Strong continuity of semigroup homomorphisms

by

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Abstract. Let \( J \) be an abelian topological semigroup and \( C \) a subset of a Banach space \( X \). Let \( L(X) \) be the space of bounded linear operators on \( X \) and \( \text{Lip}(C) \) the space of Lipschitz functions \( f : C \rightarrow C \). We exhibit a large class of semigroups \( J \) for which every weakly continuous semigroup homomorphism \( T : J \rightarrow L(X) \) is necessarily strongly continuous. Similar results are obtained for weakly continuous homomorphisms \( T : J \rightarrow \text{Lip}(C) \) and for strongly measurable homomorphisms \( T : J \rightarrow L(X) \).

1. Introduction. Throughout this note \( J \) will denote an abelian topological semigroup and \( C \) a subset of a Banach space \( X \). We consider homomorphisms \( T : J \rightarrow F(C) \) where \( F(C) \) is a semigroup under composition of functions \( f : C \rightarrow C \). In particular, we will take \( F(C) = C^C \), the space of all functions \( f : C \rightarrow C \), \( F(C) = \text{Lip}(C) \), the subsemigroup of \( C^C \) consisting of Lipschitz functions, and, with \( C = X \), \( F(C) = L(X) \), the space of bounded linear operators on \( X \).

A homomorphism \( T : J \rightarrow F(C) \) is called weakly continuous if \( (T(\cdot))(x), \varphi \) is continuous for each \( x \in C \) and \( \varphi \in X^* \). It is called strongly continuous if \( T(\cdot)(x) \) is continuous for each \( x \in C \).

Such homomorphisms have recently been considered by the authors in [2]. The results therein generalized previous work by Goldstein [6] and others who considered contractive representations \( T : [0, \infty) \rightarrow L(X) \). In the latter case, it is well known that weak continuity of \( T \) is equivalent to strong continuity. See for example [7, p. 308] and [10, p. 233].

It is therefore natural to ask

**QUESTION 1.1.** Is every weakly continuous homomorphism \( T : J \rightarrow F(C) \) strongly continuous?

The following example shows that for certain semigroups \( J \) the answer is negative.

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EXAMPLE 1.2. (a) Let $J = [1, \infty)$ and let $X$ be a separable Hilbert space with orthonormal basis $\{e_j : j \in \mathbb{N}\}$. For $k \in \mathbb{N}$ define $g_k : [0, 1] \to X$ by

\[
g_k(t) = \begin{cases} 2te_k & \text{if } 0 \leq t \leq 1/2, \\ (1 - \lambda)e_k + \lambda e_{k+1} & \text{if } t = (1 - \lambda)\left(\frac{n}{n+1}\right) + \lambda\left(\frac{n+1}{n+2}\right), \text{ where } 0 \leq \lambda \leq 1 \text{ and } n \in \mathbb{N}, \\ g_k(1) = 0. & \end{cases}
\]

Then define $g : J \to X$ by

\[
g(t) = g_n\left(n(n+1)\left(t + \frac{1}{n} - 2\right)\right)
\]

if $2 - \frac{1}{n} \leq t \leq 2 - \frac{1}{n+1}$, where $n \in \mathbb{N}$, and $g(t) = 0$ if $t \geq 2$. Finally, define $T : J \to L(X)$ by

\[
T(t)(x) = (x, e_1)g(t).
\]

Note that the sequence $(e_n)$ converges weakly to 0 in $X$ but not strongly. Moreover, $T(t)T(s)(x) = T((t+s))(x) = 0$ for all $s, t \in J$. It follows that $T$ is a contractive representation which is weakly continuous on $J$ and strongly continuous except on the set $\{2 - 1/(n+1) : n \in \mathbb{N}\}$.

(b) Note that the concrete choice of a map $g : J \to \{e_1\}^\perp$ in (a) is not important, but essential are its properties:

(i) $g$ is weakly continuous and vanishing on $[2, \infty)$,

(ii) $g$ is not strongly continuous.

Using an orthonormal decomposition $\{e_1\}^\perp = X_1 \oplus X_2 \oplus \ldots$ into infinite-dimensional subspaces one can similarly construct $g = \sum_{k=1}^\infty g_k$, $g_k(t) \in X_k$, with property (i) such that every rational $t \in [1, 2]$ is a point of discontinuity of $g$ in the strong topology.

In this note we obtain classes of semigroups $J$ for which the answer to Question 1.1 is affirmative for the codomains $L(X)$ and $\text{Lip}(C)$. The proofs are based on the following proposition which is an immediate consequence of Namioka [8, Theorem 4.1].

PROPOSITION 1.3. Let $J$ be locally compact Hausdorff and $T : J \to OC$ weakly continuous. Then for every $x \in C$ there is a dense $G_\delta$ set $A_x$ in $J$ such that $T(\cdot)(x)$ is continuous on $A_x$.

REMARK 1.4. Proposition 1.3 remains valid for the more general case of $\sigma$-well $\alpha$-favorable topological spaces $J$ as defined in Christensen [9]. In this case we use [3, Theorem 1] in place of [8, Theorem 4.1].

REMARK 1.5. Denote the set of neighborhoods of an element $t$ in $J$ by $N(t)$. Let $T : J \to \text{Lip}(C)$ be a weakly continuous homomorphism, let $x \in C$, and choose $A_x$ as in Proposition 1.3. If there exists $a \in A_x$ such that $t + N(a) \subseteq N(t + a)$ for all $t \in J$, then $T(\cdot)(a)$ is continuous on $a + J$.

Proof. Let $\varepsilon > 0$ and let $k > 0$ be a Lipschitz constant for $T(t)$. Given $\varepsilon > 0$, choose a neighborhood $U$ of $a$ such that $||T(h)(x) - T(a)(x)|| < \varepsilon/k$ whenever $h \in U$. Then $||T(t + h)(x) - T(t + a)(x)|| < \varepsilon$ for all $h \in U$. Since $t + U \in N(t + a)$ we are finished.

It follows from Remark 1.5 that if $T : [1, \infty) \to \text{Lip}(C)$ is a weakly continuous homomorphism, then $T$ is strongly continuous on $(2, \infty)$. Example 1.2 shows that the conclusions of Proposition 1.3 and Remark 1.5 cannot be greatly improved without additional restrictions on $J$. In particular, we will require that $J$ has a unit 0. These restrictions are introduced in Section 2 and our results for weakly continuous homomorphisms are in Section 3. Finally, in Section 4 we discuss strong continuity for homomorphisms $T : J \to L(X)$ which are only assumed to be locally strongly measurable and locally bounded.

2. Restrictions on semigroups. Throughout this section $J$ will denote an abelian topological semigroup with unit 0. Consider the following conditions:

(2.1) For each neighborhood $U$ of 0 in $J$, for each $t \in J$, and for each neighborhood $V$ of $t$, there exists $s \in V$ such that $s + U$ is a neighborhood of $t$.

This condition is satisfied, for example, by the additive subsemigroups of $\mathbb{R}^2$ defined by $J_1 = [0, \infty)$, $J_2 = \{(x, y) : |y| \leq x, 0 \leq x < \infty\}$, and by any topological group $G$. On the other hand, $J_3 = \{(x, y) : |y| \leq x^2, 0 \leq x < \infty\}$ and $J_4 = [0, 1] \cup \{(x, y) : |y| \leq x - 1, 1 \leq x < \infty\}$ do not satisfy (2.1).

In our next two conditions we will require that $J$ be a topological subspace of an abelian topological group $G$. We will denote this by $J \subseteq G$. The interior of $J$ in $G$ will be denoted by $J^o$. Consider:

(2.2) $J \subseteq G$ and each neighborhood of 0 in $J$ contains an open subset of $G$.

This condition is satisfied by $J_1$, $J_2$, $G$, and $J_3$, but not by $J_4$. Moreover, if $J \subseteq G$ and $J^o \neq \emptyset$ then (2.1) implies (2.2). Finally, consider:

(2.3) $J \subseteq G$ and for each $t \in J^o$ and each dense subset $A$ of $J^o$ there exists $s \in J$ such that $t - s \in A$.

Note that (2.3) is satisfied by all of $J_1$, $J_2$, $G$, $J_3$, and $J_4$. Moreover, (2.2) implies (2.3). Indeed, let $t \in J^o$ and let $A$ be a dense subset of $J^o$. This completes the proof of the following theorem:
So there is an open neighbourhood $W$ of 0 in $G$ such that $W = -W$ and $t + W \subseteq J$. By (2.2) there is an open subset $U$ of $G$ such that $U \subseteq W \cap J$. As $A$ is dense in $J^0$, there exists $h \in U \cap A$. Hence $s = t - h \in J$ and (2.3) follows.

3. Weakly continuous homomorphisms. For representations we have the following.

**Theorem 3.1.** Let $J$ be a locally compact Hausdorff abelian unital topological semigroup satisfying condition (2.1). Every weakly continuous homomorphism $T : J \to L(X)$ is strongly continuous.

**Proof.** Let $x \in X$. By Proposition 1.3, $T(\cdot)x$ is continuous on a dense $G_{\delta}$-set $A_x$ in $J$.

First we prove continuity of $T(\cdot)x$ at 0. Let $V$ be a compact neighbourhood of 0 in $J$. Since $T$ is weakly continuous, $T(V)x$ is weakly compact for all $x \in X$. By the uniform boundedness theorem $T(V)$ is bounded. Let $\kappa = \sup_{t \in V} \|T(t)\|$ and set $M = \text{co}\{T(t)x : t \in A_x\}$. The weak and norm closures of $M$ coincide, so $x \in M$. Given $\varepsilon > 0$, choose $y \in M$ such that $\|y - z\| < \varepsilon/(2\kappa + 2)$. So $y = \sum_{j=1}^m c_j T(a_j)x$ for some $a_j \in A_x$ and $c_j > 0$ with $\sum_{j=1}^m c_j = 1$. As $T(\cdot)x$ is continuous at each $a_j$, there is a neighbourhood $U$ of 0 such that $\|T(t + a_j)x - T(a_j)x\| < \varepsilon/2$ for all $t \in U$. Hence

$$\|T(t)x - x\| \leq \|T(t)x - T(h)y\| + \|T(h)y - y\| + |y - z| \leq (1 + \kappa)\|z - y\| + \left\| \sum_{j=1}^m c_j (T(t + a_j)x - T(a_j)x) \right\| < \varepsilon$$

for all $h \in U \cap V$. Hence $T(\cdot)x$ is continuous at 0.

Now let $t \in J \setminus \{0\}$. Let $V$ be a compact neighbourhood of $t$. Define $\kappa = \sup_{t \in V} \|T(t)\|$. For $\varepsilon > 0$ choose a neighbourhood $U$ of 0 such that $\|T(t)x - x\| < \varepsilon/(2\kappa + 1)$ for all $t \in U$. By (2.1) there exists $s \in V$ such that $s + U$ is a neighbourhood of $t$. Hence, for all $w \in s + U$,

$$\|T(w)x - T(t)x\| \leq \|T(w)x - T(s)x\| + \|T(s)x - T(t)x\| \leq \|T(s)||T(w)x - x| + \|x - T(w)x\| < \varepsilon$$

where $w = s + w_0$, $t = s + t_0$ for $w_0, t_0 \in U$. So $T(\cdot)x$ is continuous at $t$ and the proof is complete.

Darty and Muraz [4] obtained Theorem 3.1 under the additional assumption that $J$ is a group. Baas and Pryde [1] also obtained this result for groups, but without the assumption that $J$ is abelian.

The dual representation of a homomorphism $T : J \to L(X)$ is the homomorphism $T^* : J \to L(X^*)$ defined by $\langle T(t)x, \varphi \rangle = (T(t)x, \varphi)$ for all $x \in X$, $\varphi \in X^*$ and $t \in J$. We immediately have

**Corollary 3.2.** Let $J$ be a locally compact Hausdorff abelian unital topological semigroup satisfying condition (2.1). If $T : J \to L(X)$ is a strongly (or weakly) continuous homomorphism and $X$ is reflexive then $T^* : J \to L(X^*)$ is strongly continuous.

For non-linear operator semigroups we have

**Theorem 3.3.** Let $J$ be a unital semigroup satisfying condition (2.3) of a locally compact Hausdorff abelian topological group $G$. Let $C$ be a subset of a Banach space $X$. Every weakly continuous homomorphism $T : J \to Lip(C)$ is strongly continuous on $J^0$.

**Proof.** Let $x \in C$. By Proposition 1.3, $T(\cdot)x$ is continuous on a dense subset $A_x$ of $J^0$. Let $t \in J^0$. By (2.3) there exist $s \in J$ and $h \in A_x$ such that $t = s + h$. Let $\kappa > 0$ be a Lipschitz constant for $T(s)$. For each $\varepsilon > 0$ there is an open neighbourhood $U$ of $h$ in $G$ such that $U \subseteq J^0$ and $\|T(u)(x) - T(h)(x)\| < \varepsilon/\kappa$ for all $u \in U$. Now $s + U$ is a neighbourhood of $t$ in $J$ and for $u = s + u \in U$ we have

$$\|T(u)(x) - T(t)(x)\| = \|T(s + u)(x) - T(s + h)(x)\| \leq \kappa \|T(u)(x) - T(h)(x)\| < \varepsilon.$$

Hence $T(\cdot)x$ is continuous at $t$ as claimed.

**Corollary 3.4.** Let $G$ be a locally compact Hausdorff abelian topological group and $C$ a subset of a Banach space $X$. Every weakly continuous homomorphism $T : G \to Lip(C)$ is strongly continuous.

4. Strongly measurable homomorphisms. In this section $J$ is a closed unital subsemigroup of a locally compact abelian topological group $G$ equipped with Haar measure $\mu$. By $L^\infty(J, X)$ we denote the Banach space of strongly measurable functions $g : J \to X$ for which $\|g(x)\|_X \in L^\infty(J)$; by $X_V$ the characteristic function of a set $V$; and by $L^\infty(J, X)$ the space of functions $g : J \to X$ for which $g x_V \in L^\infty(J, X)$ for all compact subsets $V$ of $J$.

Dunford [5, Theorem] proved that every strongly measurable and locally bounded homomorphism $T : [0, \infty) \to L(X)$ is strongly continuous from the right on $(0, \infty)$. In generalizing this result, we will assume that $T : J \to L(X)$ is locally strongly measurable and locally bounded. By this we mean $T(\cdot)x \in L^\infty(J, X)$ for every $x \in X$. With the above assumptions on $J$ we have

**Lemma 4.1.** If $g \in L^\infty_{loc}(J, X)$ then

$$\lim_{h \to 0} \int_X \|g(s + h) - g(s)\| d\mu(s) = 0$$

for each compact subset $K$ of $J$. 
Proof. Extending $g$ by 0 outside $J$ we reduce the lemma to the case $J = G$. If $W$ is a relatively compact subset of $G$ then
\[
\lim_{h \to 0} \int_G |\chi_W(s + h) - \chi_W(s)| \, d\mu(s) = 0.
\]
See for example [9, 1.1.5]. It follows that
\[
\lim_{h \to 0} \int_G \|\psi(s + h) - \psi(s)\| \, d\mu(s) = 0
\]
for each step function $\psi = \sum_{j=1}^N \lambda_j \chi_{x_j}$, where $x_j \in X$ and $W_j$ is relatively compact in $G$. Given a compact subset $K$ of $G$ and $g \in L^\infty_{qs}(G, X)$, let $V$ be a compact neighbourhood of 0. Then $g$ is strongly measurable on the compact set $K + V$. So there is a sequence of step functions $\psi_j$ convergent $\mu$-a.e. to $g$ on $K + V$. By Fatou’s lemma,
\[
\int_K g(s + h) - g(s) \, d\mu(s) \leq \liminf_{j \to \infty} \int_K \psi_j(s + h) - \psi_j(s) \, d\mu(s)
\]
for each $h \in V$. The result follows.

THEOREM 4.2. Let $J$ be a closed unital subsemigroup satisfying (2.2) of a locally compact Hausdorff abelian topological group $G$. If the homomorphism $T : J \to L(X)$ is locally strongly measurable on $J$ and locally bounded, then $T$ is strongly continuous on $J^0$.

Proof. Let $t \in J^0$ and $z \in X$. Choose compact neighbourhoods $U, V$ of 0 in $G$ such that $V - V \subseteq U$ and $t + U \subseteq J$. Let $K = V \cap J^0$. Then $t + V - K \subseteq J$ and by (2.2), $\mu(K) > 0$. Since $T$ is strongly measurable and locally bounded, $T(\cdot)x$ is Bochner integrable on $K$. Set $\kappa = \sup_{s \in K} \|T(s)x\|$ and let $\varepsilon > 0$. By Lemma 4.1 there is a neighbourhood $W$ of 0 in $G$ such that $W \subseteq V$ and
\[
\int_{t - K} \|T(s + h)x - T(s)x\| \, d\mu(s) < \frac{\varepsilon \mu(K)}{\kappa + 1}
\]
for all $h \in W$. Hence, for $h \in W$,
\[
\|T(t + h)x - T(t)x\| = \frac{1}{\mu(K)} \int_{t - K} \|T(s)[T(t + h - s)x - T(t - s)x]d\mu(s)\|
\]
\[
\leq \frac{\kappa}{\mu(K)} \int_{t - K} \|T(s + h)x - T(s)x\| \, d\mu(s) < \varepsilon.
\]
So $T(\cdot)x$ is continuous on $J^0$.

This theorem is also proved in Hille and Phillips [7, Theorem 10.10.1] for the special case $J = \mathbb{R}^n_+ = \{t = (t_1, \ldots, t_n) \in \mathbb{R}^n: t_j \geq 0 \text{ for } 1 \leq j \leq n\}$. Note that if the homomorphism $T : \mathbb{R}^n_+ \to L(X)$ is strongly continuous at 0 then each of the sets $\{T(t)x : x \in X, \ t \in \mathbb{R}^n_+ \setminus \{0\}, t_j = 0 \text{ for } j \neq k\}$, where $1 \leq k \leq n$, is dense in $X$. Hille and Phillips also show that this last condition, together with strong measurability and local boundedness, implies strong continuity of $T$ at 0 and hence on all of $\mathbb{R}^n_+$ [7, Theorem 10.10.2].

**Remark 4.3.** A homomorphism $T : J \to \text{Lip}(C)$ is nonexpansive if
\[
\|T(t)(y) - T(t)(z)\| \leq \|y - z\| \quad \text{for all } y, z \in C \text{ and all } t \in J.
\]
More generally, a homomorphism $T : J \to \text{Lip}(C)$ is locally uniformly Lipschitz-valued if for each compact $K \subseteq J$,
\[
\|T(t)(y) - T(t)(z)\| \leq \kappa \|y - z\| \quad \text{for all } y, z \in C, \text{ all } t \in K \text{ and some } \kappa > 0.
\]
Theorem 4.2 remains valid, with the same proof, for a homomorphism $T : J \to \text{Lip}(C)$ which is locally strongly measurable on $J$ and locally uniformly Lipschitz-valued.

The following example shows that $J^0$ cannot be replaced by $J$ in Theorem 4.2.

**Example 4.4.** Let $J = [0, \infty)$ and let $X$ be a separable Hilbert space. Let $g_1 : [0, 1] \to X$ be as defined in Example 1.2(a). Define $g : J \to X$ by $g(t) = g_1(1 - t)$ if $0 \leq t \leq 1$ and $g(t) = 0$ for $t > 1$. Then $g$ is weakly continuous on $J$, continuous on $J^0$, discontinuous at 0, uniformly continuous on $t + J$ for each $t > 0$, and bounded. Let $B(J, X)$ be the Banach space of all bounded functions $f : J \to X$ and let $Y$ be the smallest closed subspace of $B(J, X)$ containing all the translates $g_t$. Here $t \in J$ and $g_t(s) = g(s + t)$ for $s \in J$. Define $T : J \to L(Y)$ by $T(s)g_t = g_{t+s}$. If $f \in Y$ and $t \in J^0$ then $f_t$ is uniformly continuous on $J$, which means $T(\cdot)f$ is continuous at $t$. It follows that $T$ is strongly measurable on $J$ and strongly continuous on $J^0$. However, $T$ is neither strongly continuous at 0 nor, by Theorem 3.1, weakly continuous at 0.

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**References.**


Lower bounds for Schrödinger operators in $H^1(\mathbb{R})$

by

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Abstract. We prove trace inequalities of type $\|u\|_{L^2}^2 + \sum_{j \in \mathbb{Z}} k_j |w(a_j)|^2 \geq \lambda \|u\|_{L^2}^2$
where $u \in H^1(\mathbb{R})$, under suitable hypotheses on the sequences $(a_j)_{j \in \mathbb{Z}}$ and $(k_j)_{j \in \mathbb{Z}}$, with
the first sequence increasing and the second bounded.

Introduction. In 1989, R. Strichartz proved (see [Str]) that for an increasing real sequence $(a_j)_{j \in \mathbb{Z}}$ unbounded from above and below and such that, for all $j$ in $\mathbb{Z}$, $a_{j+1} - a_j < \beta$ where $\beta$ is a fixed positive constant, the following inequality holds in $H^1(\mathbb{R})$:

$$\frac{\beta}{\sqrt{8}} \|u\|_{L^2}^2 + \sqrt{\beta} \left( \sum_{j \in \mathbb{Z}} |w(a_j)|^2 \right)^{1/2} \geq \|u\|_{L^2}.$$  

(1)

This result enables us to define operators such as $-\Delta + \lambda \sum_{j \in \mathbb{Z}} \delta_{a_j}$ with
$\lambda > 8/\beta$, where $\delta_{a_j}$ is the Dirac measure at $a_j$, as unbounded selfadjoint
operators in $L^2(\mathbb{R})$, using a theorem of [Re-Si]. This theorem (see [Re-Si],
Th. VIII.15) states that a unique selfadjoint operator can be associated with
every lower semibounded and closed quadratic form. Indeed, the form $\|u\|_{L^2}^2 + \lambda \sum_{j \in \mathbb{Z}} |w(a_j)|^2$
is lower semibounded (as sketched at the end of
the Introduction) and closed (as shown in [Pou]). In order to give a sense to
more general operators, using the same theorem, we prove the corresponding
trace inequalities.

The aim of this paper is to present inequalities similar to (1), with a
family $(k_j)_{j \in \mathbb{Z}}$ of weights attached to the points $a_j$. The improvement
is that we allow the $k_j$'s to take negative values and tend to 0 at infinity under
suitable hypotheses on the quotient $|k_j|/(a_{j+1} - a_j)$.

In Section 1, we provide the following generalizations of (1):

$$(1') \quad (\exists \lambda_1 > 0) \quad (\forall u \in H^1(\mathbb{R})) \quad \|u\|_{L^2}^2 + \sum_{j \in \mathbb{Z}} |k_j| \cdot |w(a_j)|^2 \geq \lambda_1 \|u\|_{L^2}^2,$$