c_n\Big)^q,$$

and (iii) follows from (i) via [SJ-2, Theorem 2].

Theorem 1 is now proved. We also give a proof of the implication (iii)⇒(i) which does not use [SJ-2].

We can assume that $\alpha_j\ge 0$ and $\|\alpha\|_1=1$. It is enough to consider nonnegative functions in $L^{1,q}$. Let A_j, $j\ge 1$, be measurable sets with equal finite measure $|A_j|=a$. We first give an estimate for the $L^{1,q}$ quasinorm of a finite sum $\sum_{j=1}^N\alpha_j\chi_{A_j}$.

Observe that the set where $\sum_{j=1}^N\alpha_j\chi_{A_j}\neq 0$ has measure at most Na, that $\int\sum_{j=1}^N\alpha_j\chi_{A_j}(x)\,dx=a\sum_{j=1}^N\alpha_j$ and that $\sum_{j=1}^N\alpha_j\chi_{A_j}(x)\le\sum_{j=1}^N\alpha_j$. Hence,

$$q\int_0^a\Big(t\Big(\sum_{j=1}^N\alpha_j\chi_{A_j}\Big)^*(t)\Big)^q\frac{dt}{t}\le\Big(\sum_{j=1}^N\alpha_j\Big)^q\int_0^a qt^{q-1}\,dt=\Big(\sum_{j=1}^N\alpha_j\Big)^q a^q,$$

and

$$q\int_a^{Na}\Big(t\Big(\sum_{j=1}^N\alpha_j\chi_{A_j}\Big)^*(t)\Big)^q\frac{dt}{t}\le q\Big(\int_a^{Na}\frac{dt}{t}\Big)^{1-q}\Big(\int_a^{Na}\Big(\sum_{j=1}^N\alpha_j\chi_{A_j}\Big)^*(t)\,dt\Big)^q$$

$$\le q(\log N)^{1-q}\Big(\sum_{j=1}^N\alpha_j\Big)^q a^q.$$

We thus have

$$\Big\|\sum_{j=1}^N\alpha_j\chi_{A_j}\Big\|_{1,q}^q\le C(1+\log N)^{1-q}\Big(\sum_{j=1}^N\alpha_j\Big)^q a^q.$$

For the corresponding infinite sum we get, since $L^{1,q}$ is q-normed,

$$\Big\|\sum_{j=1}^\infty\alpha_j\chi_{A_j}\Big\|_{1,q}^q\le\sum_{n=0}^\infty\Big\|\sum_{2^{1-2^{n+1}}<\alpha_j\le 2^{1-2^n}}\alpha_j\chi_{A_j}\Big\|_{1,q}^q.$$

The number of α_j exceeding $2^{1-2^{n+1}}$ is at most $2^{-1+2^{n+1}}$, and the finite case just treated implies that

$$\Big\|\sum_{j=1}^\infty\alpha_j\chi_{A_j}\Big\|_{1,q}^q\le C\sum_{n=0}^\infty 2^{(1-q)n}\Big(\sum_{2^{1-2^{n+1}}<\alpha_j\le 2^{1-2^n}}\alpha_j\Big)^q a^q.$$

This gives the required estimate for $\sum\alpha_jf_j$ when the f_j are equidistributed characteristic functions.

In the general case, all the f_j are equidistributed with some function f. We now assume that f is of the form $f=\sum_k\beta_k\chi_{E_k}$, where the E_k are disjoint with $|E_k|=2^{-k}$ and the β_k increasing. Here k ranges over $\mathbb{Z}$ or $\mathbb{N}$ when the setting is $\mathbb{R}$ or $\mathbb{T}$, respectively. Thus $f_j=\sum_k\beta_k\chi_{E_k^j}$ for each j, with sets E_k^j like the E_k just described. Since $L^{1,q}$ is q-normed, we have, in view of the case treated above,

$$\Big\|\sum_j\alpha_j\sum_k\beta_k\chi_{E_k^j}\Big\|_{1,q}^q\le\sum_k\Big\|\sum_j\alpha_j\chi_{E_k^j}\Big\|_{1,q}^q\beta_k^q$$

$$\le C\sum_{n=0}^\infty 2^{(1-q)n}\Big(\sum_{2^{1-2^{n+1}}<\alpha_j\le 2^{1-2^n}}\alpha_j\Big)^q\sum_k\beta_k^q2^{-qk}.$$

Thus

$$\Big\|\sum\alpha_jf_j\Big\|_{1,q}^q\le C\sum_{n=0}^\infty 2^{(1-q)n}\Big(\sum_{2^{1-2^{n+1}}<\alpha_j\le 2^{1-2^n}}\alpha_j\Big)^q\|f\|_{1,q}^q.$$

This last inequality now follows in the general case, since any $L^{1,q}$ function can be majorized by an $L^{1,q}$ function of the form $\sum\beta_k\chi_{E_k}$, with equivalent quasinorm. The implication (iii)⇒(i) follows. ■

2. Proof of Theorem 2.

To prove (a), first assume that $f=\sum_k\gamma_k\chi_{E_k}$, where the sum is finite and each γ_k is positive with finite dyadic expansion. Then we can write $f=\sum_j 2^j\chi_{A_j}$, where the sum is finite. Since S is linear and $L^{1,q}$ is q-normed, we thus have

$$\|Sf\|_{1,q}^q\le\sum_j(2^j\|S\chi_{A_j}\|_{1,q})^q\le Cd^q\sum_j(2^j|A_j|)^q$$

$$\le Cd^q\sum_j(2^j|\{x:|f(x)|\ge 2^j\}|)^q\le Cd^q\|f\|_{1,q}^q.$$

If $f=\sum_k\gamma_k\chi_{E_k}$, where the sum is finite and the γ_k are positive real numbers, we can find an approximation $g=\sum_k\beta_k\chi_{E_k}$, with β_k positive with finite dyadic expansion and $|\gamma_k-\beta_k|$ suitably small. Then

$$\|Sf\|_{1,q}^q\le\|Sg\|_{1,q}^q+\sum_k(|\gamma_k-\beta_k|\,\|S\chi_{E_k}\|_{1,q})^q$$

$$\le Cd^q\Big(\|g\|_{1,q}^q+\sum_k(|\gamma_k-\beta_k|\,|E_k|)^q\Big)\le Cd^q\|f\|_{1,q}^q.$$

The general case, where f is a complex-valued simple function, now follows.

To prove (b), assume that S is bounded on $L^{1,q}$. For any set E of finite measure, one has

$$\|S\chi_E\|_1 \le \|S\chi_E\|_{1,q} \le C|E|.$$

The boundedness on L^1 follows; cf. part (a).

Since a bounded operator on L^1 which commutes with translations is given by convolution with a finite Borel measure, part (c) follows from (b). ∎

3. Proof of Theorem 3. In this proof we shall denote by $\|\mu\|$ the total mass of a finite Borel measure μ on $\mathbb{R}$. Suppose that $\mu = \delta + \nu$ with δ atomic and ν continuous and not zero. We have to show that μ is not a convolutor on $L^{1,q}(\mathbb{R})$. The idea is to construct a measurable set A such that the convolution $\mu * \chi_A$ is very different from a characteristic function and has large $L^{1,q}$ quasinorm. As an intermediate step toward this goal, we construct characteristic functions χ_{A_j} such that the convolutions $\mu * \chi_{A_j}$ take very different values.

LEMMA 3.1. *If $\eta > 0$ and if $\varepsilon > 0$ is small enough, we can find ξ with $0 < \xi < \eta$ such that $I = \{x \in \mathbb{R} : \xi < |\mu * \chi_{[0,\varepsilon]}(x)| < \eta\}$ satisfies $\int_I |\mu * \chi_{[0,\varepsilon]}(x)|\,dx \ge \frac{\varepsilon}{2}\|\nu\|$.*

Proof. Suppose for simplicity ν is real, and decompose ν into $\nu_+ - \nu_-$, with ν_+ and ν_- positive and ν_+, ν_-, δ mutually singular. Suppose also that ν_+, ν_- and δ are different from zero. If any of these measures are zero, the proof simplifies.

We can find disjoint compact sets K_+, K_-, D such that

$$\nu_+(\mathbb{R}\setminus K_+) + \nu_-(\mathbb{R}\setminus K_-) + |\delta|(\mathbb{R}\setminus D) \le \tfrac{1}{8}\min\{\|\nu_+\|, \|\nu_-\|, \|\delta\|, \eta\}.$$

Then for $\varepsilon > 0$,

$$\int\limits_{K_+ + [0,\varepsilon]} \nu_+ * \chi_{[0,\varepsilon]}(x)\,dx \ge \varepsilon\nu_+(K_+) \ge \frac{7}{8}\varepsilon\|\nu_+\|,$$

and moreover, if 2ε is smaller than each of the distances between the three compact sets K_+, K_- and D, then

$$\int\limits_{K_+ + [0,\varepsilon]} (\nu_- + |\delta|) * \chi_{[0,\varepsilon]}(x)\,dx \le \frac{\varepsilon}{4}\|\nu_+\|.$$

Since ν_+ is continuous, taking ε small enough we can also obtain $\nu_+ * \chi_{[0,\varepsilon]}(x) < \eta/2$ everywhere, and in view of the above inequalities it is clear that for every x in $K_+ + [0,\varepsilon]$ we have $(\nu_- + |\delta|) * \chi_{[0,\varepsilon]}(x) < \eta/4$.

Summing up, in $K_+ + [0,\varepsilon]$ we thus obtain $|\mu * \chi_{[0,\varepsilon]}(x)| < \eta$, and $\int_{K_+ + [0,\varepsilon]} |\mu * \chi_{[0,\varepsilon]}(x)|\,dx \ge \frac{5}{8}\varepsilon\|\nu_+\|$.

Combining this with the corresponding inequality for ν_- and K_-, we finally obtain

$$\int\limits_{\{|\mu*\chi_{[0,\varepsilon]}(x)|<\eta\}} |\mu * \chi_{[0,\varepsilon]}(x)|\,dx \ge \frac{5}{8}\varepsilon\|\nu\|.$$

It then only remains to select ξ small enough, and the lemma follows.

LEMMA 3.2. *Given $\eta > 0$, we can find a compact set A with $|A| = 1$, and ξ with $0 < \xi < \eta$, such that if $B = \{x \in \mathbb{R} : \xi < |\mu * \chi_A(x)| < \eta\}$, then we have $\int_B |\mu * \chi_A(x)|dx \ge \frac{1}{3}\|\nu\|$.*

Proof. Define $A = \bigcup_{j=1}^{1/\varepsilon}[x_j, x_j + \varepsilon]$, with $1/\varepsilon$ a suitably large integer and the sequence of points $\{x_j\}$ very sparse. Then we have $\mu * \chi_A(x) = \sum_{j=1}^{1/\varepsilon} \mu * \chi_{[0,\varepsilon]}(x - x_j)$. It is clear that if the sequence $\{x_j\}$ is very sparse, then the terms in the above sum interfere very little with each other. Hence this lemma follows from the previous one. ∎

LEMMA 3.3. *Let the functions $f_k, -\infty < k < \infty$, be supported on disjoint sets C_k, let $\inf_{x\in C_k}|f_k(x)| = \xi_k$ and $\sup_{x\in C_k}|f_k(x)| = \eta_k$, with $\xi_k > 2\eta_{k+1}$. Then $\|\sum_k f_k\|_{1,q} \approx \{\sum_k \|f_k\|_{1,q}^q\}^{1,q}$.*

Proof. It can be easily verified that an equivalent quasinorm on the Lorentz space $L^{1,q}$ is defined by

$$\|f\|_{1,q}^q \approx \sum_{j=-\infty}^{\infty} (2^j|\{x : 2^j \le |f(x)| < 2^{j+1}\}|)^q.$$

Hence,

$$\Big\|\sum_k f_k\Big\|_{1,q}^q \approx \sum_{j=-\infty}^{\infty}\Big(2^j\Big|\Big\{x : 2^j \le \Big|\sum_k f_k(x)\Big| < 2^{j+1}\Big\}\Big|\Big)^q$$
$$\approx \sum_k \sum_{\xi_k/2\le 2^j\le\eta_k} (2^j|\{x : 2^j \le |f_k(x)| < 2^{j+1}\}|)^q \approx \sum_k \|f_k\|_{1,q}^q. \ ∎$$

We can now finish the proof of Theorem 3. By Lemma 3.2, it is possible to construct a sequence $\eta_1 > \xi_1 > \eta_2 > \xi_2 > \ldots$ with $\xi_k > 8\eta_{k+1}$, and compact sets $\{A_k\}_{k=1}^N$ with $|A_k| = 1$, such that if $B_k = \{x \in \mathbb{R} : \xi_k < |\mu * \chi_{A_k}(x)| < \eta_k\}$ then $\int_{B_k} |\mu * \chi_{A_k}(x)|\,dx \ge \frac{1}{3}\|\nu\|$.

Define $A = \bigcup_{k=1}^N \{x_k + A_k\}$, where again the points $\{x_k\}$ are very sparse, and denote by C_k the mutually disjoint sets $\{x \in \mathbb{R} : \xi_k/2 < |\mu * \chi_A(x)| < 2\eta_k\}$. Then, if the sequence of points $\{x_k\}$ is suitably chosen, we have $|A| = N$, and $\int_{C_k} |\mu * \chi_A(x)|\,dx \ge \frac{1}{4}\|\nu\|$. Hence $\|\chi_A\|_{1,q} = N$, but by the previous lemma, $\|\mu * \chi_A\|_{1,q} \ge cN^{1/q}\|\nu\|$. Thus the measure μ is not a convolutor on $L^{1,q}(\mathbb{R})$, and Theorem 3 is proved. ∎

4. Proof of Theorem 4. The proof of (a) is contained in Theorem 3 of [SJ-2]. To prove (b), we use a construction based on Rudin–Shapiro polynomials. See [KA, Exercise I.6.6, p. 33]. These polynomials are defined recursively by $P_0(t) = Q_0(t) = 1$ and

$$P_{m+1}(t) = P_m(t) + \exp(2\pi i 2^m t) Q_m(t),$$
$$Q_{m+1}(t) = P_m(t) - \exp(2\pi i 2^m t) Q_m(t).$$

One easily proves by induction that $P_m(t) = \sum_{j=0}^{2^m-1} \varepsilon_j \exp(2\pi i j t)$ with $\varepsilon_j = \pm 1$, and that $\|P_m\|_\infty \le 2^{(m+1)/2}$.

Let us consider the measure $\nu_m = \sum_{j=0}^{2^m-1} \varepsilon_j \delta_j$. This measure on $\mathbb{R}$ has Fourier transform $\widehat{\nu}_m(t) = \sum_{j=0}^{2^m-1} \varepsilon_j \exp(-2\pi i j t) = P_m(-t)$. Hence for each f in $L^p(\mathbb{R})$, $0 < p < 1$,

$$\|\nu_m * f\|_p = \Big\| \sum_{j=0}^{2^m-1} \varepsilon_j f(\cdot - j) \Big\|_p \le 2^{m/p} \|f\|_p,$$

and for each f in $L^2(\mathbb{R})$,

$$\|\nu_m * f\|_2 = \|\widehat{\nu}_m \cdot \widehat{f}\|_2 \le \|\widehat{\nu}_m\|_\infty \|\widehat{f}\|_2 \le 2^{(m+1)/2} \|f\|_2.$$

Thus, by interpolation, $\|\nu_m * f\|_{1,q} \le C 2^m \|f\|_{1,q}$.

Conversely, $\|\nu_m * \chi_{[0,\varepsilon]}\|_{1,q} = 2^m \|\chi_{[0,\varepsilon]}\|_{1,q}$ if ε is small. Hence the convolutor quasinorm on $L^{1,q}(\mathbb{R})$ of the measure ν_m is about 2^m, which is the total mass of ν_m.

Let $\{\mu_m\}$ be translates of $\{\nu_m\}$ with pairwise disjoint supports. Define $\mu = \sum_{m=0}^\infty \beta_m \mu_m = \sum_{j=0}^\infty \alpha_j \delta_j$, for some coefficients β_m and with α_j determined by the β_m. Since $L^{1,q}(\mathbb{R})$ is q-normed, $\sum_{m=0}^\infty 2^{qm} |\beta_m|^q < \infty$ implies that the measure μ is a convolutor of $L^{1,q}(\mathbb{R})$. On the other hand, choosing β_m nonzero only when m is a power of 2, we can have $\{2^m \beta_m\}$ in ℓ^q, but $\{\alpha_j\}$ not in the equidistribution galb of $L^{1,q}(\mathbb{R})$, because of the factor $2^{n(1-q)}$ in Theorem 1(b). ■

5. Proof of Theorem 5. We begin by proving a version of this theorem for the Lorentz sequence space $\ell^{1,q}(\mathbb{Z})$.

Let the sequence $\{\alpha_j\}$ be positive and summable, and let $\{n_j\}$ be a sequence of distinct integers. We want to give an estimate from below of the convolutor quasinorm of $\mu = \sum_j \alpha_j \delta_{n_j}$ on $\ell^{1,q}(\mathbb{Z})$.

LEMMA 5.1. *Let $\mu = \sum_{j=1}^N \alpha_j \delta_{n_j}$, with $\alpha_j > 0$ and $\sum_{j=1}^N \alpha_j = 1$. Also let $16 \sum_{j=1}^N \alpha_j^2 < p < 1$. Then, if M is large enough, there exists a set $A \subset \{0, 1, \ldots, M\}$ with $\# A \approx Mp$ and such that $\mu * \chi_A \approx p$ on a set of cardinality at least cM.*

Proof. We shall use a probabilistic argument and write $\mathbb{P}$ for probability and $\mathbb{E}$ for expectation. Let $A \subset \{0, 1, \ldots, M\}$ be a random set such that the events $j \in A$ for $0 \le j \le M$ are independent and have probability p. Let $m = \max_{1 \le j \le N} |n_j|$, and let $m \le n \le M - m$. Then, since $n - n_j$ is in A with probability p, we have

$$\mathbb{E}[\mu * \chi_A(n)] = \sum_{j=1}^N \alpha_j \mathbb{E}[\chi_A(n - n_j)] = p,$$

and

$$\begin{aligned}
\mathbb{E}[(\mu * \chi_A(n) - p)^2] &= \mathbb{E}[(\mu * \chi_A(n))^2] - p^2 \\
&= \sum_{j=1}^N \alpha_j^2 \mathbb{E}[(\chi_A(n - n_j))^2] \\
&\quad + \sum_{1 \le i \ne j \le N} \alpha_i \alpha_j \mathbb{E}[\chi_A(n - n_i) \chi_A(n - n_j)] - p^2 \\
&= p \sum_{j=1}^N \alpha_j^2 + p^2 \sum_{1 \le i \ne j \le N} \alpha_i \alpha_j - p^2 = (p - p^2) \sum_{j=1}^N \alpha_j^2.
\end{aligned}$$

By applying Chebyshev's inequality, we get

$$\begin{aligned}
\mathbb{P}\Big[\#\{n \in [m, M - m] : |\mu * \chi_A(n) - p| > p/2\} \ge \frac{M - 2m + 1}{2}\Big] \\
\le \mathbb{P}\Big[\sum_{n=m}^{M-m} (\mu * \chi_A(n) - p)^2 \ge \frac{M - 2m + 1}{8} p^2\Big] \\
\le \frac{8}{(M - 2m + 1)p^2} \mathbb{E}\Big[\sum_{n=m}^{M-m} (\mu * \chi_A(n) - p)^2\Big] \\
= 8 \frac{1 - p}{p} \sum_{j=1}^N \alpha_j^2 \le \frac{8}{p} \sum_{j=1}^N \alpha_j^2 < \frac{1}{2}.
\end{aligned}$$

Since $\mathbb{E}[\# A] = (M + 1)p$, we also get $\mathbb{P}[\# A > 2(M + 1)p] \le 1/2$. Hence, there exists a set A with $\# A \le 2(M + 1)p$ and with

$$\#\Big\{n : \frac{1}{2} p \le \mu * \chi_A(n) \le \frac{3}{2} p\Big\} \ge \frac{M - 2m + 1}{2}. \ ■$$

LEMMA 5.2. (a) *Let $\mu = \sum_{j=1}^N \alpha_j \delta_{n_j}$ with $\alpha_j > 0$ and $\sum_{j=1}^N \alpha_j = 1$, and let $\nu = \big[\frac{1}{2} \log(1 / \sum_{j=1}^N \alpha_j^2)\big]$. Then for some large a there exist sets $\{A_k\}_{k=1}^\nu$ in $\mathbb{Z}$ with $\# A_1 = \ldots = \# A_\nu = a$ such that the convolutions $\{\mu * \chi_{A_k}\}_{k=1}^\nu$*

have mutually disjoint supports and $\mu * \chi_{A_k} \ge c2^{-k}\chi_{B_k}$ *for some sets* B_k *with* $\#B_k \approx a2^k$ *for* $k = 1, \dots, \nu$.

(b) *With* μ *and* $\{A_k\}_{k=1}^{\nu}$ *as in* (a), *let* $A = \bigcup_{k=1}^{\nu} A_k$. *Then*

$$\|\mu * \chi_A\|_{\ell^{1,q}} \ge c\Big(\sum_{k=1}^{\nu}(2^{-k}\#B_k)^q\Big)^{1/q} \ge c(\#A)\Big|\log\Big(\sum_{j=1}^{N}\alpha_j^2\Big)\Big|^{1/q-1}.$$

(c) *If the sequence* $\{\alpha_j\}$ *is positive and the integers* $\{n_j\}$ *are distinct, then the convolutor quasinorm of the measure* $\mu = \sum_j \alpha_j\delta_{n_j}$ *on the Lorentz sequence space* $\ell^{1,q}(\mathbb{Z})$ *is at least*

$$c\Big(\sum_j \alpha_j\Big)\left(1 + \log\left(\frac{\sum_j \alpha_j}{\sqrt{\sum_j \alpha_j^2}}\right)\right)^{1/q-1}.$$

Proof. Part (a) is an easy consequence of the previous lemma. It is enough to choose $p \approx 2^{-k}$ and $Mp \approx a$. To obtain disjoint supports, one applies suitable translations.

Part (b) follows from (a), and (c) follows from (b). ■

Let us go back to the problem of estimating the convolutor quasinorm on $L^{1,q}(\mathbb{R})$ of a positive measure $\mu = \sum_j \alpha_j\delta_{x_j}$. We assume this sum is finite, $\mu = \sum_{j=1}^N \alpha_j\delta_{x_j}$. Since the convolutor quasinorm is invariant under dilations, we can also assume that $|x_i - x_j| > 3$ if $1 \le i \ne j \le N$.

Let $[x_j] = n_j$. We claim that the convolutor quasinorm of the measure $\mu = \sum_{j=1}^N \alpha_j\delta_{x_j}$ as an operator on $L^{1,q}(\mathbb{R})$ is bounded from below by the convolutor quasinorm of $\sum_{j=1}^N \alpha_j\delta_{n_j}$ as an operator on $\ell^{1,q}(\mathbb{Z})$. To see this, it is enough to convolve μ with characteristic functions of sets $A + [-1, 1]$, where $A \subset \mathbb{Z}$.

The proof of Theorem 5 is complete. ■

6. Proof of Theorem 6.

If the sequence $\{\alpha_j\}_{j=1}^{\infty}$ belongs to the equidistribution galb of $L^{1,q}$, then of course the measure $\mu = \sum \alpha_j\delta_j$ is a convolutor on $L^{1,q}(\mathbb{R})$. What we need to prove is that if $\{\alpha_j\}$ is decreasing and if μ is a convolutor on $L^{1,q}(\mathbb{R})$, then $\{\alpha_j\}$ belongs to the equidistribution galb of $L^{1,q}$. We shall prove this with $\mathbb{R}$ replaced by $\mathbb{Z}$. Theorem 6 then follows if one considers functions on $\mathbb{R}$ constant between the integers; cf. the end of the preceding section. Since our setting is now $\mathbb{Z}$, we write intervals as $[a, b]$, meaning $\mathbb{Z} \cap [a, b]$.

For $n \ge 0$ define

$$\mu_n = \sum_{j=2^{2^n}-1}^{2^{2^{n+1}}-2} \alpha_j\delta_j.$$

Then $\mu = \sum_{n=0}^{\infty}\mu_n$, and we need to prove that the convolutor quasinorm of this measure is bounded from below by

$$c\Big(\sum_{n=0}^{\infty} 2^{(1-q)n}\|\mu_n\|^q\Big)^{1/q}.$$

The following lemma states the existence of a (characteristic) function whose convolution with μ has large level sets for many different levels. Let

$$l_n = 2^{1+2^{n-2}+2^{n+1}}, \quad n = 2, 3, \dots$$

LEMMA 6.1. *There exists a natural number* n_0 *with the following property. For each* $n \ge n_0$ *one can find a set* $F_n \subset [1, l_n]$ *with* $\#F_n = 2^{n-1+2^{n+1}}$ *such that for all* $m \equiv n \pmod 2$ *with* $n_0 \le m \le n$ *the inequality* $\mu * \chi_{F_n} \ge 2^{-j}\|\mu_m\|$ *holds on a subset of* $\mathbb{Z}$ *of cardinality at least* $2^{j-m}\#F_n$, *for* $2^{m-3} \le j < 2^{m-2}$.

Before proving this lemma, we use it to complete the proof of Theorem 6. Take a large n, and observe that Lemma 6.1 implies

$$(1)\qquad (\mu * \chi_{F_n})^*(2^{j-m}\#F_n) \ge 2^{-j}\|\mu_m\|, \quad 2^{m-3} \le j < 2^{m-2},$$

for those values of m described in the lemma. It is easily verified that the open intervals $(2^{j-m-1}, 2^{j-m}) \subset \mathbb{R}$ are pairwise disjoint for the values of j and m occurring here. Thus we conclude from (1) that

$$\begin{aligned}\|\mu * \chi_{F_n}\|_{1,q}^q &= q\int_0^{\infty}(t(\mu * \chi_{F_n})^*(t))^q\,\frac{dt}{t}\\ &\ge c\sum_{j,m}(2^{-m}\|\mu_m\|\#F_n)^q \ge c\sum_m 2^{(1-q)m}\|\mu_m\|^q(\#F_n)^q;\end{aligned}$$

here and in the next sum j and m are again as in the lemma. Since $\|\chi_{F_n}\|_{1,q} = \#F_n$, this implies

$$\|\mu\|_{C^{1,q}}^q \ge c\sum_m 2^{(1-q)m}\|\mu_m\|^q,$$

where c is independent of n. Combining the even and odd cases, we obtain

$$\|\mu\|_{C^{1,q}}^q \ge c\sum_{m=n_0}^{\infty} 2^{(1-q)m}\|\mu_m\|^q.$$

Since it is easy to estimate the left-hand side here from below with $c(\sum_{m=1}^{n_0-1}\|\mu_m\|)^q$ and hence with $c\sum_{m=1}^{n_0-1}2^{(1-q)m}\|\mu_m\|^q$, Theorem 6 follows. ■

Proof of Lemma 6.1. To be specific, we take n even. It will be understood below that n_0 and n are sufficiently large, and we can of course assume n_0 even. We shall proceed by induction; assuming the existence of F_{n-2}, we shall construct F_n. In the starting case $n = n_0$, we can use as F_{n-2} any subset of $[1, l_{n-2}]$ whose cardinality is $2^{n-3+2^{n-1}}$.

The first part of the construction consists in placing, for each fixed j, in the interval $[1, 2^{1+2^{n+1}}]$ regularly spaced translates of F_{n-2}. Together, these translates will form a set F_n^j. Then F_n will be defined as a union of translates of these F_n^j.

Let $2^{n-3} \le j < 2^{n-2}$, and define $k_j = 2^{j-1}\#F_{n-2}/l_{n-2}$. Notice that k_j is an integer power of 2, and that

$$k_j \ge 2^{2^{n-3}-1}2^{n-3+2^{n-1}}2^{-1-2^{n-4}-2^{n-1}} > 1.$$

Further,

$$k_j l_{n-2} = 2^{j-1}\#F_{n-2} \le 2^{2^{n-2}-1}2^{n-3+2^{n-1}} < \tfrac{1}{2}2^{2^n}. \tag{2}$$

Now define, with j as above,

$$F_n^j = \bigcup (\nu k_j l_{n-2} + F_{n-2}),$$

where the union of translated sets is taken over those integer ν with

$$0 \le \nu < \frac{2^{1+2^{n+1}}}{k_j l_{n-2}}.$$

Observe that since $F_{n-2} \subset [1, l_{n-2}]$, the set F_n^j is the union of those translates of F_{n-2} by integer multiples of $k_j l_{n-2}$ which are contained in $[1, 2^{1+2^{n+1}}]$. The cardinality of F_n^j is

$$\#F_n^j = \frac{2^{1+2^{n+1}}}{k_j l_{n-2}}\#F_{n-2} = 2^{1-j}2^{1+2^{n+1}}.$$

Thus F_n^j is a subset of $[1, 2^{1+2^{n+1}}]$ of average density 2^{1-j}.

We shall now estimate $\mu * \chi_{F_n^j}$. Take a point x with $2^{2^{n+1}} < x \le 2^{1+2^{n+1}}$. Writing $\alpha(i) = \alpha_i$ for $i \ge 1$ and $\alpha(i) = 0$ for $i \le 0$, we have

$$\mu * \chi_{F_n^j}(x) = \sum_{y \in F_n^j} \alpha(x-y) \ge \sum_{\nu=1}^{2^{1+2^{n+1}}/(k_j l_{n-2})-1} \sum_{z \in F_{n-2}} \alpha(x - \nu k_j l_{n-2} - z). \tag{3}$$

Notice that when $\nu k_j l_{n-2} + l_{n-2} < x$, the argument of α here is positive for all $z \in F_{n-2}$. For such z and ν, it is clear that

$$x - \nu k_j l_{n-2} - z \le x - w$$

for any w with $(\nu - 1)k_j l_{n-2} < w \le \nu k_j l_{n-2}$.

Because of the monotonicity of the α_i, we get

$$\alpha(x - \nu k_j l_{n-2} - z) \ge \alpha(x - w)$$

for the same ν, z and w. This inequality clearly remains valid if we take mean values in z and w, so that

$$\sum_{z \in F_{n-2}} \alpha(x - \nu k_j l_{n-2} - z) \ge \frac{\#F_{n-2}}{k_j l_{n-2}} \sum_{w=(\nu-1)k_j l_{n-2}+1}^{\nu k_j l_{n-2}} \alpha(x - w).$$

We now sum this inequality over all integer ν with $\nu \ge 1$ and $\nu k_j l_{n-2} + l_{n-2} < x$. Notice that the largest value of ν included here satisfies $\nu k_j l_{n-2} \ge x - k_j l_{n-2} - l_{n-2} \ge x - 2k_j l_{n-2}$. In view of (3), we conclude that

$$\mu * \chi_{F_n^j}(x) \ge 2^{1-j} \sum_{w=1}^{x-2k_j l_{n-2}} \alpha(x - w) = 2^{1-j} \sum_{i=2k_j l_{n-2}}^{x-1} \alpha(i).$$

Because of (2), the last expression here is at least

$$2^{1-j} \sum_{2^{2^n}-1}^{2^{2^{n+1}}-2} \alpha(i) = 2^{1-j}\|\mu_n\|.$$

Summing up, we have proved that for $2^{n-3} \le j < 2^{n-2}$,

$$\mu * \chi_{F_n^j} \ge 2^{1-j}\|\mu_n\| \quad \text{in } (2^{2^{n+1}}, 2^{1+2^{n+1}}]. \tag{4}$$

We now define F_n as a union of $\sum_{j=2^{n-3}}^{2^{n-2}-1} 2^j$ translates of the F_n^j by distinct multiples of $2^{1+2^{n+1}}$. More precisely, for each j with $2^{n-3} \le j < 2^{n-2}$, we take 2^j translates of F_n^j. Since $F_n^j \subset [1, 2^{1+2^{n+1}}]$, this can be done in such a way that

$$F_n \subset \Big[1, \sum_{j=2^{n-3}}^{2^{n-2}-1} 2^j 2^{1+2^{n+1}}\Big] \subset [1, l_n].$$

The cardinality of F_n will be

$$\#F_n = \sum_{j=2^{n-3}}^{2^{n-2}-1} 2^j \#F_n^j = \sum_j 2^{2+2^{n+1}} = 2^{n-1+2^{n+1}},$$

as claimed in the lemma.

From (4) it follows that $\mu * \chi_{F_n} \ge 2^{-j}\|\mu_n\|$ in a set of cardinality $2^j 2^{2^{n+1}} = 2^{j-n+1}\#F_n$, with j as in (4). This implies the last claim of the lemma in the case $m = n$. For smaller values of m, the last claim follows from the induction assumption, since F_n is a finite union of translates of F_{n-2} with sufficient spacing.

Lemma 6.1 proved. ■

7. Proof of Theorem 7

LEMMA 7.1. *If f is a function in $L^\infty(\mathbb{T})$ and A is a measurable set, then*

$$\|f*\chi_A\|_{1,q} \le C|A|\|f\|_1\left(1+\log\left(\frac{\|f\|_\infty}{\|f\|_1}\right)\right)^{1/q-1}.$$

Proof. For any $\varepsilon>0$,

$$q\int_0^\varepsilon t^{q-1}[(\chi_A*f)^*(t)]^q\,dt \le \|\chi_A*f\|_\infty^q\int_0^\varepsilon qt^{q-1}\,dt \le |A|^q\|f\|_\infty^q\varepsilon^q,$$

and

$$q\int_\varepsilon^1 t^{q-1}[(\chi_A*f)^*(t)]^q\,dt \le q\Big(\int_\varepsilon^1(\chi_A*f)^*(t)\,dt\Big)^q\cdot\Big(\int_\varepsilon^1\frac{dt}{t}\Big)^{1-q}$$
$$\le q|A|^q\|f\|_1^q(\log(1/\varepsilon))^{1-q}.$$

Choosing $\varepsilon=\|f\|_1/\|f\|_\infty$, we obtain

$$q\int_0^1 t^{q-1}[(\chi_A*f)^*(t)]^q\,dt \le C|A|^q\|f\|_1^q\left(1+\log\left(\frac{\|f\|_\infty}{\|f\|_1}\right)\right)^{1-q}. \blacksquare$$

In particular, when $f=\chi_E$ we have

$$\|\chi_E*\chi_A\|_{1,q}\le C|A||E|(1+\log(1/|E|))^{1/q-1},$$

and Theorem 2(a) implies that the $C^{1,q}(\mathbb{T})$ quasinorm of χ_E is at most $C|E|(1+\log(1/|E|))^{1/q-1}$.

To prove conversely that the convolutor quasinorm of χ_E is at least $c|E|(1+\log(1/|E|))^{1/q-1}$ is more complicated. We can assume $|E|$ to be small, and start by approximating E by a set A which is the union of a collection of intervals of the form $[j/N,(j+1)/N]$, $j\in\{0,1,\ldots,N-1\}$. If N is large enough, we can make the symmetric difference $E\triangle A$ so small that the convolutor quasinorm of $\chi_E-\chi_A$ is much smaller than $|E|\log(1/|E|)^{1/q-1}$. It is then sufficient to estimate the convolutor quasinorm of χ_A, and to do this, we discretize the problem.

Let us associate the torus $\mathbb{T}$ with the cyclic group $\{0,1,\ldots,N-1\}$ and the set A with the corresponding subset $\widetilde{A}$ of $\{0,1,\ldots,N-1\}$. Then $|A|=\#\widetilde{A}/N$. We can assume both $\#\widetilde{A}$ and $N/(\#\widetilde{A})$ to be large.

LEMMA 7.2. *Let ν be the largest integer with $q^\nu\le N/(\#\widetilde{A})$ and take n with $1\le n\le\nu$. Then we can choose distinct integers $0\le k_1,k_2,\ldots,k_{2^n}\le N-1$ such that*

$$\sum_{\substack{1\le i,j\le 2^n\\ i\ne j}}\sum_{k=0}^{N-1}\chi_{\widetilde{A}}(k-k_i)\chi_{\widetilde{A}}(k-k_j)\le C\frac{2^{2n}(\#\widetilde{A})^2}{N}.$$

Proof. The number of possible choices of $k_1,k_2,\ldots,k_{2^n}$ is $N(N-1)\ldots(N-2^n+1)$, so that for fixed i,j and k, we have the mean

$$\frac{1}{N(N-1)\ldots(N-2^n+1)}\sum_{k_1,k_2,\ldots,k_{2^n}}\chi_{\widetilde{A}}(k-k_i)\chi_{\widetilde{A}}(k-k_j)$$
$$=\frac{(\#\widetilde{A})(\#\widetilde{A}-1)(N-2)(N-3)\ldots(N-2^n+1)}{N(N-1)\ldots(N-2^n+1)}$$
$$=\frac{(\#\widetilde{A})(\#\widetilde{A}-1)}{N(N-1)}\approx\left(\frac{\#\widetilde{A}}{N}\right)^2.$$

Hence,

$$\frac{1}{N(N-1)\ldots(N-2^n+1)}\sum_{k_1,k_2,\ldots,k_{2^n}}\sum_{\substack{1\le i,j\le 2^n\\ i\ne j}}\sum_{k=0}^{N-1}\chi_{\widetilde{A}}(k-k_i)\chi_{\widetilde{A}}(k-k_j)$$
$$\approx N2^n(2^n-1)\left(\frac{\#\widetilde{A}}{N}\right)^2\approx\frac{2^{2n}(\#\widetilde{A})^2}{N}. \blacksquare$$

LEMMA 7.3. *Let $k_1,k_2,\ldots,k_{2^n}$ be as in the previous lemma. Then*

$$\#\Big\{k:\sum_{i=1}^{2^n}\chi_{\widetilde{A}}(k-k_i)\ge1\Big\}\ge c2^n(\#\widetilde{A}).$$

Proof. Let

$$S(k)=\sum_{i=1}^{2^n}\chi_{\widetilde{A}}(k-k_i).$$

Then $S:\{0,1,\ldots,N-1\}\to\mathbb{N}$,

$$\sum_{k=0}^{N-1}S(k)=2^n(\#\widetilde{A}),$$

and

$$\sum_{k=0}^{N-1}(S(k))^2=\sum_{k=0}^{N-1}\sum_{i=1}^{2^n}\sum_{j=1}^{2^n}\chi_{\widetilde{A}}(k-k_i)\chi_{\widetilde{A}}(k-k_j)$$
$$=\sum_{k=0}^{N-1}\sum_{i=1}^{2^n}\chi_{\widetilde{A}}(k-k_i)+\sum_{k=0}^{N-1}\sum_{\substack{1\le i,j\le 2^n\\ i\ne j}}\chi_{\widetilde{A}}(k-k_i)\chi_{\widetilde{A}}(k-k_j)$$
$$\le 2^n(\#\widetilde{A})+C\frac{2^{2n}(\#\widetilde{A})^2}{N}\le C2^n(\#\widetilde{A}),$$

because $2^n \le 2^\nu \approx N/(\#\widetilde{A})$. Thus, by Chebyshev's inequality,

$$\#\{k : S(k) > t\} \le C\frac{2^n(\#\widetilde{A})}{t^2},$$

and for a large but fixed m we have

$$\sum_{\{k:S(k)>2^m\}} S(k) \le C\sum_{h\ge m} 2^h \#\{k : S(k) > 2^h\}$$
$$\le C2^n(\#\widetilde{A})\sum_{h\ge m} 2^{-h} \le \frac{1}{2}2^n(\#\widetilde{A}).$$

Hence,

$$2^n(\#\widetilde{A}) = \sum_{k=0}^{N-1} S(k) \le 2\sum_{\{k:1\le S(k)\le 2^m\}} S(k) \le 2^{m+1}\#\{k : S(k) \ge 1\},$$

and the lemma follows. ∎

Now we return from the discrete set $\{0, 1, \ldots, N-1\}$ to the torus $\mathbb{T}$, in order to complete the proof of Theorem 7.

For each $n = 0, 1, \ldots, \nu$, let $\widetilde{F}_n$ be the set $\{k_1, \ldots, k_{2^n}\}$ defined in the previous lemmas. With $\widetilde{F}_n$ we associate the subset of the torus

$$F_n = \bigcup_{j\in\widetilde{F}_n}\left[\frac{j}{N}, \frac{j+2^{-n}}{N}\right].$$

Then $|F_n| = 1/N$, and since $\chi_A * \chi_{F_n}$ is closely similar to $2^{-n}\chi_{\widetilde{A}} * \chi_{\widetilde{F}} = 2^{-n}N^{-1}\sum_{i=1}^{2^n}\chi_{\widetilde{A}}(\cdot - k_i)$, we also have

$$|\{x \in \mathbb{T} : \chi_A * \chi_{F_n}(x) \ge 2^{-n}/N\}| \ge c\frac{2^n(\#\widetilde{A})}{N} = c2^n|A|.$$

Define $G = \bigcup_{n=1}^{\nu} F_n$. Then $|G| \le \sum_{n=1}^{\nu}|F_n| = \nu/N$, and if $1 \le n \le \nu$, we have

$$|\{x \in \mathbb{T} : \chi_A * \chi_G(x) \ge 2^{-n}/N\}| \ge c2^n|A|.$$

Hence,

$$\|\chi_A * \chi_G\|_{1,q} \ge c\Big(\sum_{n=1}^{\nu}\Big(\frac{2^{-n}}{N}|\{x \in \mathbb{T} : \chi_A * \chi_G(x) \ge 2^{-n}/N\}|\Big)^q\Big)^{1/q}$$
$$\ge c\Big(\sum_{n=1}^{\nu}\Big(\frac{|A|}{N}\Big)^q\Big)^{1/q} \ge c\nu^{1/q-1}|A||G|.$$

Since $2^\nu \approx 1/|A|$, we finally obtain

$$\|\chi_A * \chi_G\|_{1,q} \ge c|A|(\log(1/|A|))^{1/q-1}|G|.$$

This ends the proof of Theorem 7. ∎

8. Proof of Theorem 8. Part (a) of the theorem easily follows from Lemma 7.1. Indeed, let us define

$$f_m(x) = \begin{cases} f(x) & \text{if } f^*(2^{-2^m}) \le |f(x)| < f^*(2^{-2^{m+1}}), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|f_m\|_\infty \le f^*(2^{-2^{m+1}}) \le 2^{1+2^{m+1}}\|f_{m+1}\|_1.$$

By Lemma 7.1, the convolutor quasinorm of f_m on $L^{1,q}(\mathbb{T})$ is at most

$$\|f_m\|_{C^{1,q}} \le C\|f_m\|_1\Big(1 + \log\Big(\frac{\|f_m\|_\infty}{\|f_m\|_1}\Big)\Big)^{1/q-1}$$
$$\le C2^{(1/q-1)m}\|f_m\|_1 + C\|f_m\|_1\Big(\log_+\Big(\frac{\|f_{m+1}\|_1}{\|f_m\|_1}\Big)\Big)^{1/q-1}.$$

The last term contributes only when $\|f_m\|_1 < \|f_{m+1}\|_1$, and then it is dominated by

$$C\|f_{m+1}\|_1\frac{\|f_m\|_1}{\|f_{m+1}\|_1}\Big(\log_+\Big(\frac{\|f_{m+1}\|_1}{\|f_m\|_1}\Big)\Big)^{1/q-1} \le C\|f_{m+1}\|_1.$$

Thus, by the q-additivity of the $C^{1,q}$-norm,

$$\Big\|\sum_{m=0}^{\infty} f_m\Big\|_{C^{1,q}}^q \le C\sum_{m=0}^{\infty}\|f_m\|_{C^{1,q}}^q \le C\sum_{m=0}^{\infty}(2^{(1/q-1)m}\|f_m\|_1 + \|f_{m+1}\|_1)^q$$
$$\le C\sum_{m=0}^{\infty}(2^{(1/q-1)m}\|f_m\|_1)^q.$$

Part (a) then follows. We now prove (b), and without loss of generality we assume that the function f has support in the interval $[0, 1/2]$.

Let ν be a large integer. We want to construct a measurable set $E \subseteq [0, 1/2]$ such that

$$\|f * \chi_E\|_{1,q} \ge c|E|\Big(\sum_{m=0}^{\nu}\Big(2^{2m(1/q-1)}\int_{2^{-2^{2m+1}}}^{2^{-2^{2m}}} f^*(t)\,dt\Big)^q\Big)^{1/q}.$$

Let $\delta_k = 2^{-2^k}$ and $\lambda_k = \delta_{2k-1}\delta_{2k} = 2^{-3\cdot 2^{2k-1}}$. In particular, $\lambda_1 = 2^{-6}$. We shall construct inductively sets $E_1 = [0, 2^{-6}] \supseteq E_2 \supseteq E_3 \supseteq \ldots$ Each E_k will be the union of a collection of intervals of length λ_k at mutual distances at least λ_k.

Assuming E_{k-1} already defined, we denote by I one of its components, so that $|I| = \lambda_{k-1}$. Divide I into intervals of length $2\delta_{2k-1}$, say $I = \bigcup A_i$, $i = 1, 2, \ldots, \lambda_{k-1}/(2\delta_{2k-1}) = 2^{2^{2k-3}-1}$, and divide again each of these intervals A_i into intervals of length λ_k, say $A_i = \bigcup_s B_{is}$. In A_1 we choose $B_{i2}, B_{i4}, B_{i6}, \ldots$ and we discard $B_{i1}, B_{i3}, B_{i5}, \ldots$ In A_2 and A_3 we choose

$B_{i4}, B_{i8}, B_{i12}, \ldots, i = 2, 3$. In general, let $j = 1, 2, \ldots, j_k$, with $j_k = 2^{2k-3}-1$. Then in A_i, $2^{j-1} \le i < 2^j$, we choose only those B_{is} with s a multiple of 2^j, and discard the rest. Finally, we discard the last A_i, i.e., the one with $i = 2^{2^{2k-3}-1}$. The B_{is} chosen together form the part of E_k which is contained in I. In this way E_k is defined.

Our goal is to estimate $|E_k|^{-1}\chi_{E_k} * f$. Since

$$|I \cap E_k| = \tfrac{1}{2}|A_i| + \tfrac{1}{4}(|A_2| + |A_3|) + \tfrac{1}{8}(|A_4| + |A_5| + |A_6| + |A_7|) + \ldots + 2^{-j_k}(|A_{2^{j_k-1}}| + \ldots + |A_{2^{j_k}-1}|) = j_k \delta_{2k-1},$$

we have

$$\frac{|E_k|}{|E_{k-1}|} = \frac{|I \cap E_k|}{|I|} = j_k \frac{\delta_{2k-1}}{\delta_{2k-3}\delta_{2k-2}}.$$

Let $A = A_i$ be one of the above-mentioned intervals of length $2\delta_{2k-1}$ and assume $2^{j-1} \le i < 2^j$ for some $1 \le j \le j_k$. Then the distance between the components of E_k in A is $(2^j - 1)\lambda_k$, and $(2^j - 1)\lambda_k < 2^{j_k}\lambda_k < 2^{-1}\delta_{2k}$. Also, the mean density of E_k in A is 2^{-j}, that is, $|A \cap E_k|/|A| = 2^{-j}$.

Since the function f is decreasing, for x in the right-hand half of A we have

$$|E_k|^{-1}\chi_{E_k} * f(x) \ge \frac{|A \cap E_k|}{|A|\,|E_k|} \int_{\delta_{2k}}^{\delta_{2k-1}} f(t)\,dt = \frac{2^{-j}}{|E_k|} \int_{\delta_{2k}}^{\delta_{2k-1}} f(t)\,dt.$$

Notice that one can obtain $|E_h|^{-1}\chi_{E_h}$ with $h > k$ from $|E_k|^{-1}\chi_{E_k}$ simply by considering the mass of $|E_k|^{-1}\chi_{E_k}$ within each interval B_{is}. Thus if x is in the right half of A, we still have

$$|E_h|^{-1}\chi_{E_h} * f(x) \ge \frac{2^{-j}}{|E_k|} \int_{\delta_{2k}}^{\delta_{2k-1}} f(t)\,dt.$$

The number of intervals A for which the above inequality holds is 2^{j-1} times the number of the components of E_{k-1}, and the number of these components is $|E_{k-1}|/\lambda_{k-1} = |E_k|/(j_k\delta_{2k-1})$. Hence,

$$\left|\left\{x \in [0,1) : |E_h|^{-1}\chi_{E_h} * f(x) \ge \frac{2^{-j}}{|E_k|} \int_{\delta_{2k}}^{\delta_{2k-1}} f(t)\,dt\right\}\right| \ge 2^{j-1}\frac{|E_k|}{j_k}.$$

We have thus proved that if $1 \le j \le j_k, h \ge k$, and $t \le 2^{j-1}|E_k|/j_k$, then

$$(|E_h|^{-1}\chi_{E_h} * f)^*(t) \ge \frac{2^{-j}}{|E_k|} \int_{\delta_{2k}}^{\delta_{2k-1}} f(t)\,dt.$$

Comparing these values of t, we see that

$$2^{j-1}\frac{|E_k|}{j_k} < \frac{1}{2}2^{r-1}\frac{|E_{k-1}|}{j_{k-1}},$$

provided that $1 \le j \le j_k - 2k$ and $1 \le r \le j_{k-1} - 2(k-1)$.

Thus all the intervals $[2^{j-2}|E_k|/j_k, 2^{j-1}|E_k|/j_k]$, $1 \le j \le j_k - 2k$, $k = 1, 2, \ldots$, are disjoint. This allows us to estimate the $L^{1,q}$ norm of $|E_h|^{-1}\chi_{E_h} * f$. Indeed,

$$\int_0^1 |t(|E_h|^{-1}\chi_{E_h} * f)^*(t)|^q \frac{dt}{t} \ge c \sum_{k\le h} \sum_{1\le j\le j_k-2k} \left(\frac{2^{-j}}{|E_k|}\int_{\delta_{2k}}^{\delta_{2k-1}} f(t)\,dt\right)^q \left(2^j\frac{|E_k|}{j_k}\right)^q \ge c\sum_{k\le h} j_k^{1-q}\Big(\int_{\delta_{2k}}^{\delta_{2k-1}} f(t)\,dt\Big)^q.$$

Since $\|\,|E_h|^{-1}\chi_{E_h}\|_{1,q} = 1$, $j_k = 2^{2k-3} - 1$, and since $\delta_k = 2^{-2^k}$, we have proved that

$$\|f\|^q_{C^{1,q}} \ge c\sum_{k\le h} 2^{2k(1-q)}\Big(\int_{2^{-2^{2k}}}^{2^{-2^{2k-1}}} f(t)\,dt\Big)^q.$$

There is clearly a similar estimate involving the integrals between $2^{-2^{2k+1}}$ and $2^{-2^{2k}}$. This ends the proof of part (b) of Theorem 8. ■

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On oscillatory integral operators with folding canonical relations

by

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Abstract. Sharp L^p estimates are proven for oscillatory integrals with phase functions $\Phi(x,y)$, $(x,y)\in X\times Y$, under the assumption that the canonical relation C_Φ projects to T^*X and T^*Y with fold singularities.

1. Introduction. Let X and Y be open sets in $\mathbb{R}^d$ and let $\Omega\subset X\times Y$ be a bounded open set whose closure is contained in $X\times Y$. We consider oscillatory integral operators $T_{\lambda,\gamma}$ given by

$$T_{\lambda,\gamma}[a,f](x)=\int e^{i\lambda\Phi(x,y,\gamma)}a_{\lambda,\gamma}(x,y)f(y)\,dy. \tag{1.1}$$

Here $\lambda\ge 1$ and γ is a parameter in a manifold Γ. We assume that $a_{\lambda,\gamma}\in C_0^\infty(\Omega)$ and that $\Phi(\cdot,\cdot,\gamma)\in C^\infty(X\times Y)$ is a real-valued phase function; moreover, $a_{\lambda,\gamma}$, $\Phi(\cdot,\cdot,\gamma)$ and their derivatives depend continuously on γ.

It may sometimes be convenient to admit some growth in λ for the (x,y) derivatives of the amplitude. Let $0\le\delta\le 1$. We shall say that the family of amplitudes $a=\{a_{\lambda,\gamma}\}$ belongs to the class $\mathfrak{S}_\delta(\Omega)$ if $\operatorname{supp}a_{\lambda,\gamma}\subset\Omega$ and if

$$\sup_{(x,y)\in\Omega}|\partial_x^\alpha\partial_y^\beta a_{\lambda,\gamma}(x,y)|\le C_{\alpha,\beta}\lambda^{\delta(|\alpha|+|\beta|)}. \tag{1.2}$$

This definition is made in analogy to the standard symbol classes $S_{\varrho,\delta}$, although there is no parameter ϱ since we do not impose differentiability conditions with respect to the parameter λ. If $\widetilde{C}_{\alpha,\beta}$ denote the best constants in (1.2) then we define

$$\|a\|_{\mathfrak{S}_\delta}=\sup\{\widetilde{C}_{\alpha,\beta}:|\alpha|,|\beta|\le N\}$$

for some large N; the choice $N=10d$ is admissible and we shall make this

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