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Lower bounds for Schrödinger operators in $H^1(\mathbb{R})$

by

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Abstract. We prove trace inequalities of type $\|u'\|^2_{L^2} + \sum_{j\in\mathbb{Z}} k_j|u(a_j)|^2 \ge \lambda\|u\|^2_{L^2}$ where $u \in H^1(\mathbb{R})$, under suitable hypotheses on the sequences $\{a_j\}_{j\in\mathbb{Z}}$ and $\{k_j\}_{j\in\mathbb{Z}}$, with the first sequence increasing and the second bounded.

Introduction. In 1989, R. Strichartz proved (see [Str]) that for an increasing real sequence $\{a_j\}_{j\in\mathbb{Z}}$ unbounded from above and below and such that, for all j in $\mathbb{Z}$, $a_{j+1} - a_j < \beta$ where β is a fixed positive constant, the following inequality holds in $H^1(\mathbb{R})$:

$$(1)\qquad \frac{\beta}{\sqrt{8}}\|u'\|_{L^2} + \sqrt{\beta}\Big(\sum_{j\in\mathbb{Z}}|u(a_j)|^2\Big)^{1/2} \ge \|u\|_{L^2}.$$

This result enables us to define operators such as $-\Delta + \lambda\sum_{j\in\mathbb{Z}}\delta_{a_j}$ with $\lambda > 8/\beta$, where δ_{a_j} is the Dirac measure at a_j, as unbounded selfadjoint operators in $L^2(\mathbb{R})$, using a theorem of [Re-Si]. This theorem (see [Re-Si], Th. VIII.15) states that a unique selfadjoint operator can be associated with every lower semibounded and closed quadratic form. Indeed, the form $\|u'\|^2_{L^2} + \lambda\sum_{j\in\mathbb{Z}}|u(a_j)|^2$ is lower semibounded (as sketched at the end of the Introduction) and closed (as shown in [Pou]). In order to give a sense to more general operators, using the same theorem, we prove the corresponding trace inequalities.

The aim of this paper is to present inequalities similar to (1), with a family $\{k_j\}_{j\in\mathbb{Z}}$ of weights attached to the points a_j. The improvement is that we allow the k_j's to take negative values and tend to 0 at infinity under suitable hypotheses on the quotient $|k_j|/(a_{j+1} - a_j)$.

In Section 1, we provide the following generalizations of (1):

$$(1')\qquad (\exists\lambda_1 > 0)\ (\forall u \in H^1(\mathbb{R}))\qquad \|u'\|^2_{L^2} + \sum_{j\in\mathbb{Z}}|k_j|\cdot|u(a_j)|^2 \ge \lambda_1\|u\|^2_{L^2},$$

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(1″) $\quad (\exists \lambda_2 > 0)\ (\forall u \in H^1(\mathbb{R})) \quad \|u'\|^2_{L^2} - \sum_{j\in\mathbb{Z}} |k_j| \cdot |u(a_j)|^2 \geq -\lambda_2 \|u\|^2_{L^2}.$

The proof is based on the approximation of a scaling function by a piecewise linear function.

Hence, given a quadratic form $\|u'\|^2_{L^2} + \sum k_j |u(a_j)|^2$, we can prove that it is lower semibounded: separate the positive and negative subsequences of $\{k_j\}_{j\in\mathbb{Z}}$ (respectively denoted $\{k_{p_j}\}_{j\in\mathbb{Z}}$ and $\{-k_{n_j}\}_{j\in\mathbb{Z}}$) and apply inequality (1″) to the quadratic form $\|u'\|_{L^2} - \sum k_{n_j} |u(a_j)|^2$ in order to obtain the result. Inequality (1′) is useful to give a first information about the spectrum of an operator defined from a quadratic form with a positive sequence of weights $\{k_j\}_{j\in\mathbb{Z}}$. However, this question will not be studied in the present paper. See [Pou] for the explicit construction of the operator and [Al-Ge] for a study of its spectrum.

In Section 2, we give a different proof of a weaker form of the second inequality, using the Poisson formula.

1. Weighted trace inequalities in $H^1(\mathbb{R})$

THEOREM 1.1. *Let $\{a_j\}_{j\in\mathbb{Z}}$ and $\{k_j\}_{j\in\mathbb{Z}}$ be two real sequences, the first one increasing and the second one positive. Suppose that there exists a uniform upper bound β on the difference $a_{j+1} - a_j$ and that*

$$\lim_{j\to-\infty} a_j = -\infty \quad \text{and} \quad \lim_{j\to+\infty} a_j = +\infty.$$

If $\liminf_{j\to\infty} k_j/(a_{j+1} - a_j) > 0$ then there exists $\lambda > 0$ such that the following inequality holds in $H^1(\mathbb{R})$:

$$\|u'\|^2_{L^2} + \sum_{j\in\mathbb{Z}} k_j |u(a_j)|^2 \geq \lambda \|u\|^2_{L^2}. \tag{2}$$

If $\lim_{j\to+\infty} k_j/(a_{j+1} - a_j) = 0$ (or if $\lim_{j\to-\infty} k_j/(a_{j+1} - a_j) = 0$) then (2) is false.

Proof. Let u be in $H^1(\mathbb{R})$. Setting $m = \inf_{j\in\mathbb{Z}} k_j/(a_{j+1} - a_j)$, we shall, in fact, prove that

$$\|u'\|^2_{L^2(\mathbb{R})} + \sum_{j\in\mathbb{Z}} k_j |u(a_j)|^2 \geq \frac{1}{2} \min\left(\frac{1}{2\beta^2}, m\right) \|u\|^2_{L^2(\mathbb{R})}.$$

First define a new sequence of weights $\{k'_j\}_{j\in\mathbb{Z}}$ by $k'_j = m(a_{j+1} - a_j)$. Then for any n in $\mathbb{N}$ and j_0 in $\mathbb{N}$ such that $j_0 > n$ we have

$$\sum_{j=n}^{j_0-1} (k'_j |u(a_j)|^2 - k'_j |u(a_{j_0})|^2) = -2\,\mathrm{Re} \int_{a_n}^{a_{j_0}} f(s) u(s) \overline{u}'(s)\, ds \tag{3}$$

where the function f is defined in $[a_n, +\infty[$ by

$$f(s) = \sum_{j\geq n} \mathbf{1}_{]a_j,+\infty[}(s) k'_j. \tag{4}$$

We can approximate f by the following function g on each interval $[a_j, a_{j+1}]$, $j > n$:

$$g(s) = f(a_j) + \frac{f(a_{j+1}) - f(a_j)}{a_{j+1} - a_j}(x - a_j),$$

which is continuous on $\mathbb{R}$, piecewise differentiable and satisfies on each $]a_j, a_{j+1}[$,

$$g'(s) = \frac{k'_j}{a_{j+1} - a_j}, \qquad |f(s) - g(s)| \leq k'_j. \tag{5}$$

Now, for any $j_0 > n$, we can write (3) as

$$\sum_{j=n}^{j_0-1} k'_j |u(a_j)|^2 = g(a_{j_0}) |u(a_{j_0})|^2 - 2\,\mathrm{Re} \int_{a_n}^{a_{j_0}} (f(s) - g(s)) u(s) \overline{u}'(s)\, ds$$
$$- 2\,\mathrm{Re} \int_{a_n}^{a_{j_0}} g(s) u(s) \overline{u}'(s)\, ds.$$

Integrating the last term by parts, we have, letting j_0 tend to infinity,

$$\sum_{j\geq n} k'_j |u(a_j)|^2 = -2\,\mathrm{Re} \int_{a_n}^{+\infty} (f(s) - g(s)) u(s) \overline{u}'(s)\, ds + \int_{a_n}^{+\infty} g'(s) |u(s)|^2\, ds \tag{6}$$

$$\geq -2 \int_{a_n}^{+\infty} |f(s) - g(s)| \cdot |u(s)| \cdot |\overline{u}'(s)|\, ds + \sum_{j\geq n} \int_{a_j}^{a_{j+1}} \frac{k'_j}{a_{j+1} - a_j} |u(s)|^2\, ds. \tag{6′}$$

On the other hand, using (5), we have

$$2 \int_{a_n}^{+\infty} |f(s) - g(s)| \cdot |u(s)| \cdot |\overline{u}'(s)|\, ds$$
$$\leq \sum_{j\geq n} \frac{1}{\gamma} \int_{a_j}^{a_{j+1}} |\overline{u}'(s)|^2\, ds + \sum_{j\geq n} \gamma (k'_j)^2 \int_{a_j}^{a_{j+1}} |u(s)|^2\, ds,$$

where γ is an arbitrary positive constant. We combine this inequality with (6′) to obtain

$$(7)\quad \sum_{j\ge n}\int_{a_j}^{a_{j+1}}|u'(s)|^2\,ds+\sum_{j\ge n}\gamma k_j'|u(a_j)|^2 \ge \sum_{j\ge n}\int_{a_j}^{a_{j+1}}\cdot\left(\frac{\gamma k_j'}{a_{j+1}-a_j}-\gamma^2(k_j')^2\right)|u(s)|^2\,ds \ge \sum_{j\ge n}\int_{a_j}^{a_{j+1}}(\gamma m-\gamma^2m^2\beta^2)|u(s)|^2\,ds.$$

Setting $\gamma=1/(2m\beta^2)$, we get

$$(8)\quad \sum_{j\ge n}\int_{a_j}^{a_{j+1}}|u'(s)|^2\,ds+\sum_{j\ge n}\frac{k_j'}{2m\beta^2}|u(a_j)|^2\ge\frac{1}{4\beta^2}\sum_{j\ge n}\int_{a_j}^{a_{j+1}}|u(s)|^2\,ds.$$

Using $k_j'<k_j$ and dividing (8) by $\max(1,1/(2m\beta^2))$, we obtain

$$\min(1,2m\beta^2)\sum_{j\ge n}\int_{a_j}^{a_{j+1}}|u'(s)|^2\,ds+\sum_{j\ge n}k_j|u(a_j)|^2 \ge\frac12\min\left(\frac{1}{2\beta^2},m\right)\sum_{j\ge n}\int_{a_j}^{a_{j+1}}|u(s)|^2\,ds.$$

It remains to let n tend to $-\infty$, applying the monotone convergence theorem.

We now prove that (2) is false if, for instance, $\lim_{j\to+\infty}k_j/(a_{j+1}-a_j)=0$. Suppose (2) is valid for a given $\lambda=C_0>0$. Let $M>0$, $C_0=C_1+C_2$ where $C_1,C_2\in\mathbb{R}^+$ are fixed constants and $\varphi\in\mathcal{D}(]-M-1,M+1[)$, such that $\varphi|_{]-M,M[}=1$ and $0\le\varphi(x)\le1$. The positive constant M is taken sufficiently large, so that we can write

$$\|\varphi'\|^2_{L^2(\mathbb{R})}<C_2\|\varphi\|^2_{L^2(\mathbb{R})}.$$

We shall prove that, for $\gamma>0$ large enough, the function $\varphi_\gamma(x)=\varphi(x-\gamma)$ does not satisfy (2) any more. By the last inequality, we must prove that there exists $A>0$ such that, for $\gamma>A$,

$$(9)\quad \sum_{j\in\mathbb{Z}}k_j|\varphi_\gamma(a_j)|^2<C_1\|\varphi_\gamma\|^2_{L^2(\mathbb{R})}.$$

Let

$$C'=\frac{C_1\|\varphi_\gamma\|^2}{2(M+1)+\beta}.$$

Note that C' is independent of γ. From $\lim_{j\to+\infty}k_j/(a_{j+1}-a_j)=0$, we deduce that for j greater than a certain j_0, we have $k_j<C'(a_{j+1}-a_j)$, hence

$$\sum_{l=j_0}^{j}k_l<C'(a_{j+1}-a_{j_0}).$$

Taking $A=a_{j_0}+M+1$, we obtain for any $\gamma>A$,

$$\sum_{j/a_j\in\operatorname{supp}(\varphi_\gamma)}k_j\le C'\sup_{a_j,a_k\in\operatorname{supp}(\varphi_\gamma)}(a_{k+1}-a_j)\le C'(2(M+1)+\beta).$$

Hence, (9) is satisfied, since $\sum_{j\in\mathbb{Z}}k_j|\varphi_\gamma(a_j)|^2\le\sum_{j/a_j\in\operatorname{supp}(\varphi_\gamma)}k_j$.

REMARKS. If $\limsup_{j\to+\infty}(a_{j+1}-a_j)=+\infty$, then inequality (2) is false, whatever the hypothesis on $k_j/(a_{j+1}-a_j)$: a counter-example is given by $\varphi_l\in\mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi_l)\subset\,]a_{j(l)},a_{j(l)+1}[$, $0\le\varphi_l(x)\le1$ and $\varphi_l|_{]a_{j(l)}+1,a_{j(l)+1}-1[}=1$. Here $a_{j(l)}$ is a subsequence of a_j which satisfies $\lim_{l\to+\infty}(a_{j(l)+1}-a_{j(l)})=+\infty$. For such functions, $\|\varphi_l'\|_{L^2}$ remains constant and $\sum_{j\in\mathbb{Z}}k_j|\varphi_l(a_j)|^2$ is zero, while $\lim_{l\to+\infty}\|\varphi_l\|_{L^2}=+\infty$.

We can assume that the sequence $\{a_j\}_{j\in\mathbb{Z}}$ converges to finite limits γ and δ as j tends to $-\infty$ and $+\infty$, respectively. Inequality (2) then remains true with norms in $L^2(]\gamma,\delta[)$.

There is a more general condition under which we can have $\liminf_{j\to+\infty}k_j/(a_{j+1}-a_j)=0$ and still prove an inequality similar to (2): Let m be a positive constant and $\{a_{j(l)}\}_{l\in\mathbb{Z}}$ a subsequence such that, for l in $\mathbb{Z}$, we have

$$a_{j(l+1)}-a_{j(l)}\le\beta\quad\text{and}\quad\frac{\sum_{m=j(l)}^{j(l+1)-1}k_m}{a_{j(l+1)}-a_{j(l)}}\ge m.$$

Then inequality (2) holds, with λ still given by $2^{-1}\min((2\beta)^{-1},m)$. The proof is the same as for Theorem 1.1, after replacing $g(x)$ by the function associated with the subsequence $\{a_{j(l)}\}_{l\in\mathbb{Z}}$.

A similar inequality with negative weights is given by

THEOREM 1.2. *Let $\{a_j\}_{j\in\mathbb{Z}}$ and $\{k_j\}_{j\in\mathbb{Z}}$ be two real sequences, respectively increasing and bounded positive. Suppose that*

$$\lim_{j\to-\infty}a_j=-\infty\quad\text{and}\quad\lim_{j\to+\infty}a_j=+\infty.$$

If $\limsup_{j\to+\infty}k_j/(a_{j+1}-a_j)<+\infty$ then there exists $\lambda>0$ such that the following inequality is true in $H^1(\mathbb{R})$:

$$(10)\quad \|u'\|^2_{L^2}-\sum_{j\in\mathbb{Z}}k_j|u(a_j)|^2\ge-\lambda\|u\|^2_{L^2}.$$

If $\lim_{j\to+\infty}k_j/(a_{j+1}-a_j)=+\infty$ (or if $\lim_{j\to-\infty}k_j/(a_{j+1}-a_j)=+\infty$) then (10) fails.

Proof. If $\{k_j\}_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ and $\limsup_{j\to+\infty} k_j/(a_{j+1}-a_j) < +\infty$, then we can find a positive constant M such that, for any j in $\mathbb{Z}$,

$$\frac{k_j}{a_{j+1}-a_j} \le M. \tag{11}$$

We prove that, in $H^1(\mathbb{R})$,

$$\sum_{j\in\mathbb{Z}} k_j|u(a_j)|^2 \le \|u'\|^2_{L^2} + (M + \|k_j\|^2_{\ell^\infty})\|u\|^2_{L^2}.$$

This is an immediate consequence of (6) applied to the sequence $\{k_j\}_{j\in\mathbb{Z}}$ (see (4) for the definition of f and g, after replacing k_j' by k_j):

$$\sum_{j\ge n} k_j|u(a_j)|^2 = -2\operatorname{Re}\int_{a_n}^{+\infty}(f(s)-g(s))u(s)\overline{u}'(s)\,ds + \int_{a_n}^{+\infty} g'(s)|u(s)|^2\,ds.$$

Hence

$$\begin{aligned}\sum_{j\ge n} k_j|u(a_j)|^2 &\le \|g'\|_{L^\infty([a_n,+\infty[)}\int_{a_n}^{+\infty}|u(s)|^2\,ds \\ &\quad + 2\|f-g\|_{L^\infty([a_n,+\infty[)}\int_{a_n}^{+\infty}|u(s)\overline{u}'(s)|\,ds \\ &\le (\|g'\|_{L^\infty([a_n,+\infty[)} + \|f-g\|^2_{L^\infty([a_n,+\infty[)})\|u\|^2_{L^2([a_n,+\infty[)} \\ &\quad + \|u'\|^2_{L^2([a_n,+\infty[)}.\end{aligned}$$

We finish the proof using $|f(s)-g(s)| \le \|k_j\|_{\ell^\infty}$ and $|g'(s)| \le M$ in $[a_n,+\infty[$ and letting n tend to $-\infty$, by the monotone convergence theorem.

Now, let us prove that (10) is false if $\lim_{j\to+\infty} k_j/(a_{j+1}-a_j) = +\infty$. As $\{k_j\}_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, we immediately deduce that $\lim_{j\to+\infty}|a_{j+1}-a_j| = 0$. For two positive constants $0 < N < M$ we can write $|a_{j+1}-a_j| \le M-N$ for $j > j_0$.

Let $A \in \mathbb{R}$ and $C = A/N$. As $\lim_{j\to+\infty} k_j/(a_{j+1}-a_j) = +\infty$, there exists a positive integer j_1 such that, for any $j > j_1$, $k_j \ge C|a_{j+1}-a_j|$.

Define $j_2 = \max(j_0, j_1)$ and $j_3 = \max\{j \mid a_j \le M + a_{j_2}\}$. As $\lim_{j\to+\infty} a_j = +\infty$, we have $j_3 < +\infty$. Moreover, $a_{j_3} \ge N + a_{j_2}$: indeed, if $a_{j_3} \le N + a_{j_2}$, then j_3 would not be the maximum of the above set, as $a_{j_3+1} \le a_{j_3} + M - N \le M + a_{j_2}$. If we set $\varphi_j(\cdot) = \varphi(\cdot - a_j)$ with $\varphi \in \mathcal{D}(]-1, M+1[)$ such that $0 \le \varphi(x) \le 1$ and $\varphi_{|]0,M[} = 1$, we have

$$\sum_{j\in\mathbb{N}} k_j|\varphi_{j_2}(a_j)|^2 \ge \sum_{j=j_2}^{j_3-1} k_j \ge C|a_{j_3}-a_{j_2}| \ge CN.$$

Hence for all $A \in \mathbb{R}$, we can find a member φ_{j_2} of the family $\{\varphi_j\}_{j\in\mathbb{N}}$ such that

$$\sum_{j\in\mathbb{N}} k_j|\varphi_{j_2}(a_j)|^2 \ge A.$$

Noticing that $\|\varphi_j'\|^2_{L^2}$ and $\|\varphi_j\|^2_{L^2}$ are finite and do not depend on j concludes the proof.

REMARKS. If $\limsup_{j\to+\infty} k_j = +\infty$, then inequality (10) fails under any hypothesis on $k_j/(a_{j+1}-a_j)$: a counter-example is provided by $\varphi \in \mathcal{D}(]-1,+1[)$, $\varphi(0) = 1$ and $\varphi_l(x) = \varphi(x - a_{j(l)})$, where $a_{j(l)}$ is the subsequence corresponding to $k_{j(l)}$, a subsequence of k_j satisfying $\lim_{l\to+\infty} k_{j(l)} = +\infty$. For such functions we have $\lim_{l\to+\infty}\sum_{j\in\mathbb{Z}} k_j|\varphi_l(a_j)|^2 = +\infty$, while $\|\varphi_l\|_{L^2}$ and $\|\varphi_l'\|_{L^2}$ remain constant.

As for Theorem 1.1, we can generalize the hypotheses on $\{a_j\}_{j\in\mathbb{Z}}$: first we can suppose that $\lim_{j\to+\infty} a_j = \gamma$ and $\lim_{j\to-\infty} a_j = \delta$. Independently of this, if there exist two positive constants α and M and a subsequence $\{a_{j(l)}\}_{l\in\mathbb{Z}}$ such that, for any l in $\mathbb{Z}$, either

$$\begin{cases} a_{j(l+1)} - a_{j(l)} \le \alpha, \\ \dfrac{\sum_{m=j(l)}^{j(l+1)-1} k_m}{a_{j(l+1)} - a_{j(l)}} \le M \end{cases} \quad\text{or}\quad \begin{cases} a_{j(l+1)} - a_{j(l)} > \alpha, \\ j(l+1) = j(l)+1, \end{cases}$$

then, for $k = \max(\alpha M, \|k_j\|^2_{\ell^\infty})$, inequality (10) holds with $\|\cdot\|_{L^2(]\gamma,\delta[)}$ and $\lambda = k(1/\alpha + k)$.

If we assume that γ and δ are finite and $\lim_{j\to+\infty} k_j/(a_{j+1}-a_j) = +\infty$, then inequality (10) holds if and only if $\{k_j\}_{j\in\mathbb{Z}} \in l^1(\mathbb{Z})$. When this condition is not satisfied, functions with compact support, locally constant at γ or δ, are simple counter-examples.

Under more restrictive hypotheses on $\{a_j\}_{j\in\mathbb{Z}}$, we will obtain an inequality without weights, similar to (10), using the Poisson formula.

2. Trace inequalities and the Poisson formula in $\mathbb{R}$

THEOREM 2.1. *Let $\{a_j\}_{j\in\mathbb{Z}}$ be a real increasing sequence. Assume that there exists a uniform lower bound α for $a_{j+1}-a_j$. Then, for any positive constant μ, the following inequality holds in $H^1(\mathbb{R})$:*

$$\|u'\|^2_{L^2} - \mu\sum_{j\in\mathbb{Z}}|u(a_j)|^2 \ge -\frac{C_1(2\alpha\mu)}{2\alpha^2}\|u\|^2_{L^2}, \tag{12}$$

where $C_1(x) = 4x(x+\pi^2)/\pi^2$. For $\mu \in [0,\mu_0]$, $\mu_0 \simeq 8.83$, we get a better lower bound given by $C_2(x) = 24x/(12-x)$, where μ_0 is the unique positive solution of $C_1(x) = C_2(x)$.

We need the following two lemmas:

LEMMA 2.2. *Let $u \in H^1(\mathbb{R})$. Then*

$$\Big(\sum_{j\in\mathbb{Z}}|u(j)|^2\Big)^{1/2} \le \sum_{k\in\mathbb{Z}}\Big(\int_k^{k+1}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}.$$

Proof. Let u be in $H^1(\mathbb{R})$. Using the Poisson formula, we can write

$$\sum_{j\in\mathbb{Z}}u(j)\overline{u}(j) = \sum_{j\in\mathbb{Z}}\widehat{u}*\widehat{\overline{u}}(j)$$
$$\le \sum_{j,k\in\mathbb{Z}}\Big(\int_k^{k+1}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}\Big(\int_k^{k+1}|\widehat{u}(\xi+j)|^2\,d\xi\Big)^{1/2}.$$

Let $C_k = (\int_k^{k+1}|\widehat{u}(\xi)|^2\,d\xi)^{1/2}$. By a change of variables in the second integral, we see that the right-hand side is $\sum_{j,k\in\mathbb{Z}}C_kC_{k+j}$, which is equal to $(\sum_{k\in\mathbb{Z}}C_k)^2$. The conclusion follows.

LEMMA 2.3. *There exist two positive functions, $C_1(\mu)$ and $C_2(\mu)$, and a constant $\mu_0 \simeq 8.83$ such that, for any u in $H^1(\mathbb{R})$,*

$$(\forall\mu\in[0,\mu_0])\qquad \|u'\|^2_{L^2} - \mu\sum_{k\in\mathbb{Z}}|u(k)|^2 \ge -C_2(\mu)\|u\|^2_{L^2},$$
$$(\forall\mu>\mu_0)\qquad \|u'\|^2_{L^2} - \mu\sum_{k\in\mathbb{Z}}|u(k)|^2 \ge -C_1(\mu)\|u\|^2_{L^2}.$$

Moreover, $C_1(\mu) = 4\mu(\mu+\pi^2)/\pi^2$ on $[\mu_0,+\infty[$ and $C_2(\mu) = 24\mu/(12-\mu)$ on $[0,\mu_0]$.

REMARK. The lower bound $C_1(\mu)$ is, in fact, valid for all $\mu\in\mathbb{R}^+$. The constant μ_0 is the unique positive solution of $C_1(x) = C_2(x)$. Its exact value is

$$\mu_0 = \frac{12-\pi^2+\sqrt{\pi^4+144}}{2}.$$

Proof (of Lemma 2.3). Let $u\in H^1(\mathbb{R})$. Then

$$(13)\qquad \sum_{k\in\mathbb{Z}}\Big(\int_k^{k+1}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}$$
$$= \sum_{k=-l}^{l-1}\Big(\int_k^{k+1}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} + \sum_{\substack{k\ge l\\ k<-l}}\Big(\int_k^{k+1}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}$$

$$\le (2l)^{1/2}\Big(\int_{-l}^{l}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} + \Big(\sum_{k\ge l}\frac{1}{|k|^2}\Big)^{1/2}\Big(\int_l^{+\infty}|\xi|^2|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}$$
$$+ \Big(\sum_{k<-l}\frac{1}{|k+1|^2}\Big)^{1/2}\Big(\int_{-\infty}^{-l}|\xi|^2|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}.$$

If $l\in\mathbb{N}$, $l>1$, we can write

$$\sum_{k\ge l}\frac{1}{|k|^2} < \int_{l-1}^{+\infty}\frac{dx}{x^2} = \frac{1}{l-1}.$$

Similarly,

$$\sum_{k<-l}\frac{1}{|k+1|^2} < \frac{1}{l-1}.$$

Combined with (13), this gives

$$(14)\qquad \sum_{k\in\mathbb{Z}}\Big(\int_k^{k+1}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} \le (2l)^{1/2}\Big(\int_{|\xi|<l}|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}$$
$$+ \frac{1}{\sqrt{l-1}}\Big[\Big(\int_{-\infty}^{-l}|\xi|^2|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}$$
$$+ \Big(\int_l^{+\infty}|\xi|^2|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}\Big].$$

For $l\in\mathbb{R}^+\setminus\mathbb{N}$, we interpolate the versions of inequality (14) for $[l]$ and $[l]+1$, where $[l]$ is the integer part of l, and use Lemma 2.2 to obtain, for any α in $[0,1]$ and $\lambda>0$,

$$\sum_{k\in\mathbb{Z}}|u(k)|^2 \le 2\Big(1+\frac{1}{\lambda}\Big)(\alpha([l]+1)+(1-\alpha)[l])\,\|u\|^2_{L^2}$$
$$+ \frac{1}{2\pi^2}(1+\lambda)\Big(\frac{\alpha}{[l]}+\frac{1-\alpha}{[l]-1}\Big)\|u'\|^2_{L^2}.$$

Hence, setting

$$\alpha = \frac{[l](f(l)+1)}{f(l)+2[l]-1}\quad\text{and}\quad \lambda = \frac{[l]+\alpha}{[l]+2f(l)-\alpha}$$

(where $f(l) = l-[l]$), we have, for all $l\ge 1$, in $H^1(\mathbb{R})$,

$$\|u'\|^2_{L^2} + 4\pi^2(l-1)l\|u\|^2_{L^2} \ge \pi^2(l-1)\sum_{k\in\mathbb{Z}}|u(k)|^2.$$

Putting $\mu = \pi^2(l-1)$, we finally obtain, for any $\mu \geq 0$,

$$\|u'\|_{L^2}^2 + \frac{4\mu(\mu+\pi^2)}{\pi^2}\|u\|_{L^2}^2 \geq \mu \sum_{k\in\mathbb{Z}} |u(k)|^2.$$

If $l = 1$, we use the equality $\sum_{k\geq 1} 1/|k|^2 = \pi^2/6$ in (13) and Lemma 2.2 to obtain

$$\Big(\sum_{k\in\mathbb{Z}} |u(k)|^2\Big)^{1/2} \leq \sqrt{2}\Big(\int_{|\xi|<1} |\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} + \frac{\pi}{\sqrt{6}}\Big[\Big(\int_{-\infty}^{-1} |\xi|^2|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2} + \Big(\int_{1}^{+\infty} |\xi|^2|\widehat{u}(\xi)|^2\,d\xi\Big)^{1/2}\Big].$$

Hence for any $\lambda > 0$,

$$\sum_{k\in\mathbb{Z}} |u(k)|^2 \leq 2\Big(1+\frac{1}{\lambda}\Big)\|u\|_{L^2}^2 + \frac{\lambda+1}{12}\|u'\|_{L^2}^2.$$

Setting $\mu = 12/(\lambda+1)$, we finally obtain the second inequality, valid in $[0, 12[$:

$$\|u'\|_{L^2}^2 + \frac{24\mu}{12-\mu}\|u\|_{L^2}^2 \geq \mu \sum_{k\in\mathbb{Z}} |u(k)|^2.$$

Proof of Theorem 2.1. We can easily define $\{b_k\}_{k\in\mathbb{Z}}$, with $\{a_k\}_{k\in\mathbb{Z}}$ a subsequence, satisfying for all k in $\mathbb{Z}$,

$$\alpha \leq b_{k+1} - b_k < 2\alpha.$$

It can be constructed by recurrence, setting $b_0 = a_0$. If the sequence is defined up to $k(j_0)$ with $b_{k(j_0)} = a_{j_0}$ $(j_0 > 0)$, we set

$$(\forall l \in \{1,\ldots,N_{j_0}\})\ b_{k(j_0)+l} = a_{j_0} + l\alpha \quad\text{and}\quad b_{k(j_0+1)} = a_{j_0+1},$$

where $N_{j_0} = [(a_{j_0+1} - a_{j_0})/\alpha] - 1$ and $k(j_0+1) = k(j_0) + N_{j_0} + 1$.

The similar process for negative indices yields the whole sequence. Then we define the function $b(x)$ in each interval $[k, k+1[$ by $b(x) = b_k + (x-k)(b_{k+1} - b_k)$; it is continuous on $\mathbb{R}$ and differentiable on each $]k, k+1[$. Let $u \in H^1(\mathbb{R})$. Since $u(b(x))$ is in $H^1(\mathbb{R})$ we can apply Lemma 2.3. For all λ in $\mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \int_k^{k+1} |u'(b(x))|^2|b'(x)|\,dx - \lambda\sum_{k\in\mathbb{Z}} |u(b_k)|^2 \geq -C(\lambda)\sum_{k\in\mathbb{Z}} \int_k^{k+1} |u(b(x))|^2\,dx.$$

By a change of variables in each integral, we obtain

$$2\alpha\|u'\|_{L^2}^2 - \lambda\sum_{k\in\mathbb{Z}} |u(b_k)|^2 \geq -\frac{C(\lambda)}{\alpha}\|u\|_{L^2}^2.$$

As $\{a_j\}_{j\in\mathbb{Z}}$ is a subsequence of $\{b_k\}_{k\in\mathbb{Z}}$, we have

$$\sum_{j\in\mathbb{Z}} |u(a_j)|^2 \leq \sum_{k\in\mathbb{Z}} |u(b_k)|^2.$$

Setting $\mu = \lambda/(2\alpha)$ yields the theorem.

REMARK. Though inequality (12) is not as sharp as (10), even in the case where all the k_j's are equal and $\{a_j\}_{j\in\mathbb{Z}}$ is the whole $\mathbb{Z}$, the interest of the proof is that it can easily be extended to the case of a lattice of points in $\mathbb{R}^n$. This result is, therefore, to be related to work of S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden [Al-Ge].

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