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Strong continuity of semigroup homomorphisms

by

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Abstract. Let J be an abelian topological semigroup and C a subset of a Banach space X. Let $L(X)$ be the space of bounded linear operators on X and $\mathrm{Lip}(C)$ the space of Lipschitz functions $f : C \to C$. We exhibit a large class of semigroups J for which every weakly continuous semigroup homomorphism $T : J \to L(X)$ is necessarily strongly continuous. Similar results are obtained for weakly continuous homomorphisms $T : J \to \mathrm{Lip}(C)$ and for strongly measurable homomorphisms $T : J \to L(X)$.

1. Introduction. Throughout this note J will denote an abelian topological semigroup and C a subset of a Banach space X. We consider homomorphisms $T : J \to F(C)$ where $F(C)$ is a semigroup under composition of functions $f : C \to C$. In particular, we will take $F(C) = C^C$, the space of all functions $f : C \to C$, $F(C) = \mathrm{Lip}(C)$, the subsemigroup of C^C consisting of Lipschitz functions, and, with $C = X$, $F(C) = L(X)$, the space of bounded linear operators on X.

A homomorphism $T : J \to F(C)$ is called *weakly continuous* if $\langle T(\cdot)(x), \varphi\rangle$ is continuous for each $x \in C$ and $\varphi \in X^*$. It is called *strongly continuous* if $T(\cdot)(x)$ is continuous for each $x \in C$.

Such homomorphisms have recently been considered by the authors in [2]. The results therein generalized previous work by Goldstein [6] and others who considered contractive representations $T : [0, \infty) \to L(X)$. In the latter case, it is well known that weak continuity of T is equivalent to strong continuity. See for example [7, p. 305] and [10, p. 233].

It is therefore natural to ask

QUESTION 1.1. *Is every weakly continuous homomorphism* $T : J \to F(C)$ *strongly continuous?*

The following example shows that for certain semigroups J the answer is negative.

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EXAMPLE 1.2. (a) Let $J = [1, \infty)$ and let X be a separable Hilbert space with orthonormal basis $\{e_j : j \in \mathbb{N}\}$. For $k \in \mathbb{N}$ define $g_k : [0,1] \to X$ by

$$g_k(t) = 2te_{k+1} \quad \text{if } 0 \le t \le 1/2,$$

$$g_k(t) = (1-\lambda)e_{k+n} + \lambda e_{k+n+1}$$

$$\text{if } t = (1-\lambda)\left(\frac{n}{n+1}\right) + \lambda\left(\frac{n+1}{n+2}\right) \text{ where } 0 \le \lambda \le 1 \text{ and } n \in \mathbb{N},$$

$$g_k(1) = 0.$$

Then define $g : J \to X$ by

$$g(t) = g_n\left(n(n+1)\left(t + \frac{1}{n} - 2\right)\right)$$

$$\text{if } 2 - \frac{1}{n} \le t \le 2 - \frac{1}{n+1}, \text{ where } n \in \mathbb{N},$$

and $g(t) = 0$ if $t \ge 2$. Finally, define $T : J \to L(X)$ by

$$T(t)(x) = \langle x, e_1 \rangle g(t).$$

Note that the sequence (e_n) converges weakly to 0 in X but not strongly. Moreover, $T(t)T(s)(x) = T(t+s)(x) = 0$ for all $s, t \in J$. It follows that T is a contractive representation which is weakly continuous on J and strongly continuous except on the set $\{2\} \cup \{2 - 1/(n+1) : n \in \mathbb{N}\}$.

(b) Note that the concrete choice of a map $g : J \to \{e_1\}^\perp$ in (a) is not important, but essential are its properties:

(i) g is weakly continuous and vanishing on $[2, \infty)$,

(ii) g is not strongly continuous.

Using an orthonormal decomposition $\{e_1\}^\perp = X_1 \oplus X_2 \oplus \ldots$ into infinite-dimensional subspaces one can similarly construct $g = \sum_{k=1}^{\infty} g_k$, $g_k(t) \in X_k$, with property (i) such that every rational $t \in [1,2]$ is a point of discontinuity of g in the strong topology.

In this note we obtain classes of semigroups J for which the answer to Question 1.1 is affirmative for the codomains $L(X)$ and $\mathrm{Lip}(C)$. The proofs are based on the following proposition which is an immediate consequence of Namioka [8, Theorem 4.1].

PROPOSITION 1.3. *Let J be locally compact Hausdorff and $T : J \to C^C$ weakly continuous. Then for every $x \in C$ there is a dense G_δ set A_x in J such that $T(\cdot)(x)$ is continuous on A_x.*

REMARK 1.4. Proposition 1.3 remains valid for the more general case of σ-well α-favourable topological spaces J as defined in Christensen [3]. In this case we use [3, Theorem 1] in place of [8, Theorem 4.1].

REMARK 1.5. Denote the set of neighbourhoods of an element t in J by $\mathcal{N}(t)$. Let $T : J \to \mathrm{Lip}(C)$ be a weakly continuous homomorphism, let $x \in C$, and choose A_x as in Proposition 1.3. If there exists $a \in A_x$ such that $t + \mathcal{N}(a) \subseteq \mathcal{N}(t+a)$ for all $t \in J$, then $T(\cdot)(x)$ is continuous on $a + J$.

Proof. Let $t \in J$ and let $\kappa > 0$ be a Lipschitz constant for $T(t)$. Given $\varepsilon > 0$, choose a neighbourhood U of a such that $\|T(h)(x) - T(a)(x)\| < \varepsilon/\kappa$ whenever $h \in U$. Then $\|T(t+h)(x) - T(t+a)(x)\| < \varepsilon$ for all $h \in U$. Since $t + U \in \mathcal{N}(t+a)$ we are finished.

It follows from Remark 1.5 that if $T : [1, \infty) \to \mathrm{Lip}(C)$ is a weakly continuous homomorphism, then T is strongly continuous on $(2, \infty)$. Example 1.2 shows that the conclusions of Proposition 1.3 and Remark 1.5 cannot be greatly improved without additional restrictions on J. In particular, we will require that J has a unit 0. These restrictions are introduced in Section 2 and our results for weakly continuous homomorphisms are in Section 3. Finally, in Section 4 we discuss strong continuity for homomorphisms $T : J \to L(X)$ which are only assumed to be locally strongly measurable and locally bounded.

2. Restrictions on semigroups. Throughout this section J will denote an abelian topological semigroup with unit 0. Consider the following condition.

(2.1) *For each neighbourhood U of 0 in J, for each $t \in J$, and for each neighbourhood V of t, there exists $s \in V$ such that $s + U$ is a neighbourhood of t.*

This condition is satisfied, for example, by the additive subsemigroups of $\mathbb{R}^2$ defined by $J_1 = [0, \infty)$, $J_2 = \{(x,y) : |y| \le x,\ 0 \le x < \infty\}$, and by any topological group G. On the other hand, $J_3 = \{(x,y) : |y| \le x^2,\ 0 \le x < \infty\}$ and $J_4 = [0,1] \cup \{(x,y) : |y| \le x - 1,\ 1 \le x < \infty\}$ do not satisfy (2.1).

In our next two conditions we will require that J be a topological subspace of an abelian topological group G. We will denote this by $J \subseteq G$. The interior of J in G will be denoted by J°. Consider:

(2.2) *$J \subseteq G$ and each neighbourhood of 0 in J contains an open subset of G.*

This condition is satisfied by J_1, J_2, G and J_3, but not by J_4. Moreover, if $J \subseteq G$ and $J^\circ \ne \emptyset$ then (2.1) implies (2.2). Finally, consider:

(2.3) *$J \subseteq G$ and for each $t \in J^\circ$ and each dense subset A of J° there exists $s \in J$ such that $t - s \in A$.*

Note that (2.3) is satisfied by all of J_1, J_2, G, J_3 and J_4. Moreover, (2.2) implies (2.3). Indeed, let $t \in J^\circ$ and let A be a dense subset of J°.

So there is an open neighbourhood W of 0 in G such that $W = -W$ and $t + W \subseteq J$. By (2.2) there is an open subset U of G such that $U \subset W \cap J$. As A is dense in J°, there exists $h \in U \cap A$. Hence $s = t - h \in J$ and (2.3) follows.

3. Weakly continuous homomorphisms. For representations we have the following.

THEOREM 3.1. *Let J be a locally compact Hausdorff abelian unital topological semigroup satisfying condition* (2.1). *Every weakly continuous homomorphism $T : J \to L(X)$ is strongly continuous.*

Proof. Let $x \in X$. By Proposition 1.3, $T(\cdot)x$ is continuous on a dense G_δ-set A_x in J.

First we prove continuity of $T(\cdot)x$ at 0. Let V be a compact neighbourhood of 0 in J. Since T is weakly continuous, $T(V)y$ is weakly compact for all $y \in X$. By the uniform boundedness theorem $T(V)$ is bounded. Let $\kappa = \sup_{t\in V} \|T(t)\|$ and set $M = \mathrm{co}\{T(t)x : t \in A_x\}$. The weak and norm closures of M coincide, so $x \in \overline{M}$. Given $\varepsilon > 0$, choose $y \in M$ such that $\|y - x\| < \varepsilon/(2\kappa + 2)$. So $y = \sum_{j=1}^{m} c_j\, T(a_j)x$ for some $a_j \in A_x$ and $c_j > 0$ with $\sum_{j=1}^{m} c_j = 1$. As $T(\cdot)x$ is continuous at each a_j, there is a neighbourhood U of 0 such that $\|T(h + a_j)x - T(a_j)x\| < \varepsilon/2$ for all $h \in U$. Hence

$$\begin{aligned}\|T(h)x - x\| &\le \|T(h)x - T(h)y\| + \|T(h)y - y\| + \|y - x\| \\ &\le (1+\kappa)\|x - y\| + \Big\| \sum_{j=1}^{m} c_j [T(h + a_j)x - T(a_j)x] \Big\| < \varepsilon\end{aligned}$$

for all $h \in U \cap V$. Hence $T(\cdot)x$ is continuous at 0.

Now let $t \in J \setminus \{0\}$. Let V be a compact neighbourhood of t. Define $\kappa = \sup_{t\in V} \|T(t)\|$. For $\varepsilon > 0$ choose a neighbourhood U of 0 such that $\|T(h)x - x\| < \varepsilon/(2\kappa + 1)$ for all $h \in U$. By (2.1) there exists $s \in V$ such that $s + U$ is a neighbourhood of t. Hence, for all $w \in s + U$,

$$\begin{aligned}\|T(w)x - T(t)x\| &\le \|T(w)x - T(s)x\| + \|T(s)x - T(t)x\| \\ &\le \|T(s)\| [\|T(w_0)x - x\| + \|x - T(t_0)x\|] < \varepsilon\end{aligned}$$

where $w = s + w_0$, $t = s + t_0$ for $w_0, t_0 \in U$. So $T(\cdot)x$ is continuous at t and the proof is complete.

Datry and Muraz [4] obtained Theorem 3.1 under the additional assumption that J is a group. Basit and Pryde [1] also obtained this result for groups, but without the assumption that J is abelian.

The dual representation of a homomorphism $T : J \to L(X)$ is the homomorphism $T^* : J \to L(X^*)$ defined by $\langle x, T^*(t)\varphi\rangle = \langle T(t)x, \varphi\rangle$ for all $x \in X$, $\varphi \in X^*$ and $t \in J$. We immediately have

COROLLARY 3.2. *Let J be a locally compact Hausdorff abelian unital topological semigroup satisfying condition* (2.1). *If $T : J \to L(X)$ is a strongly* (*or weakly*) *continuous homomorphism and X is reflexive then $T^* : J \to L(X^*)$ is strongly continuous.*

For non-linear operator semigroups we have

THEOREM 3.3. *Let J be a unital subsemigroup satisfying condition* (2.3) *of a locally compact Hausdorff abelian topological group G. Let C be a subset of a Banach space X. Every weakly continuous homomorphism $T : J \to \mathrm{Lip}(C)$ is strongly continuous on J°.*

Proof. Let $x \in C$. By Proposition 1.3, $T(\cdot)x$ is continuous on a dense subset A_x of J°. Let $t \in J^\circ$. By (2.3) there exist $s \in J$ and $h \in A_x$ such that $t = s + h$. Let $\kappa > 0$ be a Lipschitz constant for $T(s)$. For each $\varepsilon > 0$ there is an open neighbourhood U of h in G such that $U \subseteq J^\circ$ and $\|T(u)(x) - T(h)(x)\| < \varepsilon/\kappa$ for all $u \in U$. Now $s + U$ is a neighbourhood of t in J and for $v = s + u \in U$ we have

$$\begin{aligned}\|T(v)(x) - T(t)(x)\| &= \|T(s + u)(x) - T(s + h)(x)\| \\ &\le \kappa \|T(u)(x) - T(h)(x)\| < \varepsilon.\end{aligned}$$

Hence $T(\cdot)(x)$ is continuous at t as claimed.

COROLLARY 3.4. *Let G be a locally compact Hausdorff abelian topological group and C a subset of a Banach space X. Every weakly continuous homomorphism $T : G \to \mathrm{Lip}(C)$ is strongly continuous.*

4. Strongly measurable homomorphisms. In this section J is a closed unital subsemigroup of a locally compact abelian topological group G equipped with Haar measure μ. By $L^\infty(J, X)$ we denote the Banach space of strongly measurable functions $g : J \to X$ for which $\|g(\cdot)\|_X \in L^\infty(J)$; by χ_V the characteristic function of a set V; and by $L^\infty_{\mathrm{loc}}(J, X)$ the space of functions $g : J \to X$ for which $g\chi_V \in L^\infty(J, X)$ for all compact subsets V of J.

Dunford [5, Theorem] proved that every strongly measurable and locally bounded homomorphism $T : [0, \infty) \to L(X)$ is strongly continuous from the right on $(0, \infty)$. In generalizing this result, we will assume that $T : J \to L(X)$ is locally strongly measurable and locally bounded. By this we mean $T(\cdot)x \in L^\infty_{\mathrm{loc}}(J, X)$ for every $x \in X$. With the above assumptions on J we have

LEMMA 4.1. *If $g \in L^\infty_{\mathrm{loc}}(J, X)$ then*

$$\lim_{h\to 0} \int_K \|g(s + h) - g(s)\|\, d\mu(s) = 0$$

for each compact subset K of J.

Proof. Extending g by 0 outside J we reduce the lemma to the case $J = G$. If W is a relatively compact subset of G then

$$\lim_{h\to 0}\int_G |\chi_W(s+h) - \chi_W(s)|\,d\mu(s) = 0.$$

See for example [9, 1.1.5]. It follows that

$$\lim_{h\to 0}\int_G \|\psi(s+h) - \psi(s)\|\,d\mu(s) = 0$$

for each step function $\psi = \sum_{j=1}^N \chi_{W_j} x_j$, where $x_j \in X$ and W_j is relatively compact in G. Given a compact subset K of G and $g \in L^\infty_{\rm loc}(G,X)$, let V be a compact neighbourhood of 0. Then g is strongly measurable on the compact set $K+V$. So there is a sequence of step functions ψ_j convergent μ-a.e. to g on $K+V$. By Fatou's lemma,

$$\int_K \|g(s+h) - g(s)\|\,d\mu(s) \le \liminf_{j\to\infty}\int_K \|\psi_j(s+h) - \psi_j(x)\|\,d\mu(s)$$

for each $h \in V$. The result follows.

THEOREM 4.2. *Let J be a closed unital subsemigroup satisfying* (2.2) *of a locally compact Hausdorff abelian topological group G. If the homomorphism $T : J \to L(X)$ is locally strongly measurable on J and locally bounded, then T is strongly continuous on J°.*

Proof. Let $t \in J^\circ$ and $x \in X$. Choose compact neighbourhoods U, V of 0 in G such that $V - V \subseteq U$ and $t + U \subseteq J$. Let $K = V \cap J^\circ$. Then $t + V - K \subseteq J$ and by (2.2), $\mu(K) > 0$. Since T is strongly measurable and locally bounded, $T(\cdot)x$ is Bochner integrable on K. Set $\kappa = \sup_{s\in K}\|T(s)\|$ and let $\varepsilon > 0$. By Lemma 4.1 there is a neighbourhood W of 0 in G such that $W \subseteq V$ and

$$\int_{t-K} \|T(s+h)x - T(s)x\|\,d\mu(s) < \frac{\varepsilon\mu(K)}{\kappa+1}$$

for all $h \in W$. Hence, for $h \in W$,

$$\|T(t+h)x - T(t)x\| = \frac{1}{\mu(K)}\Big\|\int_K (T(s)T(t+h-s)x - T(s)T(t-s)x)\,d\mu(s)\Big\|$$
$$\le \frac{\kappa}{\mu(K)}\int_{t-K}\|T(s+h)x - T(s)x\|\,d\mu(s) < \varepsilon.$$

So $T(\cdot)x$ is continuous on J°.

This theorem is also proved in Hille and Phillips [7, Theorem 10.10.1] for the special case $J = \mathbb{R}^n_+ = \{t = (t_1,\dots,t_n) \in \mathbb{R}^n : t_j \ge 0 \text{ for } 1 \le j \le n\}$. Note that if the homomorphism $T : \mathbb{R}^n_+ \to L(X)$ is strongly continuous at 0 then each of the sets $\{T(t)x : x \in X,\ t \in \mathbb{R}^n_+ \setminus \{0\},\ t_j = 0 \text{ for } j \ne k\}$, where $1 \le k \le n$, is dense in X. Hille and Phillips also show that this last condition, together with strong measurability and local boundedness, implies strong continuity of T at 0 and hence on all of $\mathbb{R}^n_+$ [7, Theorem 10.10.2].

REMARK 4.3. A homomorphism $T : J \to \mathrm{Lip}(C)$ is *nonexpansive* if

$$\|T(t)(y) - T(t)(z)\| \le \|y - z\| \quad \text{for all } y, z \in C \text{ and all } t \in J.$$

More generally, a homomorphism $T : J \to \mathrm{Lip}(C)$ is *locally uniformly Lipschitz-valued* if for each compact $K \subset J$,

$$\|T(t)(y) - T(t)(z)\| \le \kappa\|y - z\| \quad \text{for all } y, z \in C, \text{ all } t \in K \text{ and some } \kappa > 0.$$

Theorem 4.2 remains valid, with the same proof, for a homomorphism $T : J \to \mathrm{Lip}(C)$ which is locally strongly measurable on J and locally uniformly Lipschitz-valued.

The following example shows that J° cannot be replaced by J in Theorem 4.2.

EXAMPLE 4.4. Let $J = [0,\infty)$ and let X be a separable Hilbert space. Let $g_1 : [0,1] \to X$ be as defined in Example 1.2(a). Define $g : J \to X$ by $g(t) = g_1(1-t)$ if $0 \le t \le 1$ and $g(t) = 0$ for $t > 1$. Then g is weakly continuous on J, continuous on J°, discontinuous at 0, uniformly continuous on $t + J$ for each $t > 0$, and bounded. Let $B(J,X)$ be the Banach space of all bounded functions $f : J \to X$ and let Y be the smallest closed subspace of $B(J,X)$ containing all the translates g_t. Here $t \in J$ and $g_t(s) = g(s+t)$ for $s \in J$. Define $T : J \to L(Y)$ by $T(s)g_t = g_{t+s}$. If $f \in Y$ and $t \in J^\circ$ then f_t is uniformly continuous on J, which means $T(\cdot)f$ is continuous at t. It follows that T is strongly measurable on J and strongly continuous on J°. However, T is neither strongly continuous at 0 nor, by Theorem 3.1, weakly continuous at 0.

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Lower bounds for Schrödinger operators in $H^1(\mathbb{R})$

by

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Abstract. We prove trace inequalities of type $\|u'\|_{L^2}^2 + \sum_{j\in\mathbb{Z}} k_j|u(a_j)|^2 \ge \lambda\|u\|_{L^2}^2$ where $u \in H^1(\mathbb{R})$, under suitable hypotheses on the sequences $\{a_j\}_{j\in\mathbb{Z}}$ and $\{k_j\}_{j\in\mathbb{Z}}$, with the first sequence increasing and the second bounded.

Introduction. In 1989, R. Strichartz proved (see [Str]) that for an increasing real sequence $\{a_j\}_{j\in\mathbb{Z}}$ unbounded from above and below and such that, for all j in $\mathbb{Z}$, $a_{j+1} - a_j < \beta$ where β is a fixed positive constant, the following inequality holds in $H^1(\mathbb{R})$:

$$(1) \qquad \frac{\beta}{\sqrt{8}}\|u'\|_{L^2} + \sqrt{\beta}\Big(\sum_{j\in\mathbb{Z}}|u(a_j)|^2\Big)^{1/2} \ge \|u\|_{L^2}.$$

This result enables us to define operators such as $-\Delta + \lambda\sum_{j\in\mathbb{Z}}\delta_{a_j}$ with $\lambda > 8/\beta$, where δ_{a_j} is the Dirac measure at a_j, as unbounded selfadjoint operators in $L^2(\mathbb{R})$, using a theorem of [Re-Si]. This theorem (see [Re-Si], Th. VIII.15) states that a unique selfadjoint operator can be associated with every lower semibounded and closed quadratic form. Indeed, the form $\|u'\|_{L^2}^2 + \lambda\sum_{j\in\mathbb{Z}}|u(a_j)|^2$ is lower semibounded (as sketched at the end of the Introduction) and closed (as shown in [Pou]). In order to give a sense to more general operators, using the same theorem, we prove the corresponding trace inequalities.

The aim of this paper is to present inequalities similar to (1), with a family $\{k_j\}_{j\in\mathbb{Z}}$ of weights attached to the points a_j. The improvement is that we allow the k_j's to take negative values and tend to 0 at infinity under suitable hypotheses on the quotient $|k_j|/(a_{j+1} - a_j)$.

In Section 1, we provide the following generalizations of (1):

$$(1') \qquad (\exists\lambda_1 > 0)\ (\forall u \in H^1(\mathbb{R})) \qquad \|u'\|_{L^2}^2 + \sum_{j\in\mathbb{Z}}|k_j|\cdot|u(a_j)|^2 \ge \lambda_1\|u\|_{L^2}^2,$$

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