Finally, we define $\phi : Y^{**} \to \mathbb{R}$ by $\phi(y^{**}) = \lim_n y^{**}(h_n)$. By (3), we have $\phi \in Y^{***}$. Now, by applying Lemma 4,

$$Y^{**} = \mathrm{Ker}\phi \oplus \langle x_0^{**}\rangle = N_Y \oplus \langle x_0^{**}\rangle = Y \oplus \langle x_0^{**}\rangle.$$

Thus Y is an order-one quasireflexive subspace. ∎

REMARK. With an analogous proof, the conclusion of Theorem B is also true if we suppose that $\dim(N_X/X) < \infty$ instead of $N_X = X$.

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A spectral theory for locally compact abelian groups of automorphisms of commutative Banach algebras

by
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Abstract. Let $\mathcal{A}$ be a commutative Banach algebra with Gelfand space $\Delta(\mathcal{A})$. Denote by $\mathrm{Aut}(\mathcal{A})$ the group of all continuous automorphisms of $\mathcal{A}$. Consider a $\sigma(\mathcal{A}, \Delta(\mathcal{A}))$-continuous group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ of a locally compact abelian group G by automorphisms of $\mathcal{A}$. For each $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$, the function $\varphi_a(t) := \varphi(\alpha_t a)$ ($t \in G$) is in the space $C(G)$ of all continuous and bounded functions on G. The weak-star spectrum $\sigma_{w^*}(\varphi_a)$ is defined as a closed subset of the dual group $\widehat{G}$ of G. For $\varphi \in \Delta(\mathcal{A})$ we define Λ_φ^α to be the union of all sets $\sigma_{w^*}(\varphi_a)$ where $a \in \mathcal{A}$, and Λ_α to be the closure of the union of all sets Λ_φ^α where $\varphi \in \Delta(\mathcal{A})$, and call Λ_α the unitary spectrum of α.

Starting by showing that the closure of Λ_φ^α (for fixed $\varphi \in \Delta(\mathcal{A})$) is a subsemigroup of $\widehat{G}$ we characterize the structure properties of the group representation α such as norm continuity, growth and existence of non-trivial invariant subspaces through its unitary spectrum Λ_α.

For an automorphism T of a semisimple commutative Banach algebra $\mathcal{A}$ we consider the group representation $\mathbf{T} : \mathbb{Z} \to \mathrm{Aut}(\mathcal{A})$ defined by $\mathbf{T}_n := T^n$ for all $n \in \mathbb{Z}$. It is shown that $\Lambda_{\mathbf{T}} = \sigma(T) \cap \mathbb{T}$, where $\sigma(T)$ is the spectrum of T and $\mathbb{T}$ is the unit circle. From this fact we give an easy proof of the Kamowitz–Scheinberg theorem which asserts that the spectrum $\sigma(T)$ either contains $\mathbb{T}$ or is a finite union of finite subgroups of $\mathbb{T}$.

Introduction. Let $\mathcal{A}$ be a commutative Banach algebra with Gelfand space $\Delta(\mathcal{A})$ (i.e., the space of regular maximal ideals of $\mathcal{A}$). Denote by $\mathrm{Aut}(\mathcal{A})$ the group of all continuous automorphisms of $\mathcal{A}$. For an automorphism T on $\mathcal{A}$ we consider the group representation $\mathbf{T} : \mathbb{Z} \to \mathrm{Aut}(\mathcal{A})$ given by $\mathbf{T}_n := T^n$ for all $n \in \mathbb{Z}$. For each $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$ the function $\varphi_a(n) := \varphi(T^n a)$ ($n \in \mathbb{Z}$) belongs to the space $C(\mathbb{Z})$ of all continuous and bounded functions on the group $\mathbb{Z}$. The weak-star spectrum $\sigma_{w^*}(\varphi_a)$ of φ_a, as a closed subset of the unit circle $\mathbb{T}$, is defined in the classical way (see [14] or [20]). Note that $\mathbb{T}$ is the dual group of $\mathbb{Z}$.

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Fix $\varphi \in \Delta(\mathcal{A})$. Define $\Lambda_\varphi^{\mathbf{T}}$ to be the union of all $\sigma_{w^*}(\varphi_a)$ where a runs through $\mathcal{A}$. As will be shown in Proposition 2.4, the closure of $\Lambda_\varphi^{\mathbf{T}}$ (for fixed $\varphi \in \Delta(\mathcal{A})$) is a subgroup of $\mathbb{T}$. We define $\Lambda_{\mathbf{T}}$ to be the closure of the union of all sets $\Lambda_\varphi^{\mathbf{T}}$, where φ runs through $\Delta(\mathcal{A})$, and call $\Lambda_{\mathbf{T}}$ the unitary spectrum of the group representation $\mathbf{T}$. The relation of the unitary spectrum of $\mathbf{T}$ and the spectrum $\sigma(T)$ of the generator T is as follows: If $\mathcal{A}$ is semisimple, then $\Lambda_{\mathbf{T}} = \sigma(T) \cap \mathbb{T}$. From this fact the Kamowitz–Scheinberg theorem [13] follows: Either $\sigma(T)$ contains $\mathbb{T}$ or $\sigma(T)$ is a finite union of finite subgroups of $\mathbb{T}$. The present proof gains in interest if one realizes that no deep result is needed besides the properties of weak-star spectrum.

In replacing $\mathbb{Z}$ by a locally compact abelian (LCA) group G we consider a $\sigma(\mathcal{A}, \Delta(\mathcal{A}))$-continuous group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ of G by automorphisms of $\mathcal{A}$. We define the spectral sets Λ_φ^α and the unitary spectrum Λ_α in a similar way. Our aim is to characterize the properties of the group representation α through its unitary spectrum Λ_α. The advantage of using Λ_α lies in the fact (Proposition 2.2) that the closure of Λ_φ^α (for fixed $\varphi \in \Delta(\mathcal{A})$) is a closed subsemigroup of the dual group $\widehat{G}$ of G.

The organization of the paper is as follows.

In Section 1 we recall some facts of harmonic analysis.

In Section 2 we define three spectral notions: Λ_φ^α, $\Lambda_\alpha(a)$ and Λ_α. Their properties are studied. The main results are Proposition 2.4 and Theorem 2.5.

In Section 3 we study two special cases: $G = \mathbb{Z}$ and $G = \mathbb{R}$. The result for $G = \mathbb{Z}$ is Theorem 3.2 which has the above cited Kamowitz–Scheinberg theorem as consequence (Corollary 3.3). One of the results for $G = \mathbb{R}$ (Corollary 3.5) says that if D is a closed derivation on a semisimple commutative Banach algebra $\mathcal{A}$ that generates a strongly continuous group of automorphisms, then the boundedness of $\sigma(D) \cap \mathbb{R}$ implies that $D = 0$. This result helps understand the fact that the zero operator is the unique bounded derivation on a semisimple commutative Banach algebra.

The topic of Section 4 is the so-called "Spectral Mapping Theorem" and applications of our spectral theory. Consider a $\sigma(\mathcal{A}, \Delta(\mathcal{A}))$-continuous group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$. By a "Spectral Mapping Theorem" we mean a result which describes the spectral set $\sigma(\alpha_t) \cap \mathbb{T}$ of an automorphism α_t through the unitary spectrum Λ_α. The main result is Theorem 4.2 that has a consequence (Corollary 4.3) saying that $\sigma(\alpha_t) \cap \mathbb{T} = \overline{\{\gamma(t) : \gamma \in \Lambda_\alpha\}}$ for all $t \in G$ whenever the Banach algebra $\mathcal{A}$ is semisimple. More applications of Theorem 4.2 to a group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ are given in Theorems 4.4 and 4.5, and Corollary 4.6. Theorem 4.4 describes the concrete structure of α if the unitary spectrum Λ_α contains only finitely many elements. Theorem 4.5 says that the compactness of Λ_α implies the norm continuity of α and Corollary 4.6 further asserts that if the group G is connected then a representation α whose unitary spectrum is compact is trivial, i.e., $\alpha_t = I$ for all $t \in G$. Also, we deal with the problem of existence of a non-trivial closed subspace (resp. subalgebra) of $\mathcal{A}$ which is invariant under all operators α_t. The result in Corollary 4.9 gives a positive answer to this problem, while Proposition 4.12(i) ensures the existence of a non-trivial invariant subalgebra for α if the unitary spectrum Λ_α is not a subsemigroup of $\widehat{G}$.

A spectral theory for group representations $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ where either the group G is assumed to be locally compact but non-commutative or the Banach algebra $\mathcal{A}$ is non-commutative would better match the scope of the recent topic "Non-commutative Geometry" suggested by A. Connes [7]. However, building such a spectral theory is a very difficult task. Our theory might provide a sample.

2. Preliminaries. As prerequisites, the reader is expected to be familiar with the basic facts of harmonic analysis on LCA groups and Banach algebras; we use [10], [21], [16] and [17] as basic references for these topics.

In the sequel G denotes a LCA group with identity 0. Let $L^1(G)$ and $M(G)$ be the corresponding group algebra and measure algebra on G (see [10]). Recall from [17] that the *hull* of a set $I \subset L^1(G)$ is defined as

$$\mathrm{h}(I) := \{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in I\}.$$

For each closed subset $\Lambda \subset \widehat{G}$ we define two ideals in $L^1(G)$:

$$k(\Lambda) := \{f \in L^1(G) : \widehat{f} \text{ vanishes on } \Lambda\},$$

$$j(\Lambda) := \{f \in L^1(G) : \widehat{f} \text{ vanishes in a neighbourhood of } \Lambda\}.$$

It is clear that $j(\Lambda) \subset k(\Lambda)$. A closed subset Λ of $\widehat{G}$ is called a *spectral set* if $j(\Lambda)$ is norm dense in $k(\Lambda)$. In the following theorem we collect some standard facts of harmonic analysis on LCA groups which are needed for our present purposes. The assertion (i) is referred to as "Tauberian theorem" [17, Theorem 37A, p. 148] and (ii) is a special case of [17, Theorem 25D, p. 84]. The proof of (iii) depends heavily on the fact that $L^1(G)$ satisfies Ditkin's condition (see [17, Theorem 37C, p. 151]).

THEOREM 1.1. *Let G be a LCA group with dual group $\widehat{G}$. Then:*

(i) *Let I be an ideal of $L^1(G)$. Then, $h(I) = \emptyset$ if and only if I is dense in $L^1(G)$.*

(ii) *Let Λ be a closed subset of $\widehat{G}$. If I is a closed ideal of $L^1(G)$ such that $h(I) = \Lambda$, then $j(\Lambda) \subseteq I \subseteq k(\Lambda)$.*

(iii) *If Λ is a closed subset of $\widehat{G}$ whose boundary $\partial\Lambda$ is scattered, then Λ is a spectral set.*

Let $L^\infty(G)$ be the dual of $L^1(G)$ and $C(G)$ be the space of all bounded continuous functions on G. For $\gamma \in \widehat{G}$, define $e_\gamma \in C(G)$ by $e_\gamma(t) := \gamma(t)$ ($t \in G$). For $h \in L^\infty(G)$ let $[h]$ be the subspace spanned by all translates of h, and let $[h]_{w^*}$ be the w^*-closure of $[h]$. The *weak-star spectrum* of $h \in L^\infty(G)$, denoted by $\sigma_{w^*}(h)$, is defined to be (see [20] and compare [14])

$$\sigma_{w^*}(h) := \{\gamma \in \widehat{G} : e_\gamma \in [h]_{w^*}\}.$$

To find a more useful description of weak-star spectrum we need one more spectral notion from [11] (cf. [2]).

Recall that a function $\omega : G \to [1,\infty)$ on a LCA group G is called a *weight* if it is locally bounded, Borel measurable and submultiplicative in the sense that

$$1 \le \omega(s+t) \le \omega(s)\omega(t) \quad \text{for all } s,t \in G.$$

The *Beurling algebra* $L^1_\omega(G)$ corresponding to a weight ω on G is the subalgebra of $L^1(G)$ defined by

$$L^1_\omega(G) := \Big\{f \in L^1(G) : \|f\|_\omega := \int_G |f(t)|\omega(t)\,dt < \infty\Big\}.$$

The subalgebra $M_\omega(G)$ of $M(G)$ is defined by

$$M_\omega(G) := \Big\{\mu \in M(G) : \|\mu\|_\omega := \int_G \omega(t)\,d|\mu|(t) < \infty\Big\}.$$

It follows from Lebesgue's decomposition theorem that $L^1_\omega(G)$ is an ideal in $M_\omega(G)$. The weight ω is called *non-quasianalytic* (for short, n.q.a.) if

$$\sum_{n=1}^{\infty} \frac{\log\omega(nt)}{n^2} < \infty \quad \text{for all } t \in G.$$

It is proved by Domar [8, Theorem 2.11] that ω is a non-quasianalytic weight if and only if the Beurling algebra $L^1_\omega(G)$ is a regular Banach algebra. Also, it is shown in [8] that if $L^1_\omega(G)$ is a regular Beurling algebra then $\Delta(L^1_\omega(G)) = \widehat{G}$ and the Gelfand transform on $L^1_\omega(G)$ is the usual Fourier transform.

For a (complex) Banach space X, let $\mathcal{L}(X)$ be the Banach space of all bounded linear operators on X. A *non-quasianalytic representation* of $M_\omega(G)$ on X is a continuous algebra homomorphism Φ from $M_\omega(G)$ into $\mathcal{L}(X)$, where ω is a non-quasianalytic weight, i.e., Φ satisfies

(i) $\Phi_{\mu*\nu} = \Phi_\mu\Phi_\nu$ for all $\mu,\nu \in M_\omega(G)$; and

(ii) $\Phi_{\delta_0} = I_X$, where δ_0 is the Dirac measure at the identity 0 of G and I_X is the identity operator on X.

For a n.q.a. representation $\Phi : M_\omega(G) \to \mathcal{L}(X)$ we denote by Φ_t (resp. Φ_f) the image of the Dirac measure δ_t at $t \in G$ (resp. the function $f \in L^1_\omega(G)$) under Φ. It is clear that $\Phi : L^1_\omega(G) \to \mathcal{L}(X)$ is a continuous algebra homomorphism and $\Phi : G \to \mathcal{L}(X)$ is a group homomorphism; for the latter no continuity condition is assumed. Let σ be a topology on X. We say Φ is *σ-continuous* if the group representation $\Phi : G \to \mathcal{L}(X)$ is continuous with respect to the topology σ. A systematic spectral theory for n.q.a. representations has been developed recently by the author in [11] which is based on and extends previous theory of Arveson [2]. Below we collect some facts from [2] and [11] which are needed for our present purposes.

Let $\Phi : M_\omega(G) \to \mathcal{L}(X)$ be a n.q.a. representation. The set

$$I_\Phi := \{f \in L^1_\omega(G) : \Phi_f = 0\}$$

is a closed ideal in $L^1_\omega(G)$. The *spectrum* $\mathrm{Sp}(\Phi)$ of Φ is then defined as the hull of I_Φ:

$$\mathrm{Sp}(\Phi) := \{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in I_\Phi\}.$$

For $x \in X$, let $I_x := \{f \in L^1_\omega(G) : \Phi_f(x) = 0\}$. The *local spectrum* $\mathrm{Sp}_\Phi(x)$ is defined as the hull of I_x:

$$\mathrm{Sp}_\Phi(x) := \{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in I_x\}.$$

The *spectral subspace* $X^\Phi(\Lambda)$ corresponding to a closed subset $\Lambda \subseteq \widehat{G}$ is defined by

$$X^\Phi(\Lambda) := \{x \in X : \mathrm{Sp}_\Phi(x) \subseteq \Lambda\}.$$

A n.q.a. representation $\Phi : M_\omega(G) \to \mathcal{L}(X)$ is called *non-degenerate* if the kernel

$$X_0 := \{x \in X : \Phi_f(x) = 0 \text{ for all } f \in L^1_\omega(G)\}$$

is trivial, i.e., $X_0 = \{0\}$. It is shown in [11] (cf. [2, Prop. 1.4]) that all weakly continuous and w^*-continuous representations are non-degenerate. We have the following results corresponding to Corollary 1.2.7 and Theorem 3.1.1 of [11]. The second assertion is referred to as "Spectral Decomposition Theorem".

THEOREM 1.2. *Let $\Phi : M_\omega(G) \to \mathcal{L}(X)$ be a non-degenerate n.q.a. representation. Then:*

(i) *For all $\mu \in M_\omega(G)$ such that $\mathrm{supp}\,\widehat{\mu} \cap \mathrm{Sp}(\Phi) = \emptyset$ one has $\Phi_\mu = 0$.*

(ii) *If $\mathrm{Sp}(\Phi) = E \cup F$, where E is compact and F is closed such that $E \cap F = \emptyset$, then there exists a bounded projection $P \in \{\Phi_f : f \in L^1_\omega(G)\}$ such that*

$$\mathrm{Sp}(\alpha \circ P) = E \quad \text{and} \quad \mathrm{Sp}(\alpha \circ (I-P)) = F,$$

where $\alpha \circ P$ (resp. $\alpha \circ (I-P)$) denotes the restriction of α to the α-invariant subspace PX (resp. $(I-P)X$).

Now, let $\mathcal{R}$ be the right translation group representation of G on the group algebra $L^1(G)$ given by

$$\mathcal{R}_t f(\cdot) := f(\cdot - t), \quad t \in G.$$

It is a standard fact that $\mathcal{R}$ is a strongly continuous and isometric group representation. It extends to a representation of $M(G)$ on $L^1(G)$ by convolution operators:

$$\mathcal{R}_\mu f = \mu * f \quad \text{for all } (\mu, f) \in M(G) \times L^1(G).$$

The dual of $\mathcal{R}$, denoted by $\mathcal{L}$, is a w^*-continuous representation of $M(G)$ on $L^\infty(G)$. We have

$$\langle \mathcal{L}_\mu h, f\rangle = \langle h, \mathcal{R}_\mu f\rangle = \langle h, \mu * f\rangle$$

for all $(f, h) \in L^1(G) \times L^\infty(G)$ and $\mu \in M(G)$. It is easily seen that the function $\mathcal{L}_f h$ with $f \in L^1(G)$ and $h \in L^\infty(G)$ is uniformly continuous and satisfies

$$\text{(1)} \qquad \mathcal{L}_f h(t) = \int_G f(s)h(s+t)\,ds \quad \text{for all } t \in G.$$

For $h \in L^\infty(G)$ we define the *spectrum* of h, denoted by $\mathrm{Sp}(h)$, to be the local spectrum of h with respect to $\mathcal{L}$:

$$\mathrm{Sp}(h) := \mathrm{Sp}_{\mathcal{L}}(h) = \{\gamma \in \widehat{G} : \mathcal{L}_f h = 0,\ f \in L^1(G) \Rightarrow \widehat{f}(\gamma) = 0\}.$$

PROPOSITION 1.3. (i) $\sigma_{w^*}(h) = \mathrm{Sp}(h)$ *for* $h \in L^\infty(G)$. *Hence, for a closed subset* Ω *of* $\widehat{G}$,

$$\sigma_{w^*}(h) \subseteq \Omega \Leftrightarrow \langle h, f\rangle = \int_G f(t)h(t)\,dt = 0\ \forall f \in j(\Omega).$$

(ii) *Let* $h \in L^\infty(G)$. *Then* $\sigma_{w^*}(h) = \emptyset$ *if and only if* h *is the zero vector.*

(iii) *Let* $h \in L^\infty(G)$ *be such that* $\mathrm{Sp}(h) = \{\gamma_k : 1 \le k \le n\}$. *Then there exist* $c_1, \ldots, c_n \in \mathbb{C}$ *such that* $h = \sum_{k=1}^n c_k e_{\gamma_k}$.

(iv) $\sigma_{w^*}(g \cdot h) \subseteq \sigma_{w^*}(g) + \sigma_{w^*}(h)$ *for* $g, h \in L^\infty(G)$.

Proof. (i) The identity $\sigma_{w^*}(h) = \mathrm{Sp}(h)$ has been established in [11, §3.2], while its consequence follows from the definition of $\mathrm{Sp}(h)$ combined with the representation of $\mathcal{L}_f h$ given in (1) and the translation invariance of the ideal $j(\Omega)$.

(ii)–(iv) can be proved using Theorem 1.1(i), (iii) (see [20, pp. 141–142] for more details). ■

2. The spectral sets Λ_φ^α, $\Lambda_\alpha(a)$ and Λ_α.

For a commutative Banach algebra $\mathcal{A}$ we denote by $\Delta(\mathcal{A})$ the structure space of $\mathcal{A}$, and by $\mathrm{Aut}(\mathcal{A})$ the group of all continuous algebra automorphisms on $\mathcal{A}$. We begin with the following definition.

DEFINITION 2.1. By an *a-dynamical system* we mean a triple $\{\mathcal{A}, G, \alpha\}$, where $\mathcal{A}$ is a commutative Banach algebra, G a LCA group and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ a $\sigma(\mathcal{A}, \Delta(\mathcal{A}))$-continuous group representation. More precisely, the mapping α has the following properties:

(i) $\alpha_{s+t} = \alpha_s\alpha_t$ for all $s, t \in G$;

(ii) for each pair $(a, \varphi) \in \mathcal{A} \times \Delta(\mathcal{A})$ the function $\varphi_a(t) := \varphi(\alpha_t a)$ $(t \in G)$ is continuous.

Given an a-dynamical system $\{\mathcal{A}, G, \alpha\}$, let $\varphi \in \Delta(\mathcal{A})$. Since $\alpha_t^*\Delta(\mathcal{A}) \subseteq \Delta(\mathcal{A})$, we find that $\|\alpha_t^*\varphi\| \le 1$ for all $t \in G$ and thus

$$|\varphi_a(t)| = |\langle \alpha_t^*\varphi, a\rangle| \le \|a\| \quad \text{for all } a \in \mathcal{A}.$$

Therefore, the functions φ_a are in $C(G)$ and their w^*-spectra $\sigma_{w^*}(\varphi_a)$ are closed subsets of $\widehat{G}$ (see Section 1). Let

$$\Lambda_\varphi^\alpha := \bigcup_{a\in\mathcal{A}} \sigma_{w^*}(\varphi_a).$$

PROPOSITION 2.2. *Let* $\{\mathcal{A}, G, \alpha\}$ *be an a-dynamical system. Let* $\varphi \in \Delta(\mathcal{A})$. *Then:*

(i) *Let* $\gamma \in \widehat{G}$. *If* $\gamma \in \Lambda_\varphi^\alpha$, *then there exists a net* $(a_d) \subset \mathcal{A}$ *such that* $\varphi_{a_d} \xrightarrow{w^*} e_\gamma$, *i.e.*,

$$\lim_d \int_G f(t)\varphi(\alpha_t a_d)\,dt = \widehat{f}(\gamma) \quad \textit{for all } f \in L^1(G).$$

Conversely, if there exists a net $(a_d) \subset \mathcal{A}$ *such that* $\varphi_{a_d} \xrightarrow{w^*} e_\gamma$, *then* $\gamma \in \overline{\Lambda_\varphi^\alpha}$.

(ii) *The closure* $\overline{\Lambda_\varphi^\alpha}$ *is a closed subsemigroup of* $\widehat{G}$.

Proof. (i) Let $\gamma \in \Lambda_\varphi^\alpha$. Then $\gamma \in \sigma_{w^*}(\varphi_a)$ for some $a \in \mathcal{A}$, i.e., $e_\gamma \in [\varphi_a]_{w^*}$. Therefore, there exists a net $(h_d) \subset [\varphi_a]$ such that $h_d \xrightarrow{w^*} e_\gamma$. Each $h_d \in [\varphi_a]$ can be written as

$$h_d = \sum_{j\in J_d} b_j \mathcal{L}_{t_j}\varphi_a,$$

where J_d is a finite subset of $\mathbb{N}$, $\{b_j : j \in J_d\} \subset \mathbb{C}$ and $\{t_j : j \in J_d\} \subset G$. Let $f \in L^1(G)$. Then by (1) we have

$$\begin{aligned}\langle h_d, f\rangle = \mathcal{L}_f h_d(0) &= \sum_{j\in J_d} b_j \int_G f(t)\varphi_a(t+t_j)\,dt \\ &= \sum_{j\in J_d} b_j \int_G f(t)\varphi(\alpha_t\alpha_{t_j}a)\,dt = \int_G f(t)\varphi(\alpha_t a_d)\,dt,\end{aligned}$$

where $a_d := \sum_{j\in J_d} b_j\alpha_{t_j}a \in \mathcal{A}$. It follows that

$$\int_G f(t)\varphi(\alpha_t a_d)\,dt \to \langle e_\gamma, f\rangle = \widehat{f}(\gamma).$$

Thus, for the net $(a_d) \subset \mathcal{A}$ we have $\varphi_{a_d} \xrightarrow{w^*} e_\gamma$.

For the converse part of (i) assume that $\varphi_{a_d} \xrightarrow{w^*} e_\gamma$. Let $f \in j(\overline{\Lambda_\varphi^\alpha})$ and $a \in \mathcal{A}$. Then $\sigma_{w^*}(\varphi_a) \subseteq \overline{\Lambda_\varphi^\alpha}$ and thus by Proposition 1.3(i), $\langle f, \varphi_a\rangle = 0$. This implies in particular that

$$\widehat{f}(\gamma) = \lim_d \langle f, \varphi_{a_d}\rangle = 0.$$

Hence, $\gamma \in \mathrm{h}(j(\overline{\Lambda_\varphi^\alpha})) = \overline{\Lambda_\varphi^\alpha}$, proving (i).

(ii) It suffices to verify that $\Lambda_\varphi^\alpha + \Lambda_\varphi^\alpha \subseteq \overline{\Lambda_\varphi^\alpha}$.

To this end, let $\gamma_1, \gamma_2 \in \Lambda_\varphi^\alpha$. By (i) we can choose two nets $\{a_d : d \in D_1\}$ and $\{b_d : d \in D_2\}$ in $\mathcal{A}$ such that w^*-$\lim_{d\in D_1} \varphi_{a_d} = e_{\gamma_1}$ and w^*-$\lim_{d\in D_2} \varphi_{b_d} = e_{\gamma_2}$. For $d := (d_1, d_2) \in D_1 \times D_2$ let $c_d := a_{d_1} b_{d_2}$. Then for all $t \in G$ we have

$$\varphi_{c_d}(t) = \langle \alpha_t^*\varphi, a_{d_1} b_{d_2}\rangle = \langle \alpha_t^*\varphi, a_{d_1}\rangle \cdot \langle \alpha_t^*\varphi, b_{d_2}\rangle = \varphi_{a_{d_1}}(t)\varphi_{b_{d_2}}(t),$$

where we use the fact that $\alpha_t^*\varphi \in \Delta(\mathcal{A})$ in an obvious way. Let $f \in j(\Lambda_\varphi^\alpha)$. Note that $\sigma_{w^*}(\varphi_{c_d}) \subseteq \Lambda_\varphi^\alpha$. It follows from Proposition 1.3(i) that

$$0 = \langle \varphi_{c_d}, f\rangle = \int_G f(t)\varphi_{a_{d_1}}(t)\varphi_{b_{d_2}}(t)\,dt.$$

Hence,

$$0 = \lim_{d_1\in D_1} \lim_{d_2 \in D_2} \langle \varphi_{c_d}, f\rangle = \int_G f(t) e_{\gamma_1}(t) e_{\gamma_2}(t)\,dt = \widehat{f}(\gamma_1+\gamma_2).$$

It follows that $\gamma_1 + \gamma_2 \in \mathrm{h}(j(\Lambda_\varphi^\alpha)) = \overline{\Lambda_\varphi^\alpha}$. ∎

For the sake of clarity we define once again the functions φ_a and the spectral sets Λ_φ^α.

Definition 2.3. Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system. For $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$ define $\varphi_a \in C(G)$ by $\varphi_a(t) := \varphi(\alpha_t a)$ for all $t \in G$ and two subsets of $\widehat{G}$ by

$$\Lambda_\alpha(a) := \Big(\bigcup_{\psi\in\Delta(\mathcal{A})} \sigma_{w^*}(\psi_a)\Big)^- \quad \text{and} \quad \Lambda_\varphi^\alpha := \bigcup_{b\in\mathcal{A}} \sigma_{w^*}(\varphi_b),$$

where the bar denotes the closure in $\widehat{G}$. Define

$$\Lambda_\alpha := \Big(\bigcup_{\varphi\in\Delta(\mathcal{A})} \Lambda_\varphi^\alpha\Big)^-.$$

The closed set Λ_α is called the *unitary spectrum* of the group representation α, and $\Lambda_\alpha(a)$ is called the *local unitary spectrum* of α at the point $a \in \mathcal{A}$.

It is evident that

$$\Lambda_\alpha := \Big(\bigcup_{a\in\mathcal{A}} \Lambda_\alpha(a)\Big)^-.$$

The unitary spectrum of a group representation is an automorphism invariant. More precisely, if $\{\mathcal{A}, G, \alpha\}$ and $\{\mathcal{A}, G, \beta\}$ are two a-dynamical systems such that there exists $U \in \mathrm{Aut}(\mathcal{A})$ satisfying

$$\alpha_t = U\beta_t U^{-1} \quad \text{for all } t \in G,$$

then $\Lambda_\alpha = \Lambda_\beta$. Below we give some more properties of this notion.

Proposition 2.4. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system.*

(i) *Λ_α is a cyclic subset of $\widehat{G}$ in the sense that $\gamma \in \Lambda_\alpha \Rightarrow n\gamma \in \Lambda_\alpha$ for all $n \in \mathbb{N}$.*

(ii) *If Λ_α is compact, then for each $\varphi \in \Delta(\mathcal{A})$ the closure $\overline{\Lambda_\varphi^\alpha}$ is a compact subgroup of $\widehat{G}$. As a consequence, for each $\gamma \in \Lambda_\alpha$ the closed subsemigroup $\overline{\mathbb{N}\cdot\gamma}$ is a compact subgroup of $\widehat{G}$.*

(iii) *If the Banach algebra $\mathcal{A}$ is self-adjoint, then Λ_α is symmetric in the sense that $-\Lambda_\alpha = \Lambda_\alpha$.*

Proof. (i) follows from Proposition 2.2(ii). Under the assumption of (ii) each semigroup $\overline{\Lambda_\varphi^\alpha}$ for each $\varphi \in \Delta(\mathcal{A})$ is compact. Since a compact commutative semigroup is actually a group (see [10, Theorem 9.16, p. 99]) we obtain the assertion of (ii).

Note that the Banach algebra $\mathcal{A}$ is self-adjoint if for each $\varphi \in \Delta(\mathcal{A})$ the formula $\overline{\varphi}(a) := \overline{\varphi(a)}$ $(a \in \mathcal{A})$ defines an element $\overline{\varphi} \in \Delta(\mathcal{A})$, where the bar denotes complex conjugation (see [17, p. 132]). Using the fact that $\overline{e}_\gamma = e_{-\gamma}$ for all $\gamma \in \widehat{G}$ it is easily verified that $\Lambda_{\overline{\varphi}} = -\Lambda_\varphi^\alpha$ for each $\varphi \in \Delta(\mathcal{A})$. This proves (iii). ∎

Theorem 2.5. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system. Let Ω be a closed subset of $\widehat{G}$ and $a \in \mathcal{A}$. Then $\Lambda_\alpha(a) \subseteq \Omega$ if and only if*

$$(*) \qquad \int_G f(t)\varphi(\alpha_t a)\,dt = 0 \quad \textit{for all } f \in j(\Omega) \textit{ and } \varphi \in \Delta(\mathcal{A}).$$

Hence, the set

$$\mathcal{A}^\alpha(\Omega) := \{a \in \mathcal{A} : \Lambda_\alpha(a) \subseteq \Omega\}$$

is an α-invariant closed subspace of $\mathcal{A}$. Moreover, if Ω is a closed subsemigroup of $\widehat{G}$, then $\mathcal{A}^\alpha(\Omega)$ is a closed subalgebra of $\mathcal{A}$.

Proof. From the definition, $\Lambda_\alpha(a) \subseteq \Omega$ if and only $\sigma_{w^*}(\varphi_a) \subseteq \Omega$ for all $\varphi \in \Delta(\mathcal{A})$. By Proposition 1.3(i) the latter is equivalent to $\langle \varphi_a, f\rangle = 0$ for all $f \in j(\Omega)$ and $\varphi \in \Delta(\mathcal{A})$. This is just $(*)$. The closedness of $\mathcal{A}^\alpha(\Omega)$ follows directly from $(*)$ while its α-invariance follows from the observation that $\varphi_{\alpha_t a} = (\alpha_t^*\varphi)_a$ and $\alpha_t^*\varphi \in \Delta(\mathcal{A})$ for all $t \in G$ and $\varphi \in \Delta(\mathcal{A})$.

To prove the "moreover" part assume Ω to be a closed subsemigroup. Let $a, b \in \mathcal{A}^\alpha(\Omega)$. Then for all $\varphi \in \Delta(\mathcal{A})$ we have $\sigma_{w^*}(\varphi_a) \cup \sigma_{w^*}(\varphi_b) \subseteq \Omega$.

Since

$$\varphi_{ab}(t) = \varphi(\alpha_t(ab)) = \varphi((\alpha_t a)(\alpha_t b)) = \varphi(\alpha_t a)\varphi(\alpha_t b),$$

we find that $\varphi_{ab} = \varphi_a \cdot \varphi_b$. Hence, by Proposition 1.3(iv),

$$\sigma_{w^*}(\varphi_{ab}) \subseteq \sigma_{w^*}(\varphi_a) + \sigma_{w^*}(\varphi_b) \subseteq \Omega + \Omega \subseteq \Omega.$$

Therefore, $ab \in \mathcal{A}^\alpha(\Omega)$. ∎

The subspace $\mathcal{A}^\alpha(\Omega)$ is called the *unitary spectral subspace* of α corresponding to the closed subset $\Omega \subseteq \widehat{G}$.

The spectral notion of this section should be compared with the one given in Section 1. To this end, consider an a-dynamical system $\{\mathcal{A}, G, \alpha\}$. Assume further that α is weakly continuous. We call α a *non-quasianalytic group representation* if there exists a n.q.a. weight ω on G such that

$$\|\alpha_t\| \le \omega(t) \quad \text{for all } t \in G.$$

It follows from [2, Prop. 1.2] that for each $\mu \in M_\omega(G)$ there exists an operator $\Phi_\mu \in \mathcal{L}(\mathcal{A})$ satisfying

$$\langle \varrho, \Phi_\mu a\rangle = \int_G \langle \varrho, \alpha_t a\rangle \, d\mu(t) \quad \text{for all } (a, \varrho) \in \mathcal{A} \times \mathcal{A}^*.$$

The map $\mu \mapsto \Phi_\mu$ is a continuous algebra homomorphism from $M_\omega(G)$ into $\mathcal{L}(\mathcal{A})$ which extends the group representation α in the sense that $\Phi_t = \alpha_t$ for all $t \in G$. We refer to [11, §1.3] for more details.

Let Φ^* be the representation on the dual $\mathcal{A}^*$ given by

$$\Phi^*_\mu := (\Phi_\mu)^* \quad \text{for all } \mu \in M_\omega(G).$$

Recall that the local spectrum of Φ at $a \in \mathcal{A}$ is defined as

$$\mathrm{Sp}_\Phi(a) = \{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in L^1_\omega(G) \text{ with } \Phi_f(a) = 0\}.$$

Analogously, for $\varphi \in \Delta(\mathcal{A})$,

$$\mathrm{Sp}_{\Phi^*}(\varphi) = \{\gamma \in \widehat{G} : \widehat{f}(\gamma) = 0 \text{ for all } f \in L^1_\omega(G) \text{ with } \Phi^*_f(\varphi) = 0\}.$$

THEOREM 2.6. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system, where the commutative Banach algebra $\mathcal{A}$ is semisimple and the group representation α is weakly continuous and non-quasianalytic. Let ω be a n.q.a. weight on G such that $\|\alpha_t\| \le \omega(t)$ for all $t \in G$. Let $\Phi : M_\omega(G) \to \mathcal{L}(\mathcal{A})$ be the extension of α as above. Then*

$$\Lambda_\alpha = \mathrm{Sp}(\Phi).$$

Moreover, for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$ we have

$$\Lambda_\alpha(a) = \mathrm{Sp}_\Phi(a) \quad \textit{and} \quad \overline{\Lambda^\alpha_\varphi} = \mathrm{Sp}_{\Phi^*}(\varphi).$$

Proof. Since the proof needs some more facts from [11], we only give its outline.

Let $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. Let $f \in L^1_\omega(G)$. Then for all $t \in G$ we have

$$\varphi(\Phi_f(\alpha_t a)) = \int_G f(s)\varphi(\alpha_{t+s} a)\, ds.$$

It follows from (1) that

$$\varphi(\Phi_f(\alpha_t a)) = \mathcal{L}_f \varphi_a(t) \quad \text{for all } t \in G,$$

where $\mathcal{L}$ is the left translation representation on $L^\infty(G)$. Fix $a \in \mathcal{A}$. Let $f \in L^1_\omega(G)$. Then from the semisimplicity of $\mathcal{A}$ we find that

$$\mathcal{L}_f \varphi_a = 0 \ \forall \varphi \in \Delta(\mathcal{A}) \Leftrightarrow \Phi_f(a) = o.$$

It follows from the definition that $\Lambda_\alpha(a) = \mathrm{Sp}_\Phi(a)$.

Analogously, for $\varphi \in \Delta(\mathcal{A})$ and $f \in L^1_\omega(G)$ we have

$$\Phi^*_f(\varphi) = 0 \Leftrightarrow \mathcal{L}_f \varphi_a = 0 \ \forall a \in \mathcal{A}.$$

This also implies that $\mathrm{Sp}_{\Phi^*}(\varphi) = \overline{\Lambda^\alpha_\varphi}$. Therefore,

$$\mathrm{Sp}(\Phi) = \Big(\bigcup_{a \in \mathcal{A}} \mathrm{Sp}_\Phi(a)\Big)^- = \Big(\bigcup_{a \in \mathcal{A}} \Lambda_\alpha(a)\Big)^- = \Lambda_\alpha. \ ∎$$

The identity $\Lambda_\alpha = \mathrm{Sp}(\Phi)$ implies that the unitary spectrum Λ_α defined in this section for a weakly continuous and bounded group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ coincides with the usual Arveson spectrum of α (see [2]).

3. Spectrum of automorphisms and derivations. In this section we study two special cases: $G = \mathbb{Z}$ and $G = \mathbb{R}$. As usual, $\mathbb{T}$ denotes the unit circle group. It is standard that $\widehat{\mathbb{Z}} = \mathbb{T}$ and $\widehat{\mathbb{R}} = \mathbb{R}$, where the group duality is implemented by

$$z(n) := z^n \qquad \text{for all } (n, z) \in \mathbb{Z} \times \mathbb{T},$$
$$s(t) := e^{-ist} \qquad \text{for all } (t, s) \in \mathbb{R} \times \mathbb{R}.$$

The Fourier transform in $L^1(\mathbb{Z})$ and $L^1(\mathbb{R})$ is given by

$$\widehat{f}(z) := \sum_{n \in \mathbb{Z}} a_n z^n, \qquad z \in \mathbb{T},\ f = (a_n)_{n \in \mathbb{Z}} \in L^1(\mathbb{Z}),$$

$$\widehat{f}(s) := \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt, \quad s \in \mathbb{R},\ f \in L^1(\mathbb{R}).$$

We denote $L^1(\mathbb{Z})$ by l^1, and l^∞ stands for its dual. The dual of $L^1(\mathbb{R})$ is $L^\infty(\mathbb{R})$.

We need the notion of Carleman spectrum. Let $\mathbf{x} := (x_n)_{n \in \mathbb{Z}} \in l^\infty$ be a bounded sequence. Then

$$\widetilde{\mathbf{x}}(z) := \begin{cases} \sum_{n=1}^{\infty} x_n z^n, & |z| < 1, \\ -\sum_{n=-\infty}^{0} x_n z^n, & |z| > 1, \end{cases}$$

defines a holomorphic function in $\mathbb{C}\setminus\mathbb{T}$. We call $\widetilde{\mathbf{x}}$ the *Carleman transform* of $\mathbf{x}$. A point $z_0\in\mathbb{C}$ is called *regular* for $\widetilde{\mathbf{x}}$ if $\widetilde{\mathbf{x}}$ has a holomorphic extension in a neighbourhood of z_0. The *Carleman spectrum* $\sigma(\mathbf{x})$ of $\mathbf{x}$ is defined as the complement of the set of all regular points for $\widetilde{\mathbf{x}}$. Clearly, $\sigma(\mathbf{x})\subseteq\mathbb{T}$.

Analogously, the *Carleman transform* of a function $h\in L^\infty(\mathbb{R})$ is defined as the two-side Laplace transform

$$\widetilde{h}(\lambda):=\begin{cases}\int_0^\infty e^{i\lambda t}h(t)\,dt, & \operatorname{Im}\lambda>0,\\ -\int_{-\infty}^0 e^{i\lambda t}h(t)\,dt, & \operatorname{Im}\lambda<0.\end{cases}$$

It is a holomorphic function in $\mathbb{C}\setminus\mathbb{R}$. A point $\lambda_0\in\mathbb{C}$ is called *regular* for $\widetilde{h}$ if $\widetilde{h}$ has a holomorphic extension in a neighbourhood of λ_0. The *Carleman spectrum* $\sigma(h)$ is the complement of the set of all regular points for $\widetilde{h}$. It is a closed subset of $\mathbb{R}$.

In general, we have the following spectral relation (see [11, Prop. 4.2.3 and 4.2.4]).

Lemma 3.1. *The following identities are true*:

$$\sigma_{w^*}(\mathbf{x})=\sigma(\mathbf{x})^{-1}\quad \textit{for all } \mathbf{x}\in l^\infty,$$
$$\sigma_{w^*}(h)=\sigma(h)\quad \textit{for all } h\in L^\infty(\mathbb{R}).$$

Let $(A,\mathcal{D}(A))$ be a closed operator in a Banach space X. The *resolvent set* $\varrho(A)$ is the set of all $\lambda\in\mathbb{C}$ such that the operator $\lambda I-A:\mathcal{D}(A)\to X$ is injective and surjective. The complement $\sigma(A):=\mathbb{C}\setminus\varrho(A)$ is called the *spectrum* of A. Let $R(\lambda,A):=(\lambda I-A)^{-1}$ for $\lambda\in\varrho(A)$. It is a holomorphic function. For $x\in X$, let $\varrho_A(x)$ be the set of all $\lambda_0\in\mathbb{C}$ for which there exists a neighbourhood U_0 of λ_0 and a holomorphic function $x(\cdot):U_0\to\mathcal{D}(A)$ such that

$$(\lambda-A)x(\lambda)=x\quad\text{for all }\lambda\in U_0.$$

Note that there might exist more than one such function satisfying this identity. If there is exactly one, then the operator A is said to have the *single-valued extension property* (SVEP) (see [9] and [5]). Let

$$\sigma_A(x):=\mathbb{C}\setminus\varrho_A(x).$$

The set $\varrho_A(x)$ (resp. $\sigma_A(x)$) is called the *local resolvent set* (resp. *local spectrum*) of A at x. We have

$$\sigma_A(x)\subseteq\sigma(A)\quad\text{for all }x\in X.$$

Recall that an isolated point $\lambda_0\in\sigma(A)$ is called a *simple pole of the resolvent* if the spectral projection P corresponding to $\{\lambda_0\}$ satisfies

$$APx=\lambda_0Px\quad\text{for all }x\in X.$$

We refer to [5], [18, Chapter A-III] and [9] for a detailed discussion of the spectral theory for bounded and unbounded operators. Our definition of local spectrum for any closed operator should be compared with the one in [9] and [5] defined for operators having SVEP.

Theorem 3.2. *Let $\mathcal{A}$ be a commutative Banach algebra and T a continuous automorphism on $\mathcal{A}$. Let $\mathbf{T}$ be the group representation of $\mathbb{Z}$ on $\mathcal{A}$ given by*

$$\mathbf{T}_n:=T^n\quad\textit{for all } n\in\mathbb{Z}.$$

Then:

(i) $\Lambda_{\mathbf{T}}(a)\subseteq\sigma_T(a)\cap\mathbb{T}$ *and* $\Lambda_\varphi^{\mathbf{T}}\subseteq\sigma_{T^*}(\varphi)\cap\mathbb{T}$ *for all* $a\in\mathcal{A}$ *and* $\varphi\in\Delta(\mathcal{A})$.

(ii) $\Lambda_{\mathbf{T}}\subseteq\sigma(T)\cap\mathbb{T}$.

(iii) *Let* $\varphi\in\Delta(\mathcal{A})$. *If* $\sigma_{T^*}(\varphi)\cap\mathbb{T}$ *is a proper subset of* $\mathbb{T}$, *then there exists* $m\in\mathbb{N}$ *such that* $T^{*m}\varphi=\varphi$.

(iv) *Assume* $\Delta(\mathcal{A})\neq\emptyset$. *Then either there exists* $m\in\mathbb{N}$ *such that* $T^{*m}\varphi=\varphi$ *for all* $\varphi\in\Delta(\mathcal{A})$ *or* $\Lambda_{\mathbf{T}}\supseteq\mathbb{T}$.

Proof. (i) Let $a\in\mathcal{A}$ and $z_0\in\mathbb{T}\cap\varrho_T(a)$. Then there exists a neighbourhood U_0 of z_0 and a holomorphic function $a(\cdot):U_0^{-1}\to\mathcal{A}$ such that

$$(T^{-1}-z)a(z)=a\quad\text{for all }z\in U_0^{-1}.$$

Let $\varphi\in\Delta(\mathcal{A})$. We want to compute the Carleman spectrum of $\varphi_a\in l^\infty$. If $z\in U_0^{-1}$ with $|z|<1$, then

$$\begin{aligned}\widetilde{\varphi}_a(z)&=\sum_{n=1}^\infty z^n\varphi(T^n(T^{-1}-z)a(z))\\&=\sum_{n=1}^\infty z^n\varphi(T^{n-1}a(z))-\sum_{n=1}^\infty z^{n+1}\varphi(T^na(z))=z\varphi(a(z)).\end{aligned}$$

Analogously, for $z\in U_0^{-1}$ with $|z|>1$ we have $\widetilde{\varphi}_a(z)=z\varphi(a(z))$. This implies that $\widetilde{\varphi}_a$ can be holomorphically extended to U_0^{-1} by defining $\widetilde{\varphi}_a(z):=z\varphi(a(z))$ for all $z\in U_0^{-1}$. It follows that $U_0^{-1}\cap\sigma(\varphi_a)=\emptyset$ and thus by Lemma 3.1, $U_0\cap\sigma_{w^*}(\varphi_a)=\emptyset$ for all $\varphi\in\Delta(\mathcal{A})$. This implies that $z_0\notin\Lambda_{\mathbf{T}}(a)$.

The proof of $\Lambda_\varphi^{\mathbf{T}}\subseteq\sigma_{T^*}(\varphi)\cap\mathbb{T}$ for all $\varphi\in\Delta(\mathcal{A})$ is similar.

(ii) follows from (i) combined with the observation that $\sigma_T(a)\subseteq\sigma(T)$ for all $a\in\mathcal{A}$.

(iii) Observe that $\overline{\Lambda_\varphi^{\mathbf{T}}}$ is a closed subgroup of $\mathbb{T}$ (see Proposition 2.4(ii)). Hence, by (i), $\overline{\Lambda_\varphi^{\mathbf{T}}}\subseteq\sigma_{T^*}(\varphi)$ is a proper closed subgroup of $\mathbb{T}$, and thus there exists $m\in\mathbb{N}$ such that $\Lambda_\varphi^{\mathbf{T}}=\{z\in\mathbb{T}:z^m=1\}$. This implies that

$$\sigma_{w^*}(\varphi_a)\subseteq\{z\in\mathbb{T}:z^m=1\}\quad\text{for all }a\in\mathcal{A}.$$

Fix $a\in\mathcal{A}$. It follows from Proposition 1.3(iii) that there exist $c_1,\ldots,c_m\in\mathbb{C}$

and $z_1, \ldots, z_m \in \mathbb{T}$ with $z_j^m = 1$ such that

$$\varphi_a(n) = \sum_{j=1}^{m} c_j z_j^n \quad \text{for all } n \in \mathbb{Z}.$$

This implies that

$$\varphi(T^m a - a) = \varphi_a(m) - \varphi_a(0) = \sum_{j=1}^{m} c_j z_j^m - \sum_{j=1}^{m} c_j = 0.$$

Therefore, $T^{*m}\varphi = \varphi$.

(iv) Assume that $\sigma(T) \not\supseteq \mathbb{T}$. Then

$$\sigma(T^*) \cap \mathbb{T} = \sigma(T) \cap \mathbb{T} \neq \mathbb{T}.$$

Fix $\varphi \in \Delta(\mathcal{A})$. Then by (i) we have

$$\Lambda_\varphi^{\mathbf{T}} \subseteq \sigma_{T^*}(\varphi) \cap \mathbb{T} \neq \mathbb{T}.$$

It follows from Proposition 2.4(ii) that $\overline{\Lambda_\varphi^{\mathbf{T}}}$ is a proper closed subgroup of $\mathbb{T}$. Therefore, $\Lambda_\varphi^{\mathbf{T}}$ is closed and there exists $n_\varphi \in \mathbb{N}$ such that

$$\Lambda_\varphi^{\mathbf{T}} = \{z \in \mathbb{T} : z^{n_\varphi} = 1\}.$$

Note that $\Lambda_{\mathbf{T}} \subseteq \sigma(T) \cap \mathbb{T}$ is a proper subset of $\mathbb{T}$ and contains each subgroup $\Lambda_\varphi^{\mathbf{T}}$. It follows that

$$k_T := \sup\{n_\varphi : \varphi \in \Delta(\mathcal{A})\} < \infty.$$

Thus, for $m := k_T!$ we have

$$\bigcup_{\varphi \in \Delta(\mathcal{A})} \Lambda_\varphi^{\mathbf{T}} \subseteq \{z \in \mathbb{T} : z^m = 1\}.$$

From the proof of (iii) we see that the above spectral condition implies that $T^{*m}\varphi = \varphi$ for all $\varphi \in \Delta(\mathcal{A})$. ■

COROLLARY 3.3. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra and T a continuous automorphism on $\mathcal{A}$. Let $\mathbf{T}$ be the group representation generated by T. Then:*

(i) *$\Lambda_{\mathbf{T}} = \sigma(T) \cap \mathbb{T}$. Hence, either $\sigma(T) \supseteq \mathbb{T}$ or $\sigma(T)$ is a finite union of finite subgroups of $\mathbb{T}$. Moreover, in the latter case there exists $m \in \mathbb{N}$ such that $T^m = I$ and bounded projections $P_1, \ldots, P_k$ on $\mathcal{A}$ and $z_1, \ldots, z_k \in \mathbb{T}$ such that*

$$T = \sum_{j=1}^{k} z_j P_j \quad \text{and} \quad \sigma(T) = \{z_j : 1 \le j \le k\}.$$

(ii) *If $T^n \neq I$ for all $n \in \mathbb{N}$, then the spectrum $\sigma(T)$ is connected.*

(iii) *Let $a \in \mathcal{A}$. If there exists $m \in \mathbb{N}$ such that $\sigma_T(a) \cap \mathbb{T} \subseteq \{z \in \mathbb{T} : z^m = 1\}$, then $T^m a = a$.*

Proof. (i) If $\Lambda_{\mathbf{T}} = \mathbb{T}$ then Theorem 3.2(ii) yields that $\sigma(T) \cap \mathbb{T} = \mathbb{T} = \Lambda_{\mathbf{T}}$. Consider the case $\Lambda_{\mathbf{T}} \neq \mathbb{T}$. Then by Theorem 3.2(iv) there exists $m \in \mathbb{N}$ such that $T^{*m}\varphi = \varphi$ for all $\varphi \in \Delta(\mathcal{A})$. It follows from the semisimplicity of $\mathcal{A}$ that $T^m = I$. Hence, $\sigma(T)$ is a finite subset of $\mathbb{T}$.

Write $\sigma(T) = \{z_1, \ldots, z_k\}$. We have $z_j^m = 1$ for all $1 \le j \le k$. For each j let P_j be the spectral projection corresponding to $\{z_j\}$. Then $P_1 + \ldots + P_k = I$. Fix j. Let $0 \neq a \in P_j\mathcal{A}$. We have

$$\sigma_T(a) \subseteq \sigma(T_{|P_j\mathcal{A}}) = \{z_j\}.$$

Theorem 3.2(i) shows that $\Lambda_{\mathbf{T}}(a) \subseteq \{z_j\}$. Hence, for each $\varphi \in \Delta(\mathcal{A})$ there exists $c_\varphi \in \mathbb{C}$ such that $\varphi(T^n a) = c_\varphi z_j^n$ for all $n \in \mathbb{Z}$. This implies that $\varphi(Ta - z_j a) = 0$ for all $\varphi \in \Delta(\mathcal{A})$. By the semisimplicity of $\mathcal{A}$ again, $Ta = z_j a$ and thus $z_j \in \Lambda_{\mathbf{T}}(a)$. This establishes the inclusion $\sigma(T) \subseteq \Lambda_{\mathbf{T}}$ and, in fact, the identity $\Lambda_{\mathbf{T}} = \sigma(T)$. Moreover, we have proved that

$$T = \sum_{j=1}^{k} TP_j = \sum_{j=1}^{k} z_j P_j.$$

(ii) The assumption that $T^n \neq I$ for all $n \in \mathbb{N}$ implies by (i) that $\sigma(T) \supseteq \mathbb{T}$. Hence, if $\sigma(T)$ were not connected, then at least one of the following sets would be non-empty and disjoint from the rest of $\sigma(T)$:

$$U_1 := \{|z| < 1\} \cap \sigma(T), \quad U_2 := \{|z| > 1\} \cap \sigma(T).$$

By taking T^{-1} in place of T if necessary, we may assume $U_1 \neq \emptyset$. Then $\sigma(T)$ is the disjoint union of U_1 and $\sigma(T) \setminus U_1$. Let P be the spectral projection corresponding to U_1. Take $0 \neq a \in P\mathcal{A}$. Then

$$\sigma_T(a) \subseteq \sigma(T_{|P\mathcal{A}}) = U_1.$$

It follows from Theorem 3.2(i) that

$$\Lambda_{\mathbf{T}}(a) \subseteq \sigma_T(a) \cap \mathbb{T} \subseteq U_1 \cap \mathbb{T} = \emptyset.$$

This implies by Proposition 1.3(ii) that $\varphi_a \equiv 0$ for all $\varphi \in \Delta(\mathcal{A})$ and thus $a = 0$ by the semisimplicity of $\mathcal{A}$, a contradiction.

(iii) Let $\Omega := \{z \in \mathbb{T} : z^m = 1\}$ and S be the restriction of T to the subalgebra $\mathcal{A}^{\mathbf{T}}(\Omega)$. Theorem 2.5 shows that $\Lambda_{\mathbf{S}} \subseteq \Omega$. Hence, from the above proof of (i) we see that S^m is the identity operator on $\mathcal{A}^{\mathbf{T}}(\Omega)$. Thus $T^m a = a$, since $a \in \mathcal{A}^{\mathbf{T}}(\Omega)$ by Theorem 3.2(i). ■

REMARK. The assertion (i) of Corollary 3.3 was proved by Kamowitz and Scheinberg [13] with complex analysis methods. Johnson [12] gave a different proof using distribution theory. A third proof for $\mathcal{A}$ being an abelian C^*-algebra was given by Akemann and Ostrand [1]. Our proof reveals the exact structure of those automorphisms T such that $\sigma(T) \cap \mathbb{T} \neq \mathbb{T}$.

The assertion (ii) of Corollary 3.3 has been established by Scheinberg [22]. Our proof is a little different. It seems that the third result in Corollary 3.3 is new. As seen in Corollary 3.3, the unitary spectrum of an automorphism of a commutative Banach algebra is very simple, but it should be pointed out that Scheinberg ([22] and [23]) has shown that the structure of an automorphism outside the unit circle might be rather complicated.

Below we consider the spectral structure of derivations. For this purpose, assume D to be a closed operator in a commutative Banach algebra $\mathcal{A}$ such that $-iD$ generates a strongly continuous one-parameter group of automorphisms $\mathbf{D}_t = e^{-itD}$ ($t \in \mathbb{R}$). Then the definition domain $\mathcal{D}(D)$ of D is a dense subalgebra of $\mathcal{A}$ and

$$D(ab) = (Da)b + a(Db) \quad \text{for all } a, b \in \mathcal{D}(D).$$

We call D a *derivation.* Recall that the *adjoint* of D, denoted by D^*, has the definition domain $\mathcal{D}(D^*)$ consisting of all $x^* \in \mathcal{A}^*$ such that there exists $y^* \in \mathcal{A}^*$ satisfying

$$\langle y^*, a\rangle = \langle x^*, Da\rangle \quad \text{for all } a \in \mathcal{D}(D).$$

It is known that $\mathcal{D}(D^*)$ coincides with the set of all $x^* \in \mathcal{A}^*$ such that

$$\sigma(\mathcal{A}^*, \mathcal{A})\text{-}\lim_{t\to 0} \frac{1}{t}(\mathbf{D}_t^* x^* - x^*)$$

exists, and D^*x^* is just this limit (see [18, p. 16]). The adjoint D^* is a closed operator. Their spectra are related by $\sigma(D^*) = \sigma(D)$. The following Theorem 3.4 and Corollary 3.5 are analogous to Theorem 3.2 and Corollary 3.3.

THEOREM 3.4. *Let $\mathcal{A}$ be a commutative Banach algebra. Assume that D is a derivation on $\mathcal{A}$ generating a strongly continuous one-parameter group of automorphisms $\mathbf{D}_t := e^{-itD}$ ($t \in \mathbb{R}$). Then:*

(i) $\Lambda_{\mathbf{D}}(a) \subseteq \sigma_D(a) \cap \mathbb{R}$ *and* $\Lambda_\varphi^{\mathbf{D}} \subseteq \sigma_{D^*}(\varphi) \cap \mathbb{R}$ *for all* $a \in \mathcal{A}$ *and* $\varphi \in \Delta(\mathcal{A})$.

(ii) $\Lambda_{\mathbf{D}} \subseteq \sigma(D) \cap \mathbb{R}$.

(iii) *Let* $\varphi \in \Delta(\mathcal{A})$ *be such that* $\sigma_{D^*}(\varphi) \cap \mathbb{R}$ *is bounded. Then* $\varphi \in \mathcal{D}(D^*)$ *and* $D^*\varphi = 0$.

(iv) *If* $\Delta(\mathcal{A}) \neq \emptyset$, *then* $\sigma(D) \cap \mathbb{R}$ *contains a closed subsemigroup of* $\mathbb{R}$.

Proof. (i) Let $\lambda_0 \in \mathbb{R} \setminus \sigma_D(a)$. Then there exists a neighbourhood U_0 of λ_0 and a holomorphic function $a(\cdot) : U_0 \to \mathcal{D}(D)$ such that $(\lambda - D)a(\lambda) = a$ for all $\lambda \in U_0$. Let $\varphi \in \Delta(\mathcal{A})$. To compute the Carleman transform of $\varphi_a \in C(\mathbb{R})$ take $\lambda \in U_0$ with $\operatorname{Im}\lambda > 0$. Then

$$\begin{aligned}\widetilde{\varphi}_a(\lambda) &= \int_0^\infty e^{i\lambda t}\varphi(\mathbf{D}_t a)\,dt = \int_0^\infty e^{i\lambda t}\varphi(\mathbf{D}_t(\lambda - D)a(\lambda))\,dt \\ &= \lim_{r\to\infty}\int_0^r e^{i\lambda t}\varphi(\mathbf{D}_t(\lambda - D)a(\lambda))\,dt \\ &= -i\lim_{r\to\infty}(e^{i\lambda r}\varphi(\mathbf{D}_r a(\lambda)) - a(\lambda)) = i\varphi(a(\lambda)),\end{aligned}$$

where we have used the following basic fact (see [18, p. 14]):

$$\int_0^r e^{i\lambda t - itD}(i\lambda - iD)b\,dt = e^{i\lambda r - irD}b - b \quad \text{for all } b \in \mathcal{D}(D).$$

Analogously, $\widetilde{\varphi}_a(\lambda) = i\varphi(a(\lambda))$ for all $\lambda \in U_0$ with $\operatorname{Im}\lambda < 0$. This implies that $\widetilde{\varphi}_a$ can be holomorphically extended to U_0 by defining $\widetilde{\varphi}_a(\lambda) := i\varphi(a(\lambda))$ for all $\lambda \in U_0$. Hence, $U_0 \cap \sigma_{w^*}(\varphi_a) = U_0 \cap \sigma(\varphi_a) = \emptyset$ by Lemma 3.1. Thus $\lambda_0 \notin \Lambda_{\mathbf{D}}(a)$. This proves the inclusion $\Lambda_{\mathbf{D}}(a) \subseteq \sigma_D(a)$. The proof of $\Lambda_\varphi^{\mathbf{D}} \subseteq \sigma_{D^*}(\varphi)$ for all $\varphi \in \Delta(\mathcal{A})$ is similar. We omit the details.

(ii) Note that $\sigma_D(a) \subseteq \sigma(D)$ for all $a \in \mathcal{A}$. It follows from (i) that $\Lambda_\alpha(a) \subseteq \sigma(D) \cap \mathbb{R}$ for all $a \in \mathcal{A}$ and thus $\Lambda_{\mathbf{D}}$, as the closure of the union of all $\Lambda_\alpha(a)$, is also contained in $\sigma(D) \cap \mathbb{R}$.

(iii) Since $\overline{\Lambda_\varphi^{\mathbf{D}}}$ is a closed subsemigroup of $\mathbb{R}$, the boundedness of $\Lambda_\varphi^{\mathbf{D}}$ implies that $\Lambda_\varphi^{\mathbf{D}} = \{0\}$. If $a \in \mathcal{A}$, then $\sigma_{w^*}(\varphi_a) \subseteq \Lambda_\varphi^{\mathbf{D}} = \{0\}$. Applying Proposition 1.3(iii) to φ_a we find that $\varphi(\mathbf{D}_t a) = \varphi(a)$ for all $t \in \mathbb{R}$. This implies that $\varphi \in \mathcal{D}(D^*)$ and $D^*\varphi = 0$.

(iv) follows from the fact that $\overline{\Lambda_\varphi^{\mathbf{D}}} \subseteq \sigma(D) \cap \mathbb{R}$ is a closed subsemigroup of $\mathbb{R}$. ∎

COROLLARY 3.5. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Assume that D is a derivation in $\mathcal{A}$ generating a strongly continuous one-parameter group of automorphisms $\mathbf{D}_t := e^{-itD}$ ($t \in \mathbb{R}$). Then:*

(i) *If* $\sigma(D) \cap \mathbb{R}$ *is bounded, then* $D = 0$.

(ii) *If* $\sigma(D)$ *decomposes into disjoint closed subsets* E *and* F *where* E *is compact, then* $E \cap \mathbb{R} \neq \emptyset$. *Hence, all isolated points of* $\sigma(D)$ *lie in* $\mathbb{R}$.

(iii) *If* $\lambda_0 \in \sigma(D) \cap \mathbb{R}$ *is an isolated point of* $\sigma(D)$, *then* λ_0 *is a simple pole of the resolvent.*

(iv) *Let* $a \in \mathcal{A}$. *If there exist finitely many positive numbers* $r_1, \ldots, r_n$ *such that*

$$\sigma_D(a) \cap \mathbb{R} \subseteq \bigcup_{k=1}^n r_k^{-1} 2\pi\mathbb{Z},$$

then

$$\prod_{k=1}^{n}(I-e^{-ir_kD})a=0.$$

Proof. (i) If $\sigma(D)\cap\mathbb{R}$ is bounded, then Theorem 3.4(ii) shows that $\Lambda_{\mathbf{D}}$ is bounded. Note that for each $\varphi\in\Delta(\mathcal{A})$ the spectral set $\overline{\Lambda_\varphi^{\mathbf{D}}}\subseteq\Lambda_{\mathbf{D}}$ is a subsemigroup of $\mathbb{R}$ (see Proposition 2.4(ii)). It follows that $\Lambda_\varphi^{\mathbf{D}}\subseteq\{0\}$ for all $\varphi\in\Delta(\mathcal{A})$. By Proposition 1.3(iii) again, φ_a is constant for each $\varphi\in\Delta(\mathcal{A})$ and $a\in\mathcal{A}$. Therefore, $\varphi(\mathbf{D}_ta-a)\equiv0$. By the semisimplicity of $\mathcal{A}$ we find that $\mathbf{D}_ta=a$ for all $t\in\mathbb{R}$ and $a\in\mathcal{A}$. This implies that $D=0$.

(ii) Let P be the spectral projection corresponding to the compact subset E. Then

$$\sigma(D_{|P\mathcal{A}})=E$$

(see [18, Cor. 3.5, p. 71]). Choose $0\neq a\in P\mathcal{A}$. Then $\sigma_D(a)\subseteq\sigma(D_{|P\mathcal{A}})=E$. Since $\mathcal{A}$ is semisimple and $a\neq0$, we can find $\varphi\in\Delta(\mathcal{A})$ such that $\varphi(a)\neq0$. Therefore, the continuous function φ_a does not vanish everywhere in $\mathbb{R}$ and thus $\sigma_{w^*}(\varphi_a)\neq\emptyset$ by Proposition 1.3(ii). This implies that

$$\emptyset\neq\sigma_{w^*}(\varphi_a)\subseteq\Lambda_{\mathbf{D}}(a)\subseteq\sigma_D(a)\cap\mathbb{R}=E\cap\mathbb{R}.$$

(iii) Let P be the spectral projection corresponding to $\{\lambda_0\}$. Then, as seen in the proof of (ii), we have

$$\sigma_{w^*}(\varphi_a)\subseteq\{\lambda_0\}\quad\text{for all } a\in P\mathcal{A},\ \varphi\in\Delta(\mathcal{A}).$$

It follows from Proposition 1.3(iii) that

$$\varphi(e^{it(\lambda_0-D)}a-a)=0\quad\text{for all } t\in\mathbb{R},\ a\in P\mathcal{A},\ \varphi\in\Delta(\mathcal{A}).$$

Again, by the semisimplicity of $\mathcal{A}$ we find that $e^{it(\lambda_0-D)}a=a$ for all $t\in\mathbb{R}$ and $a\in P\mathcal{A}$. This implies that $P\mathcal{A}\subseteq\mathcal{D}(D)$ and $Da=\lambda_0a$ for all $a\in P\mathcal{A}$. Hence, λ_0 is a simple pole of the resolvent.

(iv) Consider the spectral subspace $X:=\{h\in L^\infty(\mathbb{R}):\mathrm{Sp}(h)\subseteq\sigma_D(a)\cap\mathbb{R}\}$. Let Φ be the restriction of $\mathcal{L}$ to X. Then $\mathrm{Sp}(\Phi)\subseteq\sigma_D(a)\cap\mathbb{R}$. Theorem 1.2(i) shows that

$$\mathcal{L}_fh=\Phi_fh=0\quad\text{for all } h\in X,\ f\in j(\sigma_D(a)\cap\mathbb{R}).$$

Consider $\varphi\in\Delta(\mathcal{A})$. Then by Theorem 3.2(i) we have

$$\mathrm{Sp}(\varphi_a)=\sigma_{w^*}(\varphi_a)\subseteq\Lambda_{\mathbf{D}}(a)\subseteq\sigma_D(a)\cap\mathbb{R}.$$

Hence, $\varphi_a\in X$ and thus

$$\mathcal{L}_f\varphi_a=0\quad\text{for all } f\in j(\sigma_D(a)\cap\mathbb{R}).$$

By our assumption, $\sigma_D(a)\cap\mathbb{R}$ is a closed discrete subset of $\mathbb{R}=\widehat{\mathbb{R}}$. Hence, it is a spectral set by Theorem 1.1(iii). This implies that

$$(*)\qquad \mathcal{L}_f\varphi_a=0\quad\text{for all } f\in k(\sigma_D(a)\cap\mathbb{R}).$$

Let $g\in L^1(G)$. Consider

$$f:=g*\prod_{k=1}^{n}(\delta_0-\delta_{r_k}).$$

Then $f\in L^1(G)$ and for $s\in\mathbb{R}$ we have

$$\widehat{f}(s)=\widehat{g}(s)\cdot\prod_{k=1}^{n}(1-e^{-ir_ks}).$$

It follows that $\widehat{f}$ vanishes in $\sigma_D(a)\cap\mathbb{R}$ and thus by $(*)$ we have $0=\mathcal{L}_f\varphi_a=\mathcal{L}_gh$, where $h:=\prod_{k=1}^n(I-\mathcal{L}_{r_k})\varphi_a$ is a continuous function. The identity $\mathcal{L}_gh=0$ for all $g\in L^1(G)$ implies that h has empty w^*-spectrum. Hence, by Proposition 1.3(ii), h is the zero vector and thus $h\equiv0$ by the continuity of h. In particular, $h(0)=0$. By an easy computation we obtain

$$h(0)=\varphi\Big(\prod_{k=1}^{n}(I-\mathbf{D}_{r_k})a\Big).$$

The semisimplicity of $\mathcal{A}$ again yields $\prod_{k=1}^n(I-\mathbf{D}_{r_k})a=0$. This is the desired result. ■

In Theorems 3.2 and 3.4 we have established the inclusions $\Lambda_{\mathbf{T}}(a)\subseteq\sigma_T(a)$ or $\Lambda_{\mathbf{D}}(a)\subseteq\sigma_D(a)$ for the local spectrum and a similar result for $\varphi\in\Delta(\mathcal{A})$. There remains the problem of when equality holds. The following shows that this is the case under certain growth conditions.

PROPOSITION 3.6. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Then:*

(i) *Let $T\in\mathrm{Aut}(\mathcal{A})$ be such that*

$$\sum_{n=-\infty}^{\infty}\frac{\log(1+\|T^n\|)}{1+n^2}<\infty.$$

Let $\mathbf{T}$ be the group representation generated by T. Then $\Lambda_{\mathbf{T}}(a)=\sigma_T(a)$ and $\overline{\Lambda_\varphi^{\mathbf{T}}}=\sigma_{T^*}(\varphi)$ for all $a\in\mathcal{A}$ and $\varphi\in\Delta(\mathcal{A})$.

(ii) *Let D be a derivation in $\mathcal{A}$ generating a strongly continuous one-parameter group of automorphisms $\mathbf{D}_t:=e^{-itD}$ $(t\in\mathbb{R})$. If*

$$\int_{-\infty}^{\infty}\frac{\log(1+\|\mathbf{D}_t\|)}{1+t^2}\,dt<\infty,$$

then $\Lambda_{\mathbf{D}}(a)=\sigma_D(a)$ and $\overline{\Lambda_\varphi^{\mathbf{D}}}=\sigma_{D^*}(\varphi)$ for all $a\in\mathcal{A}$ and $\varphi\in\Delta(\mathcal{A})$.

Proof. (i) Consider

$$\omega_T(n):=1+\|T^n\|,\quad n\in\mathbb{Z}.$$

Our assumption implies that ω_T is a n.q.a. weight. $\mathbf{T}$ extends to a continuous algebra homomorphism Φ from $M_{\omega_T}(\mathbb{Z})$ into $\mathcal{L}(\mathcal{A})$. Theorem 2.6 yields that

$$\Lambda_{\mathbf{T}}(a) = \mathrm{Sp}_\Phi(a) \quad \text{and} \quad \overline{\Lambda_\varphi^{\mathbf{T}}} = \mathrm{Sp}_{\Phi^*}(\varphi)$$

for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$, and it has been proved in [11, Prop. 1.3.8] that $\mathrm{Sp}_\Phi(a) = \sigma_T(a)$ and $\mathrm{Sp}_{\Phi^*}(\varphi) = \sigma_{T^*}(\varphi)$.

(ii) Consider

$$\omega_D(t) := 1 + \|\mathbf{D}_t\| \quad \text{for all } t \in \mathbb{R}.$$

The strong continuity of $\mathbf{D}$ implies the lower semicontinuity of ω_D and thus ω_D is a weight on $\mathbb{R}$. The growth condition implies that ω_D is a n.q.a. weight. Hence, $\mathbf{D}$ extends to an algebra homomorphism Ψ from $M_{\omega_D}(\mathbb{R})$ into $\mathcal{L}(\mathcal{A})$. Theorem 2.6 again yields that

$$\Lambda_{\mathbf{D}}(a) = \mathrm{Sp}_\Psi(a) \quad \text{and} \quad \overline{\Lambda_\varphi^{\mathbf{D}}} = \mathrm{Sp}_{\Psi^*}(\varphi)$$

for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$, and by [11, Prop. 1.3.9], $\mathrm{Sp}_\Psi(a) = \sigma_D(a)$ and $\mathrm{Sp}_{\Psi^*}(\varphi) = \sigma_{D^*}(\varphi)$. ■

The growth conditions in Proposition 3.6 are automatically satisfied in the class of commutative Banach algebras whose spectral norm is equivalent to the original norm. Recall that the *spectral norm* of a commutative Banach algebra $\mathcal{A}$ is given by

$$\|a\|_{\mathrm{sp}} := \sup\{|\varphi(a)| : \varphi \in \Delta(\mathcal{A})\} \quad \text{for all } a \in \mathcal{A}.$$

The condition that the spectral norm is equivalent to the original norm on $\mathcal{A}$ is equivalent to the Gelfand representation $\widehat{\mathcal{A}}$ of $\mathcal{A}$ being a uniformly closed subalgebra of $C_0(\Delta(\mathcal{A}))$ or to the existence of a constant $K > 0$ such that $\|a\|^2 \le K\|a^2\|$ for all $a \in \mathcal{A}$ (see [17, Theorem 24C, p. 77]). Typical examples are the commutative C^*-algebras $C_0(\Omega)$ where Ω is a locally compact Hausdorff topological space, and their uniformly closed subalgebras. A Banach algebra with this property is semisimple.

Let $\mathcal{A}$ be a commutative Banach algebra such that $\|\cdot\|_{\mathrm{sp}} \approx \|\cdot\|$. Let S be a commutative semigroup and $\alpha : S \to \mathrm{Aut}(\mathcal{A})$ a semigroup homomorphism. Then $\sup\{\|\alpha_s\| : s \in S\} < \infty$. In fact, let $s \in S$. Since $\alpha_s^*\Delta(\mathcal{A}) = \Delta(\mathcal{A})$, we have

$$\|\alpha_s a\|_{\mathrm{sp}} = \sup\{|\alpha_s^*\varphi(a)| : \varphi \in \Delta(\mathcal{A})\} = \|a\|_{\mathrm{sp}}.$$

Hence, $\sup\{\|\alpha_s\|_{\mathrm{sp}} : s \in S\} = 1$ and the uniform boundedness of $\|\alpha_s\|$ follows from the equivalence of the spectral norm and the original norm. In particular,

$$\sup_{n\in\mathbb{Z}} \|T^n\| < \infty \quad \text{for all } T \in \mathrm{Aut}(\mathcal{A})$$

and any strongly continuous one-parameter group of automorphisms of $\mathcal{A}$ is uniformly bounded.

4. The spectral mapping theorem and applications. As before, we consider an a-dynamical system $\{\mathcal{A}, G, \alpha\}$, i.e., $\mathcal{A}$ is a commutative Banach algebra, G a LCA group and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ a $\sigma(\mathcal{A}, \Delta(\mathcal{A}))$-continuous group representation. Let H be another LCA group and $\phi : H \to G$ a continuous group homomorphism. Let $\beta := \alpha \circ \phi$, i.e.,

$$\beta_s := \alpha_{\phi s} \quad \text{for all } s \in H.$$

Then β is a $\sigma(\mathcal{A}, \Delta(\mathcal{A}))$-continuous group representation and thus we obtain an a-dynamical system $\{\mathcal{A}, H, \alpha \circ \phi\}$.

The aim of this section is to compute the unitary spectrum of β through the unitary spectrum of α.

We need some preliminaries. Note that ϕ induces a group homomorphism $\widehat{\phi} : \widehat{G} \to \widehat{H}$, where for each $\gamma \in \widehat{G}$ the image $\widehat{\phi}\gamma \in \widehat{H}$ is given by

$$(\widehat{\phi}\gamma)(s) := \gamma(\phi s) \quad \text{for all } s \in H. \tag{2}$$

ϕ also induces a continuous algebra homomorphism from $C_0(G)$ to $C_0(H)$ through

$$(\phi f)(s) := f(\phi s) \quad \text{for all } s \in H,\ f \in C_0(G).$$

Since the measure algebra $M(G)$ (resp. $M(H)$) is the dual of the C^*-algebra $C_0(G)$ (resp. $C_0(H)$), the algebra homomorphism $\phi : C_0(G) \to C_0(H)$ induces a continuous algebra homomorphism $\phi : M(H) \to M(G)$ through the following relation:

$$\langle \phi\mu, f\rangle = \langle \mu, \phi f\rangle \quad \text{for all } (f, \mu) \in C_0(G) \times M(H).$$

Fix $f \in C(G)$ and $\mu \in M(H)$. Let $\nu := \phi\mu$. Given $\varepsilon > 0$. Choose a compact subset $K \subseteq G$ such that

$$\int_{G\setminus K} |f(t)|\, d|\nu|(t) + \int_{H\setminus\phi^{-1}(K)} |f(\phi s)|\, d|\mu|(s) < \varepsilon.$$

By Urysohn's lemma [17, p. 6] we can find $g \in C_0(G)$ such that $0 \le g \le 1$ and $g(K) = 1$. This implies that $f \cdot g \in C_0(G)$ and thus

$$\int_G f(t)g(t)\, d\nu(t) = \int_H f(\phi s)g(\phi s)\, d\mu(s).$$

It follows that

$$\begin{aligned}
\Big|\int_G f(t)\, d\nu(t) - \int_H f(\phi s)\, d\mu(s)\Big|
&= \Big|\int_G f(t)(1 - g(t))\, d\nu(t) - \int_H f(\phi s)(1 - g(\phi s))\, d\mu(s)\Big| \\
&\le \int_{G\setminus K} |f(t)|\, d|\nu|(t) + \int_{H\setminus\phi^{-1}(K)} |f(\phi s)|\, d|\mu|(s) < \varepsilon.
\end{aligned}$$

Thus,

$$\int_G f(t)\,d\phi\mu(t) = \int_H f(\phi s)\,d\mu(s) \quad \text{for all } (f,\mu)\in C(G)\times M(H). \tag{3}$$

Let $\mu \in M(H)$ and $\gamma \in \widehat{G}$. From (2) and (3) we find

$$\widehat{\phi\mu}(\gamma) = \int_H \gamma(t)\,d\phi\mu(t) = \int_H \gamma(\phi s)\,d\mu(s) = \widehat{\mu}(\widehat{\phi}\gamma).$$

Therefore,

$$\widehat{\phi\mu} = \widehat{\mu}\circ\widehat{\phi} \quad \text{for all } \mu\in M(H). \tag{4}$$

An easy computation yields that

$$\phi\delta_s = \delta_{\phi s} \quad \text{for all } s\in H. \tag{5}$$

Now let $\Phi : M(G)\to\mathcal{L}(X)$ be a representation. The composition $\Phi\circ\phi$ yields a representation of $M(H)$ on X. The following "Spectral Mapping Theorem" in [11, Theorem 1.3.12] (cf. [19, Prop. 8.1.13]) gives an easy way to compute the spectrum of $\Phi\circ\phi$ through the spectrum of Φ.

THEOREM 4.1. *Let $\Phi : M(G)\to\mathcal{L}(X)$ be a non-degenerate representation and let $\phi : H\to G$ be a continuous group homomorphism. Then*

$$\mathrm{Sp}(\Phi\circ\phi) = \overline{\widehat{\phi}(\mathrm{Sp}(\Phi))},$$

where the closure is taken in $\widehat{H}$.

We write $\mathcal{L}^G$ (resp. $\mathcal{L}^H$) for the w^*-continuous left translation representation of G (resp. H) on $L^\infty(G)$ (resp. on $L^\infty(H)$); see Section 1. Let α,ϕ and β be as at the beginning of this section. Let $a\in\mathcal{A}$ and $\varphi\in\Delta(\mathcal{A})$. We want to prove that

$$\overline{\Lambda_\varphi^\beta} = \overline{\widehat{\phi}(\Lambda_\varphi^\alpha)} \quad\text{and}\quad \overline{\Lambda_\beta(a)} = \overline{\widehat{\phi}(\Lambda_\alpha(a))}. \tag{6}$$

Once (6) is established, we have the following "Spectral Mapping Theorem" for unitary spectra:

$$\Lambda_{\alpha\circ\phi} = \overline{\widehat{\phi}(\Lambda_\alpha)}.$$

To prove (6), fix $a\in\mathcal{A}$ and $\varphi\in\Delta(\mathcal{A})$. Consider the functions

$$g_{a,\varphi}(t) := \varphi(\alpha_t a),\quad t\in G;\qquad h_{a,\varphi}(s) := \varphi(\beta_s a),\quad s\in H.$$

Then $g_{a,\varphi}\in C(G)$ and $h_{a,\varphi}\in C(H)$. Let

$$\Lambda_1 := \sigma_{w^*}(g_{a,\varphi}) \quad\text{and}\quad \Lambda_2 := \overline{\widehat{\phi}(\Lambda_1)}.$$

We want to prove the inclusion

$$\sigma_{w^*}(h_{a,\varphi}) \subseteq \Lambda_2. \tag{7}$$

To this end, consider the spectral subspace $Y := \{h\in L^\infty(G) : \sigma_{w^*}(h)\subseteq\Lambda_1\}$. Clearly, $g_{a,\varphi}\in Y$. For the restriction Φ of $\mathcal{L}^G$ to Y we have $\mathrm{Sp}(\Phi)=\Lambda_1$. Applying Theorem 4.1 to Φ and ϕ we find that $\mathrm{Sp}(\Phi\circ\phi)=\Lambda_2$. Let $f\in j(\Lambda_2)$. Then $(\Phi\circ\phi)_f g_{a,\varphi}=0$ by Theorem 1.2(i). Hence, for $\mu:=\phi f\in M(G)$,

$$\mathcal{L}_\mu^G g_{a,\varphi} = \Phi_\mu g_{a,\varphi} = (\Phi\circ\phi)_f g_{a,\varphi} = 0.$$

Given $\varepsilon>0$, choose a compact subset $K\subseteq H$ such that $\int_{H\setminus K}|f(s)|\,ds<\varepsilon$. Since $\phi(K)$ is compact in G and the function $t\mapsto\varphi(\alpha_t a)$ is continuous, it is uniformly continuous on $\phi(K)$. Hence, there exists a neighbourhood U of 0 in G such that $|U|<\infty$ and

$$\sup_{t\in U}\sup_{s\in K}|\varphi(\alpha_t\alpha_{\phi s}a - \alpha_{\phi s}a)| < \varepsilon. \tag{8}$$

Set $F := |U|^{-1}\chi_U \in L^1(G)$. Then $\mathcal{L}_{F*\mu}^G g_{a,\varphi} = \mathcal{L}_F^G\mathcal{L}_\mu^G g_{a,\varphi} = 0$. Since $F*\mu\in L^1(G)$, the function $\mathcal{L}_{F*\mu}^G g_{a,\varphi}$ is uniformly continuous and thus $\mathcal{L}_F^G\mathcal{L}_\mu^G g_{a,\varphi}(0)=0$. It follows from (1) that

$$\begin{aligned}
0 = \mathcal{L}_F^G\mathcal{L}_\mu^G g_{a,\varphi}(0) &= \int_G F(t)\int_G g_{a,\varphi}(s+t)\,d\mu(s)\,dt \\
&= \int_G F(t)\int_H \varphi(\alpha_{t+\phi s}a)f(s)\,ds\,dt \qquad \text{(by (3))} \\
&= \int_H f(s)\frac{1}{|U|}\int_U \varphi(\alpha_{t+\phi s}a)\,dt\,ds \qquad \text{(by Fubini's Theorem)} \\
&= \int_H f(s)\varphi(\alpha_{\phi s}a)\,ds + I_1 + I_2,
\end{aligned}$$

where

$$\begin{aligned}
|I_1| &= \Big|\int_{H\setminus K} f(s)\frac{1}{|U|}\int_U \varphi(\alpha_{t+\phi s}a - \alpha_{\phi s}a)\,dt\,ds\Big| \\
&\le 2\|a\|\int_{H\setminus K}|f(s)|\,ds < 2\varepsilon\|a\|, \\
|I_2| &= \Big|\int_K f(s)\frac{1}{|U|}\int_U \varphi(\alpha_{t+\phi s}a - \alpha_{\phi s}a)\,dt\,ds\Big| \\
&\le \varepsilon\int_K |f(s)|\,ds \qquad \text{(by (8))}.
\end{aligned}$$

Here in the estimate of $|I_1|$ we have used in an obvious way the fact that $|\varphi(\alpha_t a)|\le\|a\|$ for all $t\in G$. It follows that

$$\Big|\int_H f(s)\varphi(\alpha_{\phi s}a)\,ds\Big| < 2\varepsilon\|a\| + \varepsilon\|f\|_1.$$

Thus, we have proved that for all $f \in j(\Lambda_2)$,

$$\int_H f(s) h_{a,\varphi}(s)\, ds = \int_H f(s)\varphi(\alpha_{\phi s} a)\, ds = 0.$$

By Proposition 1.3(i) this implies (7). From this we obtain part of (6):

$$\overline{\Lambda_\varphi^\beta} \subseteq \overline{\widehat{\phi}(\Lambda_\varphi^\alpha)} \quad \text{and} \quad \overline{\Lambda_\beta(a)} \subseteq \overline{\widehat{\phi}(\Lambda_\alpha(a))}.$$

To show the converse inclusions, we note that $h_{a,\alpha_t^*\varphi} = h_{\alpha_t a,\varphi}$ for all $t \in G$. Assume $f \in L^1(H)$ to be such that

$$\mathcal{L}_f^H h_{a,\alpha_t^*\varphi} = \mathcal{L}_f^H h_{\alpha_t a,\varphi} = 0 \quad \text{for all } t \in G. \tag{9}$$

Fix $t \in G$. Then by (1) we have

$$0 = \mathcal{L}_f^H h_{a,\alpha_t^*\varphi}(0) = \int_H f(s)\varphi(\alpha_{t+\phi s} a)\, ds.$$

Let $\mu := \phi f \in M(G)$. Then (3) gives

$$\int_G g_{a,\varphi}(s+t)\, d\mu(s) = \int_H f(s)\varphi(\alpha_{t+\phi s} a)\, ds = 0.$$

This implies that $\mathcal{L}_\mu^G g_{a,\varphi} = 0$. Let $F \in L^1(G)$. Then $F * \mu \in L^1(G)$ and $\Phi_{F*\mu} g_{a,\varphi} = \mathcal{L}_F^G \mathcal{L}_\mu^G g_{a,\varphi} = 0$. By the definition of $\sigma_{w^*}(g_{a,\varphi})$ we find that $F * \mu \in k(\sigma_{w^*}(g_{a,\varphi}))$ for all $F \in L^1(G)$. Therefore, $\widehat{\mu}$ vanishes on $\sigma_{w^*}(g_{a,\varphi})$. Let $\gamma \in \sigma_{w^*}(g_{a,\varphi})$. By (4) we have $\widehat{f}(\widehat{\phi}\gamma) = \widehat{\mu}(\gamma) = 0$. Thus,

$$f \in k(\widehat{\phi}(\sigma_{w^*}(g_{a,\varphi}))). \tag{10}$$

We are now in a position to finish the proof of (6). Assume $f \in L^1(H)$ to be such that $\mathcal{L}_f^H h_{a,\psi} = 0$ for all $\psi \in \Delta(\mathcal{A})$. Then f satisfies (9) for all $\psi \in \Delta(\mathcal{A})$ and thus by (10),

$$\widehat{f}^{-1}(0) \supseteq \Big(\bigcup_{\psi \in \Delta(\mathcal{A})} \widehat{\phi}(\sigma_{w^*}(g_{a,\psi}))\Big)^- = \overline{\widehat{\phi}(\Lambda_\alpha(a))}.$$

This implies by the definition of $\Lambda_\beta(a)$ that $\overline{\Lambda_\beta(a)} \supseteq \overline{\widehat{\phi}(\Lambda_\alpha(a))}$.

Analogously, let $f \in L^1(H)$ be such that $\mathcal{L}_f^H h_{b,\varphi} = 0$ for all $b \in \mathcal{A}$. Then f satisfies (9) for all $b \in \mathcal{A}$ and thus by (10),

$$\widehat{f}^{-1}(0) \supseteq \Big(\bigcup_{b \in \mathcal{A}} \widehat{\phi}(\sigma_{w^*}(g_{b,\varphi}))\Big)^- = \overline{\widehat{\phi}(\Lambda_\varphi^\alpha)}.$$

This implies that $\overline{\Lambda_\varphi^\beta} \supseteq \overline{\widehat{\phi}(\Lambda_\varphi^\alpha)}$. Therefore, we have established (6) as well as the following result.

THEOREM 4.2. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system. Let H be a LCA group and $\phi : H \to G$ a continuous group homomorphism. Then the a-dynamical system $\{\mathcal{A}, H, \alpha \circ \phi\}$ satisfies*

$$\Lambda_{\alpha\circ\phi} = \overline{\widehat{\phi}(\Lambda_\alpha)}.$$

Moreover, for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$,

$$\overline{\Lambda_{\alpha\circ\phi}(a)} = \overline{\widehat{\phi}(\Lambda_\alpha(a))} \quad \text{and} \quad \overline{\Lambda_\varphi^{\alpha\circ\phi}} = \overline{\widehat{\phi}(\Lambda_\varphi^\alpha)}.$$

Theorem 4.2 has several important applications. Corollary 4.3 below says that the unitary spectrum Λ_α is large enough to determine the unitary spectrum $\sigma(\alpha_t) \cap \mathbb{T}$ of each automorphism α_t. Theorem 4.4 gives an exact representation for those group representations whose unitary spectrum contains only finitely many elements. Theorem 4.5 and Corollary 4.6 concern continuity and growth of group representations whose unitary spectrum is compact. Corollary 4.7 describes the unitary spectrum of a representation in a more intrinsic way.

COROLLARY 4.3. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system with $\mathcal{A}$ semisimple. Then for each $t \in G$,*

$$\sigma(\alpha_t) \cap \mathbb{T} = \overline{\{\gamma(t) : \gamma \in \Lambda_\alpha\}}.$$

Moreover, if the operator α_t satisfies the growth condition

$$\sum_{n\in\mathbb{Z}} \frac{\log(1 + \|\alpha_{nt}\|)}{1+n^2} < \infty,$$

then for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$,

$$\sigma_{\alpha_t}(a) = \overline{\{\gamma(t) : \gamma \in \Lambda_\alpha(a)\}} \quad \text{and} \quad \sigma_{\alpha_t^*}(\varphi) = \overline{\{\gamma(t) : \gamma \in \Lambda_\varphi^\alpha\}}.$$

Proof. Fix $t \in G$ and consider the group homomorphism $\phi : \mathbb{Z} \to G$ given by $\phi n := nt$ for all $n \in \mathbb{Z}$. Let $\gamma \in \widehat{G}$. Then for all $n \in \mathbb{Z}$,

$$\widehat{\phi}\gamma(n) = \gamma(\phi n) = \gamma(nt) = \gamma(t)^n.$$

It follows that $\widehat{\phi}\gamma = \gamma(t)$. Let $T := \alpha_t$ and $\mathbf{T}$ be the group representation generated by T. We have

$$\mathbf{T}_n = T^n = \alpha_{nt} = \alpha_{\phi n} = (\alpha \circ \phi)_n.$$

Hence, $\mathbf{T} = \alpha \circ \phi$ and by Theorem 4.2,

$$\Lambda_{\mathbf{T}} = \overline{\widehat{\phi}(\Lambda_\alpha)} = \overline{\{\gamma(t) : \gamma \in \Lambda_\alpha\}}.$$

On the other hand, by Corollary 3.3 we have $\sigma(T) \cap \mathbb{T} = \Lambda_{\mathbf{T}}$. This is the first assertion.

Further, if the operator $T = \alpha_t$ satisfies the given growth condition then Proposition 3.6(i) yields that $\sigma_T(a) = \Lambda_{\mathbf{T}}(a)$ and $\sigma_{T^*}(\varphi) = \overline{\Lambda_\varphi^{\mathbf{T}}}$ for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. By Theorem 4.2 we see that $\Lambda_{\mathbf{T}}(a)$ (resp. $\overline{\Lambda_\varphi^{\mathbf{T}}}$) is the closure of $\{\gamma(t) : \gamma \in \Lambda_\alpha(a)\}$ (resp. $\{\gamma(t) : \gamma \in \Lambda_\varphi^\alpha\}$). This proves the "moreover" part. ∎

THEOREM 4.4. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system with $\mathcal{A}$ semisimple. If the unitary spectrum Λ_α contains only finitely many elements, $\Lambda_\alpha = \{\gamma_k : 1 \le k \le n\}$ say, then there exist bounded projections $P_1, \ldots, P_n$ on $\mathcal{A}$ such that*

$$\alpha_t = \sum_{k=1}^{n} \gamma_k(t) P_k \quad \textit{for all } t \in G.$$

Proof. Let $t \in G$. By Theorem 4.2 we have

$$\sigma(\alpha_t) \cap \mathbb{T} = \{\gamma_k(t) : 1 \le k \le n\}.$$

It follows from Corollary 3.3(i) that $\alpha_t^{n_t} = I$ for some $n_t \in \mathbb{N}$. This implies that

$$\sup_{n\in\mathbb{Z}} \|\alpha_{nt}\| < \infty \quad \text{for all } t \in G. \tag{11}$$

On the other hand, let G_{d} be G with discrete topology. The identity mapping $\mathrm{id} : G_{\mathrm{d}} \to G$ is a continuous group homomorphism. We have $\widehat{\mathrm{id}}\gamma = \gamma$ for all $\gamma \in \widehat{G}$. The composition $\beta := \alpha \circ \mathrm{id} : G_{\mathrm{d}} \to \mathrm{Aut}(\mathcal{A})$ is norm continuous and (11) implies that it is a n.q.a. group representation. Let

$$\omega(t) := 1 + \|\alpha_t\|, \quad t \in G,$$

and let $\Phi : M_\omega(G_{\mathrm{d}}) \to \mathcal{L}(\mathcal{A})$ be the algebra representation generated by β. By Theorem 2.6, $\mathrm{Sp}(\Phi) = \Lambda_\beta$. Applying Theorem 1.2(ii) to Φ we obtain bounded projections $P_1, \ldots, P_n$ on $\mathcal{A}$ such that $P_1 + \ldots + P_n = I$ and

$$\mathrm{Sp}(\Phi \circ P_k) = \{\gamma_k\} \quad \text{for } k = 1, \ldots, n.$$

Fix k. Consider $a \in P_k\mathcal{A}$. Then, by Theorem 2.6 again,

$$\Lambda_\beta(a) = \mathrm{Sp}_\Phi(a) \subseteq \mathrm{Sp}(\Phi \circ P_k) = \{\gamma_k\}. \tag{12}$$

Let $\varphi \in \Delta(\mathcal{A})$. Consider

$$h(t) := \varphi(\beta_t a) = \varphi(\alpha_t a), \quad t \in G_{\mathrm{d}}.$$

Then $h \in L^\infty(G_{\mathrm{d}})$ and by (12) the weak-star spectrum of h is contained in $\{\gamma_k\}$. Proposition 1.3(iii) yields $c_\varphi \in \mathbb{C}$ such that $h(t) = c_\varphi \gamma_k(t)$ for all $t \in G_{\mathrm{d}}$. Hence, $\varphi(\alpha_t a - \gamma_k(t)a) = 0$ for all $t \in G_{\mathrm{d}}$ and $\varphi \in \Delta(\mathcal{A})$. Thus $\alpha_t a = \gamma_k(t)a$ for all $t \in G$ by the semisimplicity of $\mathcal{A}$. Therefore, $\alpha_t P_k = \gamma_k(t) P_k$. This implies that

$$\alpha_t = \sum_{k=1}^{n} \gamma_k(t) P_k \quad \text{for all } t \in G. \ ■$$

THEOREM 4.5. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system with $\mathcal{A}$ semisimple. If the unitary spectrum Λ_α is compact, then the group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is norm continuous, i.e., $\lim_{t\to 0} \|\alpha_t - I\| = 0$. Furthermore, if Λ_α is compact and scattered, then for each $t \in G$ there exists $n_t \in \mathbb{N}$ such that $\alpha_t^{n_t} = I$. Hence, in this case α is a norm continuous n.q.a. group representation. Moreover, $\int_G f(t)\alpha_t\,dt = 0$ for all $f \in k(\Lambda_\alpha)$ such that $\int_G |f(t)|\,\|\alpha_t\|\,dt < \infty$.*

Proof. Let U be a neighbourhood of 0 in G such that $\overline{U}$ is compact. It suffices to prove

$$\lim_{t\in U,\, t\to 0} \|\alpha_t - I\| = 0.$$

To this end, let G_1 be the closed subgroup of G generated by U. By the Basic Structure Theorem [10, Theorem 9.8, p. 90] there exist $m, n \ge 0$ and a compact group K such that G_1 is isomorphic to $H := \mathbb{R}^m \times \mathbb{Z}^n \times K$. Let $\phi : H \to G_1$ be the corresponding group isomorphism, and $\beta := \alpha \circ \phi$. By Theorem 4.2 the unitary spectrum Λ_β is equal to the closure of $\widehat{\phi}(\Lambda_\alpha)$ in $\widehat{H}$. Thus, Λ_β is compact since λ_α is. Consider the group injections $\phi_1 : \mathbb{R}^m \to H$, $\phi_2 : \mathbb{Z}^n \to H$ and $\phi_3 : K \to H$. Then $\beta \circ \phi_2$ is already norm continuous, since $\mathbb{Z}^n$ is discrete. By Theorem 4.2,

$$\Lambda_{\beta\circ\phi_1} = \overline{\widehat{\phi_1}(\Lambda_\beta)} \quad \text{and} \quad \Lambda_{\beta\circ\phi_3} = \overline{\widehat{\phi_3}(\Lambda_\beta)}.$$

Since Λ_β is compact, so are both $\Lambda_{\beta\circ\phi_1}$ and $\Lambda_{\beta\circ\phi_3}$. Note that $\Lambda_{\beta\circ\phi_1}$ is the closure of the union of closed subsemigroups of $\widehat{\mathbb{R}^m} = \mathbb{R}^m$. This implies that $\Lambda_{\beta\circ\phi_1} = \{0\}$ and thus $(\beta \circ \phi_1)_t = I$ for all $t \in \mathbb{R}^m$ by Theorem 4.4. Thus, $\beta \circ \phi_1$ is norm continuous. Note that $\widehat{K}$ is discrete. It follows that the compact and discrete set $\Lambda_{\beta\circ\phi_3}$ contains at most finitely many elements. Hence, by Theorem 4.4 again, $\beta \circ \phi_3$ is also norm continuous. In conclusion, β itself and the subrepresentation $\alpha : G_1 \to \mathrm{Aut}(\mathcal{A})$ are norm continuous.

Assume further that Λ_α is scattered. Let $t \in G$. Then, by a result of [15, p. 30], the compact set $\{\gamma(t) : \gamma \in \Lambda_\alpha\}$ is countable. Hence, Corollary 4.3 shows that $\sigma(\alpha_t) \cap \mathbb{T} = \{\gamma(t) : \gamma \in \Lambda_\alpha\}$ is countable. It follows from Corollary 3.3(i) that $\alpha_t^{n_t} = I$ for some $n_t \in \mathbb{N}$. This implies that $\Omega(t) := 1 + \|\alpha_t\|$ ($t \in G$) is a continuous n.q.a. weight on G. Let $f \in L^1_\omega(G)$ with $\widehat{f} \equiv 0$ on Λ_α. Let $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$. Since the scattered set Λ_α is a spectral set (see Theorem 1.1(iii)), by Theorem 1.2(i) we have $\mathcal{L}_f\varphi_a = 0$. Note that by (1),

$$0 = \mathcal{L}_f\varphi_a(0) = \int_G f(t)\varphi(\alpha_t a)\,dt = \Big\langle \varphi, \Big(\int_G f(t)\alpha_t\,dt\Big)a \Big\rangle,$$

and the semisimplicity of $\mathcal{A}$ yields $\int_G f(t)\alpha_t\,dt = 0$. ■

In the proof of Theorem 4.5 we have seen that group representations of $\mathbb{R}^m$ on a semisimple Banach algebra whose unitary spectrum is compact are trivial. More generally, the following result holds. Recall from [10, Cor.

24.19, p. 383] that a LCA group G is connected if and only if its dual group $\widehat{G}$ does not contain any proper closed subgroup.

COROLLARY 4.6. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system with $\mathcal{A}$ semi-simple and G connected. If Λ_α is compact, then $\alpha_t = I$ for all $t \in G$.*

Proof. It suffices to prove that $\Lambda_\alpha = \{0\}$. Once this is established, the result follows from Theorem 4.4. If there were $0 \neq \gamma \in \Lambda_\alpha$, then Proposition 2.4(ii) implies that $H := \overline{\gamma \cdot \mathbb{N}}$ is a compact subgroup of $\widehat{G}$. Clearly, $H \neq \widehat{G}$. This contradicts the above cited criterion of connectedness of G. ∎

The following gives an alternative description of unitary spectrum and local unitary spectrum.

COROLLARY 4.7. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system with $\mathcal{A}$ semi-simple. Let $\ker\alpha := \{t \in G : \alpha_t = I\}$. Then $\Lambda_\alpha^\perp = \ker\alpha$, where $\Lambda_\alpha^\perp$ denotes the annihilator of Λ_α in $\widehat{G}$. Moreover, for all $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$,*

$$\Lambda_\alpha(a)^\perp = \{t \in G : \alpha_t a = a\} \quad \text{and} \quad (\Lambda_\varphi^\alpha)^\perp = \{t \in G : \alpha_t^* \varphi = \varphi\}.$$

Proof. First we establish that

$$\Lambda_\alpha(a)^\perp = \{t \in G : \alpha_t a = a\} \quad \text{for all } a \in \mathcal{A}.$$

To this end, fix $a \in \mathcal{A}$ and $t \in G$. Let $\phi : \mathbb{Z} \to G$ be the group homomorphism defined by $\phi n := nt$ ($n \in \mathbb{Z}$). Let $T := \alpha_t$ and $\mathbf{T}$ be the representation generated by T. We have $\mathbf{T} = \alpha \circ \phi$. It follows from Theorem 4.2 that

$$\Lambda_{\mathbf{T}}(a) = \overline{\widehat{\phi}(\Lambda_\alpha(a))}. \tag{13}$$

If $t \in \Lambda_\alpha^\perp$, then $\Lambda_{\mathbf{T}}(a) = \{1\}$. This implies that a belongs to the $\mathbf{T}$-invariant closed subalgebra $\mathcal{B} := \mathcal{A}^{\mathbf{T}}(\{1\})$. Hence, $\Lambda_{\mathbf{T}|\mathcal{B}} = \{1\}$. From Corollary 3.3(ii) we see that $Ta = a$, i.e., $\alpha_t a = a$.

Conversely, assume $\alpha_t a = a$. Then $\Lambda_{\mathbf{T}}(a) = \{1\}$. Hence, by (13) we have $\gamma(t) = 1$ for all $\gamma \in \Lambda_\alpha(a)$, thus $t \in \Lambda_\alpha(a)^\perp$. From this we immediately derive that $\Lambda_\alpha^\perp = \ker\alpha$.

The result for φ is proved similarly. ∎

It is a general problem as to when a commutative class of operators on a Banach space has a non-trivial invariant closed subspace. In the following we tackle the corresponding problem of when a group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ has a non-trivial unitary spectral subspace.

THEOREM 4.8. *Let $\{\mathcal{A}, G, \alpha\}$ be an a-dynamical system with $\mathcal{A}$ semi-simple. If there exists $t \in G$ such that the automorphism α_t has a non-trivial unitary spectral subspace, then so does α itself.*

Proof. Let $T := \alpha_t$. Consider the group homomorphism $\phi : \mathbb{Z} \to G$ given by $\phi n := nt$ for all $n \in \mathbb{Z}$. Then $\mathbf{T} = \alpha \circ \phi$ is the representation generated by T. The assumption implies that there exists $0 \neq a \in \mathcal{A}$ such that $\Lambda_{\mathbf{T}}(a) \neq \Lambda_{\mathbf{T}}$. We need to show that $\Lambda_\alpha(a) \neq \Lambda_\alpha$. If $\Lambda_\alpha(a)$ were Λ_α, then by Theorem 4.2 we would have

$$\Lambda_{\mathbf{T}}(a) = \overline{\{\gamma(t) : \gamma \in \Lambda_\alpha(a)\}} = \overline{\{\gamma(t) : \gamma \in \Lambda_\alpha\}} = \Lambda_{\mathbf{T}}.$$

This is a contradiction. ∎

COROLLARY 4.9. *Let $\{\mathcal{A}, G, \alpha\}$ be a non-trivial a-dynamical system with $\mathcal{A}$ semisimple. Assume that there exist $t \in G$ and $0 \neq a \in \mathcal{A}$ such that $\alpha_t \neq I$ and*

$$\sum_{n\in\mathbb{Z}} \frac{\log(1 + \|\alpha_{nt} a\|)}{1 + n^2} < \infty.$$

Then α has a non-trivial unitary spectral subspace.

For the proof we need an auxiliary result of Atzmon [3, Lemma 2.1].

LEMMA 4.10. *Let $(\omega(n))_{n\in\mathbb{Z}}$ be a sequence of positive numbers such that there exists $K > 0$ satisfying*

$$1 \le \omega(n) \quad \text{and} \quad K^{-1}\omega(n) \le \omega(n+1) \le K\omega(n) \quad \text{for all } n \in \mathbb{Z}.$$

Assume further that

$$\sum_{n\in\mathbb{Z}} \frac{\log \omega(n)}{1 + n^2} < \infty.$$

Then for every $0 \le \theta_0 < \theta_1 < 2\pi$ there exists a sequence $0 \neq f = (f_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ such that

$$\sum_{n\in\mathbb{Z}} |f_n|\omega(n) < \infty$$

and the support of the function

$$\widehat{f}(z) := \sum_{n\in\mathbb{Z}} f_n z^n, \qquad z \in \mathbb{T},$$

is contained in the arc $\{z \in \mathbb{T} : \theta_0 < \arg z < \theta_1\}$.

Proof of Corollary 4.9. If $\Lambda_\alpha(a) \neq \Lambda_\alpha$ then we are done. Assume $\Lambda_\alpha(a) = \Lambda_\alpha$. As usual, let $T := \alpha_t$ and $\mathbf{T}$ be the representation generated by T. By Theorem 4.2,

$$\Lambda_{\mathbf{T}}(a) = \overline{\{\gamma(t_0) : \gamma \in \Lambda_\alpha(a)\}} = \overline{\{\gamma(t_0) : \gamma \in \Lambda_\alpha\}} = \Lambda_{\mathbf{T}}. \tag{14}$$

Since $T \neq I$, $\Lambda_{\mathbf{T}}(a) = \Lambda_{\mathbf{T}}$ contains more than one point by Corollary 3.3(i). By Theorem 4.8 we have to show that T has non-trivial unitary spectral subspaces.

Consider

$$\omega(n) := 1 + \|T^n a\| \quad \text{for } n \in \mathbb{Z}.$$

Our growth assumption implies that $(\omega(n))_{n\in\mathbb{Z}}$ satisfies the condition of Lemma 4.10. Therefore, there exists $0 \neq f = (f_n)_{n\in\mathbb{Z}} \in l^1$ such that

$$\sum_{n\in\mathbb{Z}} |f_n|(1 + \|T^n a\|) < \infty$$

and the Fourier transform $\widehat{f}(z) := \sum_{n\in\mathbb{Z}} f_n z^n$, $z \in \mathbb{T}$, satisfies

$$(15) \qquad \operatorname{supp} \widehat{f} \cap \Lambda_{\mathbf{T}}(a) \neq \emptyset \quad \text{but} \quad \operatorname{supp} \widehat{f} \cap \Lambda_{\mathbf{T}}(a) \neq \Lambda_{\mathbf{T}}(a).$$

The absolutely convergent series $\sum_{n\in\mathbb{Z}} f_n T^n a$ defines an element in $\mathcal{A}$, denoted by b. For $\varphi \in \Delta(\mathcal{A})$, let

$$g_\varphi(n) := \varphi(T^n a) \quad \text{and} \quad h_\varphi(n) := \varphi(T^n b) \quad \text{for } n \in \mathbb{Z}.$$

Then for all $g = (g_n)_{n\in\mathbb{N}} \in l^1$ and $k \in \mathbb{Z}$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} g_n h_\varphi(n+k) &= \sum_{n\in\mathbb{Z}} g_n \sum_{m\in\mathbb{Z}} f_m \varphi(T^{m+n+k}a) \\ &= \sum_{n\in\mathbb{Z}} g_n \sum_{m\in\mathbb{Z}} f_{m-n} \varphi(T^{m+k}a) = \sum_{m\in\mathbb{Z}} (g * f)_m \varphi(T^{m+k}a).\end{aligned}$$

This implies that $\mathcal{L}_g h_\varphi = \mathcal{L}_{g*f} g_\varphi$, and hence $\sigma_{w^*}(h_\varphi) \subseteq \operatorname{supp} \widehat{f} \cap \Lambda_{\mathbf{T}}(a)$. Therefore,

$$(16) \qquad \Lambda_{\mathbf{T}}(b) = \Big(\bigcup_{\varphi\in\Delta(\mathcal{A})} \sigma_{w^*}(h_\varphi)\Big)^{-} \subseteq \operatorname{supp} \widehat{f} \cap \Lambda_{\mathbf{T}}(a).$$

If b were zero, then $h_\varphi = 0$ and thus $\mathcal{L}_{g*f} g_\varphi = 0$ for all $\varphi \in \Delta(\mathcal{A})$ and $g \in l^1$. This implies that $\Lambda_{\mathbf{T}}(a) \subseteq \operatorname{supp} \widehat{f}$, contradicting (15). Hence, $b \neq 0$ and by (16), $\Lambda_{\mathbf{T}}(b)$ is a proper subset of $\Lambda_{\mathbf{T}}(a)$. Hence, $\Lambda_\alpha(b) \neq \Lambda_\alpha$ by (14). It follows from Theorem 4.8 that T and α have non-trivial unitary spectral subspaces. ■

A related question is the existence of a non-trivial unitary spectral subspace which is itself a Banach algebra. We call an a-dynamical system $\{\mathcal{A}, G, \alpha\}$ *ergodic* if there is no α-invariant closed subalgebra besides $\{0\}$ and $\mathcal{A}$. The following example justifies the name "ergodic" for this property.

EXAMPLE. Let X be a locally compact Hausdorff space and G be a LCA group. Assume that G acts continuously on X, i.e., there is a group $\{\tau_t : t \in G\}$ of continuous homeomorphisms of X such that $\tau_{s+t} = \tau_s\tau_t$ for all $s, t \in G$ and for each $x \in X$ the mapping $t \mapsto \tau_t x$ is continuous. On the C^*-algebra $\mathcal{A} := C_0(X)$ we define

$$\alpha_t f(x) := f(\tau_t x), \qquad f \in C_0(X),\ x \in X,\ t \in G.$$

Then α is a strongly continuous group representation of G by isometries. It follows from Corollary 4.9 that either α is trivial or α has a non-trivial unitary spectral subspace.

In ergodic theory the action τ is called *ergodic* if there exists no τ-invariant closed subset of X besides the empty set and X (see [25]). The following gives an alternative description of ergodicity:

$$\alpha \text{ is ergodic} \Leftrightarrow \tau \text{ is ergodic}.$$

In fact, if Ω is a τ-invariant, non-empty and proper subset of X, then $\{f \in C_0(X) : f \equiv 0 \text{ on } \Omega\}$ is a non-trivial α-invariant unitary spectral subspace which is also a subalgebra of $C_0(X)$. Hence, α is not ergodic. Conversely, if $\mathcal{B}$ is an α-invariant closed proper subalgebra of $\mathcal{A}$, then $\{x \in X : f(x) = 0\ \forall f \in \mathcal{B}\}$ is a non-empty, closed and τ-invariant proper subset of X. Thus τ is not ergodic. This establishes the equivalence of ergodicity of α and τ.

Assume α to be ergodic. Then, by Theorem 2.5, $\overline{\Lambda_\varphi^\alpha} = \Lambda_\alpha$ for each $\varphi \in \Delta(C_0(X)) = X$. But Proposition 2.4(ii) says that each $\overline{\Lambda_\varphi^\alpha}$ is a closed subsemigroup of $\widehat{G}$. It follows that Λ_α is a closed subsemigroup of $\widehat{G}$. On the other hand, since $C_0(X)$ is self-adjoint, Proposition 2.4(iii) shows that Λ_α is a symmetric subset of $\widehat{G}$. Combining these two facts we see that Λ_α is a closed subgroup of $\widehat{G}$.

In general, the following assertions hold true.

PROPOSITION 4.11. *Let $\{\mathcal{A}, G, \alpha\}$ be an ergodic a-dynamical system with $\mathcal{A}$ semisimple. Then:*

(i) *$\overline{\Lambda_\varphi^\alpha} = \Lambda_\alpha$ for all $\varphi \in \Delta(\mathcal{A})$ and hence Λ_α is a closed subsemigroup of $\widehat{G}$.*

(ii) *If either $\mathcal{A}$ is self-adjoint or Λ_α is compact, then Λ_α is a closed subgroup of $\widehat{G}$ isomorphic to the dual of $G/\ker\alpha$.*

(iii) *If $G = \mathbb{Z}$ then $\Lambda_\alpha = \mathbb{T}$ unless $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

(iv) *If $\mathcal{A}$ is self-adjoint and $G = \mathbb{R}$, then $\Lambda_\alpha = \mathbb{R}$ unless $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Proof. (i) follows from Proposition 2.5 which implies that each unitary spectral subspace $\mathcal{A}^\alpha(\overline{\Lambda_\varphi^\alpha})$ is a closed subalgebra of $\mathcal{A}$.

(ii) Λ_α is a group by Proposition 2.4 combined with (i). In Corollary 4.7 we have proved that $\Lambda_\alpha^\perp = \ker\alpha$. It follows from [21, Theorem 2.1.2, p. 35] that Λ_α is isomorphic to the dual of $G/\ker\alpha$.

(iii) It suffices to prove that if $\Lambda_\alpha \neq \mathbb{T}$ then $\alpha_1 = I$ and hence $\mathcal{A} \cong \mathbb{C}$. In fact, if $\Lambda_\alpha \neq \mathbb{T}$ then by Corollary 3.3(i) we see that $\sigma(\alpha_1) = \Lambda_\alpha$ and $1 \in \sigma(\alpha_1)$ is an eigenvalue of α_1. Note that $\mathcal{B} := \{a \in \mathcal{A} : \alpha_1 a = a\}$ is a non-zero, α-invariant closed subalgebra of $\mathcal{A}$. The ergodicity of α yields $\mathcal{B} = \mathcal{A}$ and hence $\alpha_1 = I$. By the ergodicity again, $\mathcal{A} \cong \mathbb{C}$.

(iv) Since $\mathcal{A}$ is self-adjoint, Λ_α is a closed subgroup of $\mathbb{R}$ by (ii). Note that Λ_α is one of the following sets:

$$\Lambda_\alpha = \{0\}; \quad \Lambda_\alpha = 2\pi\lambda^{-1}\cdot\mathbb{Z} \text{ for some } \lambda > 0; \quad \Lambda_\alpha = \mathbb{R}.$$

If $\Lambda_\alpha = \{0\}$ then by Corollary 4.6 we have $\alpha_t = I$ for all $t \in \mathbb{R}$ and thus $\mathcal{A} \cong \mathbb{C}$ by the ergodicity of α. If $\Lambda_\alpha = 2\pi\lambda^{-1}\cdot\mathbb{Z}$ for some $\lambda > 0$, then $\alpha_\lambda = I$ by Corollary 4.7. Let $r := \lambda/2$. Consider

$$\mathcal{B}_\pm := \{a \in \mathcal{A} : \alpha_r a = \pm a\}.$$

Since $\mathcal{B}_+$ is an α-invariant closed subalgebra of $\mathcal{A}$, by the ergodicity of α we have either $\mathcal{B}_+ = \{0\}$ or $\mathcal{B}_+ = \mathcal{A}$. If the first case occurs, then $\mathcal{B}_- = \mathcal{A}$ and thus for all $a \in \mathcal{A}$,

$$-a^2 = \alpha_r(a^2) = (\alpha_r a)\cdot(\alpha_r a) = a^2.$$

Hence, $a^2 = 0$ and thus $a = 0$ by the semisimplicity. This is impossible. In conclusion, we have $\Lambda_\alpha = \mathbb{R}$ unless $\mathcal{A} \cong \mathbb{C}$. ■

REMARK. Related results about the spectra of ergodic actions can be found in [24].

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