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Banach spaces with a supershrinking basis

by

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Abstract. We prove that a Banach space X with a supershrinking basis (a special type of shrinking basis) without c_0 copies is somewhat reflexive (every infinite-dimensional subspace contains an infinite-dimensional reflexive subspace). Furthermore, applying the c_0-theorem by Rosenthal, it is proved that X contains order-one quasireflexive subspaces if X is not reflexive. Also, we obtain a characterization of the usual basis in c_0.

1. Introduction. We use standard Banach space facts and terminology. Let us just recall that a basis $\{e_n\}$ in a Banach space X is called *shrinking* iff $X^* = \overline{\text{lin}}\{f_n : n \in \mathbb{N}\}$, where $\{f_n\}$ is the sequence of biorthogonal functionals.

With a separation argument, it is easy to see that $\{e_n\}$ is shrinking iff $\{x^{**} \in X^{**} : x^{**}(f_n) = 0 \ \forall n \in \mathbb{N}\} \subset X$.

We will say that $\{e_n\}$ is *supershrinking* iff

$$(N_X =) \ \{x^{**} \in X^{**} : \lim_n x^{**}(f_n) = 0\} \subset X.$$

In fact, if the basis $\{e_n\}$ is seminormalized, one can replace the above inclusion by equality. Also, it is possible give a definition only in terms of the basis: a shrinking basis $\{e_n\}$ is supershrinking provided whenever scalars $\{c_n\}$ satisfy $\lim_n c_n = 0$ and $\sup_n \|\sum_{k=1}^n c_k e_k\|$ is finite, then $\sum_n c_n e_n$ converges in the norm topology.

In [4], it was proved that the Radon–Nikodym and Krein–Milman properties are equivalent in Banach spaces with a supershrinking basis. Also, the supershrinking bases are used in [6] to prove that every Banach space with the point of continuity property and separable dual is somewhat order-one quasireflexive (every non-reflexive subspace contains an order-one quasireflexive subspace). A Banach space is *order-one quasireflexive* if it has codimension one in its bidual.

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Examples of Banach spaces with a supershrinking basis are the reflexive spaces with basis, c_0, J (the James space) and the natural predual of JT (the James tree space) (see [4]). In a similar way, it can be seen that the natural predual of JT_∞ has a supershrinking basis (see Th. IV.2 in [2]).

The aim of this note is to study the structure of Banach spaces with a supershrinking basis. Our main results are the following:

THEOREM A. *Let X be a Banach space with a normalized and shrinking basis $\{e_n\}$, and associated functionals $\{f_n\}$, without infinite-dimensional reflexive subspaces. Then the following are equivalent:*

(i) *$\{e_n\}$ is supershrinking.*

(ii) *N_X is separable.*

(iii) *A w-closed and bounded subset K of X is w-compact whenever $\lim_n f_n(k) = 0$ uniformly in $k \in K$.*

(iv) *A closed and bounded subset K of X is compact whenever $\lim_n f_n(k) = 0$ uniformly in $k \in K$.*

(v) *$\{e_n\}$ is equivalent to the usual basis of c_0.*

THEOREM B. *Let X be a non-reflexive Banach space with a normalized supershrinking basis $\{e_n\}$ and associated functionals $\{f_n\}$, without subspaces isomorphic to c_0. Then X contains order-one quasireflexive subspaces. In fact, every non-reflexive subspace contains an order-one quasireflexive subspace.*

Main results. We begin by recalling some concepts and known results.

A basic sequence $\{x_n\}$ in a Banach space X is said to be *strongly summing* (s.s.) if $\{x_n\}$ is weakly Cauchy and

$$\sum_n c_n \text{ converges} \quad \text{whenever} \quad \sup_n \Big\|\sum_{k=1}^n c_k x_k\Big\| < \infty.$$

The *usual basis* of c_0 is the basis of unit vectors denoted by $\{b_n\}$, and the *summing basis* of c_0 is $\{\sum_{i=1}^n b_i\}_n$.

THEOREM ([5], c_0-Rosenthal). *Let X be a Banach space and $\{x_n\}$ a weakly Cauchy and not weakly convergent sequence in X. Then $\{x_n\}$ has either a strongly summing basic subsequence or a convex basic block equivalent to the summing basis of c_0.*

THEOREM ([1], Elton). *Let $\{x_n\}$ be a normalized and weakly null sequence in a Banach space, without subsequences equivalent to the usual basis of c_0. Then $\{x_n\}$ has a basic subsequence $\{x'_n\}$ such that $\lim_n \|\sum_{k=1}^n c_i y_i\| = \infty$ for every subsequence $\{y_n\}$ of $\{x'_n\}$ and for every $\{c_n\} \notin c_0$.*

Our first result characterizes the equality $N_X = X$.

THEOREM 1. *Let X be a Banach space with a normalized and shrinking basis $\{e_n\}$ and associated functionals $\{f_n\}$. Then the following are equivalent:*

(i) *$N_X \neq X$.*

(ii) *There is a non-reflexive subspace Y of X with $\lim_n \|f_{n|Y}\|_{Y^*} = 0$.*

(iii) *There is a bounded and non-relatively weakly compact subset K of X such that $\lim_n \sup_{k\in K} f_n(k) = 0$.*

Proof. (i)$\Rightarrow$(ii). Let $x_0^{**} \in N_X \setminus X$. Then the sequence $\{z_n\} = \{\sum_{k=1}^n x_0^{**}(f_k)e_k\}_n$ is weakly Cauchy and not weakly convergent. In fact, it converges to x_0^{**} in X^{**}, for the weak-* topology.

By applying the c_0-Rosenthal Theorem, either there is an increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\{z_{\sigma(n)}\}$ is a basic strongly summing sequence, or $\{z_n\}$ has a convex basic block equivalent to the summing basis. In the first case we set $y_n = \sum_{k=1}^{\sigma(n)} x_0^{**}(f_k)e_k$ for $n \in \mathbb{N}$.

Assume, without loss of generality, that $x_0^{**}(f_n) \neq 0$ for every $n \in \mathbb{N}$.

We put $Y = \overline{\text{lin}}\{y_n : n \in \mathbb{N}\}$ and let $\{g_n\}$ be the sequence of associated functionals of $\{y_n\}$. Then

$$g_n = \frac{f_{\sigma(n)|Y}}{x_0^{**}(f_{\sigma(n)})} - \frac{f_{\sigma(n+1)|Y}}{x_0^{**}(f_{\sigma(n+1)})} \quad \forall n \in \mathbb{N}.$$

Furthermore, $x_0^{**} \in Y^{**}$. So, Y is not reflexive.

If $y^{**} \in Y^{**}$, then $\sup_n \|\sum_{k=1}^n y^{**}(g_k)y_k\| < \infty$ and so the series $\sum_n y^{**}(g_n)$ converges since $\{y_n\}$ is strongly summing. But

$$\sum_{n=1}^N y^{**}(g_n) = \frac{y^{**}(f_{\sigma(1)|Y})}{x_0^{**}(f_{\sigma(1)})} - \frac{y^{**}(f_{\sigma(N+1)|Y})}{x_0^{**}(f_{\sigma(N+1)})} \quad \forall N \in \mathbb{N}.$$

Hence, the following limit exists for every $y^{**} \in Y^{**}$:

$$\lim_{n\to\infty} \frac{y^{**}(f_{\sigma(n)|Y})}{x_0^{**}(f_{\sigma(n)})}. \tag{1}$$

Let $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} x_0^{**}(f_k)e_k$ for $n \in \mathbb{N}$ ($\sigma(0) = 0$). Then $Y = \overline{\text{lin}}\{v_n : n \in \mathbb{N}\}$ and $\{v_n\}$ is a basic block of $\{e_n\}$, so it is a shrinking basis of Y. Let $\{h_n\}$ be the sequence of associated functionals of $\{v_n\}$. Then $h_n = f_{k|Y}/x_0^{**}(f_k)$ whenever $\sigma(n-1)+1 \le k \le \sigma(n)$ and $x_0^{**}(f_k) \neq 0$. But $\sup_n \|h_n\| < \infty$ by (1). So, $\lim_n \|f_{n|Y}\|_{Y^*} = 0$, and Y is non-reflexive since $x_0^{**} \in Y^{**}$.

Now, assume that $\{z_n\}$ has a convex basic block equivalent to the summing basis. Then there is an increasing $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence $\{\lambda_n\}$ of positive real numbers with $\sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k = 1$ for every $n \in \mathbb{N}$ such that $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k z_k$ is equivalent to the summing basis.

Let $Y = \overline{\mathrm{lin}}\{v_n : n \in \mathbb{N}\}$ and let $\{g_n\}$ be the sequence of associated functionals of $\{v_n\}$. Then

$$(2)\quad \lambda_k g_n = \frac{f_{k|Y}}{x_0^{**}(f_k)} - \frac{f_{k+1|Y}}{x_0^{**}(f_{k+1})} \quad \text{whenever } \sigma(n-1)+1 \le k \le \sigma(n).$$

Therefore $M_1 = \sup_n \|g_n\| < \infty$.

Let $y_1 = v_1$ and $y_n = v_n - v_{n-1}$ for $n \ge 2$. Then $\{y_n\}$ is a basis of Y equivalent to the usual basis of c_0, and

$$y_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k z_k - \sum_{k=\sigma(n-2)+1}^{\sigma(n-1)} \lambda_k z_k \quad \forall n \ge 2.$$

Let $\{h_n\}$ be the sequence of associated functionals of $\{y_n\}$. Then $h_n = f_{\sigma(n-1)+1|Y}/x_0^{**}(f_{\sigma(n-1)+1})$ for every $n \in \mathbb{N}$ and $M_2 = \sup_n \|h_n\|$ is finite.

By (2), we obtain

$$\frac{f_{k+1|Y}}{x_0^{**}(f_{k+1})} = \frac{f_{k|Y}}{x_0^{**}(f_k)} - \lambda_k g_n \quad \text{whenever } \sigma(n-1)+1 \le k \le \sigma(n).$$

Inductively, we have

$$\frac{f_{k|Y}}{x_0^{**}(f_k)} = h_n - g_n \sum_{i=\sigma(n-1)+1}^{k-1} \lambda_i \quad \text{whenever } \sigma(n-1)+2 \le k \le \sigma(n)+1.$$

So, $\|f_{k|Y}/x_0^{**}(f_k)\|_{Y^*} \le M_2 + M_1$ for every $k \in \mathbb{N}$ and hence $\lim_{n\to\infty} \|f_{n|Y}\|_{Y^*} = 0$.

(ii)⇒(iii). This is clear, by considering $K = B_Y$.

(iii)⇒(i). Let $k^{**} \in \overline{K}^{w^*} \setminus K$. Then $k^{**} \in X^{**} \setminus X$ and $\lim_n k^{**}(f_n) = 0$, since $\lim_n \sup_{k\in K} f_n(k) = 0$. So, $y^{**} \in N_X \setminus X$ and $N_X \ne X$. ■

To prove Theorem A we need the following lemmas:

LEMMA 2. *Let X a Banach space with a normalized basis $\{e_n\}$ and let $y_p = \sum_{n=1}^\infty a_n^p e_n$ for every $p \in \mathbb{N}$ be a sequence in X such that $\lim_k a_n^k = 0$ uniformly in $n \in \mathbb{N}$ and $\|y_p\| \ge 1$ for every $p \in \mathbb{N}$. Then there is a basic subsequence $\{y_{\tau(p)}\}$ of $\{y_p\}$ equivalent to a basic block $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ for every $n \in \mathbb{N}$ such that $\inf_n \|v_n\| > 0$ and $\lim_k \lambda_k = 0$.*

Proof (analogous to Prop. 1.a.11 of [3]). Let us see that for every $\varepsilon > 0$ and $p \in \mathbb{N}$ there is $k_p \in \mathbb{N}$ such that $\|\sum_{n=1}^p a_n^{k_p} e_n\| < \varepsilon$.

If $\varepsilon > 0$ and $p \in \mathbb{N}$, then there is $k_0 \in \mathbb{N}$ such that $|a_n^k| < \varepsilon/p$ whenever $k \ge k_0$ and $1 \le n \le p$. Then

$$\Big\|\sum_{n=1}^p a_n^k e_n\Big\| \le \sum_{n=1}^p |a_n^k| < \varepsilon \quad \forall k \ge k_0.$$

Hence it suffices to put $k_p = k \ge k_0$.

Now, we construct $\{y_{\tau(p)}\}$ and $\{v_n\}$ inductively. Put $\tau(1) = 1$ and $y_1 = \sum_{n=1}^\infty a_n^1 e_n$.

Let $p_1 \in \mathbb{N}$ be such that $\|y_1 - v_1\| < \frac14 M$, where $v_1 = \sum_{n=1}^{p_1} a_n^1 e_n$ and M is the basic constant of $\{e_n\}$.

Let $\tau(2) \in \mathbb{N}$ be such that $\tau(2) > \tau(1)$ and $\|\sum_{n=1}^{p_1} a_n^{\tau(2)} e_n\| < \frac12 \frac{1}{4^2} M$ and let $p_2 \in \mathbb{N}$ with $p_2 > p_1$ and $\|\sum_{n=p_2+1}^{\infty} a_n^{\tau(2)} e_n\| < \frac12 \frac{1}{4^2} M$.

Then $\|y_{\tau(2)} - v_2\| < \frac{1}{4^2} M$, where $v_2 = \sum_{n=p_1+1}^{p_2} a_n^{\tau(2)} e_n$.

In this way we obtain a basic block $\{v_n\}$ equivalent to $\{y_{\tau(n)}\}$. So, $\inf_n \|v_n\| > 0$.

Furthermore, if $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ for every $n \in \mathbb{N}$, then $\lambda_k = a_k^{\tau(n)}$ whenever $\sigma(n-1)+1 \le k \le \sigma(n)$.

So, $\lim_k \lambda_k = 0$, since $\lim_k a_n^k = 0$ uniformly in $n \in \mathbb{N}$. ■

LEMMA 3. *Let X be a Banach space with a seminormalized supershrinking basis $\{e_n\}$. If X does not contain infinite-dimensional reflexive subspaces, then $\{e_n\}$ is equivalent to the usual basis of c_0.*

Proof. Let $\{f_n\}$ be the sequence of biorthogonal functionals. Assume that $\{e_n\}$ is not equivalent to the usual basis $\{b_n\}$ of c_0. We put

$$X_0 = \Big\{\{a_n\} \in \mathbb{R}^{\mathbb{N}} : \sum_n a_n e_n \text{ converges in } X\Big\}.$$

Then X_0 is a subspace of c_0, non-closed in general. Define $T : X_0 \to X$ by

$$T(\{a_n\}) = \sum_{n=1}^\infty a_n e_n \quad \forall \{a_n\} \in X_0.$$

Then T is linear and bijective. Furthermore, $T^{-1}(x) = \{f_n(x)\}_n$ for every $x \in X$. If $n \in \mathbb{N}$ and $x \in X$ we obtain

$$|f_n(x)| \le M\|f_n(x)e_n\| = M\|P_n(x) - P_{n-1}(x)\| \le 2KM\|x\|,$$

where $\|e_n\| \ge 1/M > 0$ for all $n \in \mathbb{N}$, K is the basic constant and $\{P_n\}$ is the sequence of the projections of the basis.

Thus, T^{-1} is continuous. But $T(b_n) = e_n$ for all $n \in \mathbb{N}$, so T is not continuous, that is, we can assume that for every $k \in \mathbb{N}$ there is $\{a_n^k\} \subset X_0$ such that $\lim_k a_n^k = 0$ uniformly in $n \in \mathbb{N}$ and $\|\sum_{n=1}^\infty a_n^k e_n\| \ge 1$ for every $k \in \mathbb{N}$.

If we set $y_p = \sum_{n=1}^\infty a_n^p e_n$ for every $p \in \mathbb{N}$, we can apply Lemma 2 to obtain a basic block $\{v_n\}$ with $\inf_n \|v_n\| > 0$ and $\lim_k \lambda_k = 0$, where $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ for every $n \in \mathbb{N}$.

We put $Y = \overline{\mathrm{lin}}\{v_n : n \in \mathbb{N}\}$. Then Y is not reflexive and $\lim_{n\to\infty} \|f_{n|Y}\|_{Y^*} = 0$, since $\lim_k \lambda_k = 0$.

By Theorem 1, we conclude that $N_X \ne X$. ■

Proof of Theorem A. The equivalence between (i) and (iii) is general and it is proved in Theorem 1. The implications (i)⇒(ii), (v)⇒(iv) and (iv)⇒(iii) are clear, and the implication (i)⇒(v) is Lemma 3.

For (ii)⇒(i), assume that $\{e_n\}$ is not supershrinking, so $N_X \neq X$. By Theorem 1, there is a non-reflexive subspace Y of X such that $Y^{**} \subset N_X$. Now, by Theorem 1.b.14 of [3], Y^{**} is not separable, since X does not contain infinite-dimensional reflexive subspaces. Then N_X is not separable. ■

Before proving Theorem B, we show what happens if one assumes the equality $N_X = X$ without c_0 copies.

PROPOSITION 3. *Let X be a Banach space with a normalized basis $\{e_n\}$ without subspaces isomorphic to c_0 and assume that $N_X = X$. Then*

(i) *Every infinite-dimensional subspace of X contains an infinite-dimensional reflexive subspace, that is, X is somewhat reflexive.*

(ii) *Every subsequence of $\{e_n\}$ has a further subsequence whose closed linear span is a reflexive subspace.*

Proof. (i) Let Y be an infinite-dimensional subspace of X. Then by Proposition 1.a.11 of [3], there is a basic sequence $\{y_n\}$ in Y equivalent to a seminormalized basic block $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ for every $n \in \mathbb{N}$, with $\{\lambda_k\}$ bounded.

As X does not contain c_0 copies, there is $\{t_n\} \subset c_0$ such that $\sum_n t_n v_n$ is not convergent.

Then there is $\varepsilon > 0$ and natural numbers $p_1 < q_1 < \ldots < p_n < q_n < \ldots$ such that $\|\sum_{k=p_n}^{q_n} t_n v_n\| \geq \varepsilon$ for every $n \in \mathbb{N}$.

In this way, we obtain a basic block

$$w_n = \sum_{k=\tau(n-1)+1}^{\tau(n)} \alpha_k e_k = \sum_{k=p_n}^{q_n} t_k v_k \quad \forall n \in \mathbb{N},$$

with $\inf_n \|w_n\| > 0$ and $\lim_k \alpha_k = 0$.

Let $Z = \overline{\text{lin}}\{w_n : n \in \mathbb{N}\}$. Then $\lim_{n\to\infty} \|f_{n|Z}\|_{Z^*} = 0$, since $\lim_k \alpha_k = 0$. So, Z is a reflexive subspace isomorphic to a subspace of Y, by Theorem 1.

(ii) It is clear that it suffices to prove that $\{e_n\}$ has a subsequence whose closed linear span is a reflexive subspace.

For this, we apply the Elton Theorem to obtain a basic subsequence $\{e_{\sigma(n)}\}$ of $\{e_n\}$ such that

$$\lim_k \Big\|\sum_{i=1}^{k} a_i e_{\sigma(i)}\Big\| = \infty \quad \forall \{a_i\} \notin c_0.$$

We put $Y = \overline{\text{lin}}\{e_{\sigma(n)} : n \in \mathbb{N}\}$. To see that Y is reflexive it suffices to prove that $\{e_{\sigma(n)}\}$ is a boundedly complete basic sequence in Y. (Observe that the assumption $N_X = X$ implies that $\{e_n\}$ is shrinking.)

Let $\{\lambda_n\} \subset \mathbb{R}$ be such that $\sup_n \|\sum_{k=1}^n \lambda_k e_{\sigma(k)}\| < \infty$. Then $\{\lambda_n\} \in c_0$ and so $\sum_n \lambda_n e_{\sigma(n)}$ converges, that is, Y is reflexive. ■

LEMMA 4. *Let X be a Banach space with a normalized basis $\{e_n\}$ and associated functionals $\{f_n\}$ such that $N_X = X$. If $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ is a basic block of $\{e_n\}$ with $\{\lambda_n\}$ bounded and $Y = \overline{\text{lin}}\{v_n : n \in \mathbb{N}\}$, then $N_Y = Y$.*

Proof. Let $\{g_n\}$ be the sequence of associated functionals of $\{v_n\}$. Then $f_k = \lambda_k g_n$ whenever $\sigma(n-1)+1 \leq k \leq \sigma(n)$.

If $y^{**} \in Y^{**}$ with $\lim_n y^{**}(g_n) = 0$ then $\lim_n y^{**}(f_n) = 0$, since $\{\lambda_n\}$ is bounded. So, $y^{**} \in N_X = X$ and $N_Y = Y$. ■

Proof of Theorem B. Let Z be a non-reflexive subspace of X. Then, by Proposition 1.a.11 of [3], it is possible to find a basic sequence $\{z_n\}$ in Z which is weakly Cauchy, not weakly convergent, and whose difference sequence $\{z_{n+1} - z_n\}$ is equivalent to a seminormalized block basis of $\{e_n\}$. The subspace generated by the sequence $\{z_{n+1} - z_n\}$ is non-reflexive and satisfies the same hypotheses as X, by Lemma 4, so it is sufficient to see that X contains an order-one quasireflexive subspace if X is not reflexive.

So, assume that X is not reflexive and let $x_0^{**} \in X^{**} \setminus X$. Now, the sequence $\{\sum_{k=1}^n x_0^{**}(f_k)e_k\}_n$ is weakly Cauchy and not weakly convergent. By the c_0-Rosenthal Theorem we find $\sigma : \mathbb{N} \to \mathbb{N}$ increasing such that the sequence

$$y_n = \sum_{k=1}^{\sigma(n)} x_0^{**}(f_k)e_k$$

is basic and strongly summing. We put $Y = \overline{\text{lin}}\{y_n : n \in \mathbb{N}\}$. We can suppose that $x_0^{**}(f_{\sigma(n)}) \neq 0$ for all $n \in \mathbb{N}$.

Let $\{g_n\}$ be the sequence of associated functionals of $\{y_n\}$. Then

$$g_n = \frac{f_{\sigma(n)|Y}}{x_0^{**}(f_{\sigma(n)})} - \frac{f_{\sigma(n+1)|Y}}{x_0^{**}(f_{\sigma(n+1)})} \quad \forall n \in \mathbb{N}.$$

If $y^{**} \in Y^{**}$ then $\sup_n \|\sum_{k=1}^n y^{**}(g_k)y_k\| < \infty$. So, the series $\sum_n y^{**}(g_n)$ converges. Therefore, the following limit exists, for every $y^{**} \in Y^{**}$:

$$\lim_{n\to\infty} \frac{y^{**}(f_{\sigma(n)})}{x_0^{**}(f_{\sigma(n)})}. \tag{3}$$

We put $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} x_0^{**}(f_k)e_k$ and $h_n = f_{\sigma(n)}/x_0^{**}(f_{\sigma(n)})$ for every $n \in \mathbb{N}$. Hence $\{v_n\}$ is a shrinking basis of Y with associated functionals $\{h_n\}$.

Finally, we define $\phi : Y^{**} \to \mathbb{R}$ by $\phi(y^{**}) = \lim_n y^{**}(h_n)$. By (3), we have $\phi \in Y^{***}$. Now, by applying Lemma 4,

$$Y^{**} = \mathrm{Ker}\phi \oplus \langle x_0^{**} \rangle = N_Y \oplus \langle x_0^{**} \rangle = Y \oplus \langle x_0^{**} \rangle.$$

Thus Y is an order-one quasireflexive subspace. ∎

REMARK. With an analogous proof, the conclusion of Theorem B is also true if we suppose that $\dim(N_X/X) < \infty$ instead of $N_X = X$.

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A spectral theory for locally compact abelian groups of automorphisms of commutative Banach algebras

by

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Abstract. Let $\mathcal{A}$ be a commutative Banach algebra with Gelfand space $\Delta(\mathcal{A})$. Denote by $\mathrm{Aut}(\mathcal{A})$ the group of all continuous automorphisms of $\mathcal{A}$. Consider a $\sigma(\mathcal{A}, \Delta(\mathcal{A}))$-continuous group representation $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ of a locally compact abelian group G by automorphisms of $\mathcal{A}$. For each $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$, the function $\varphi_a(t) := \varphi(\alpha_t a)$ ($t \in G$) is in the space $C(G)$ of all continuous and bounded functions on G. The weak-star spectrum $\sigma_{w^*}(\varphi_a)$ is defined as a closed subset of the dual group $\widehat{G}$ of G. For $\varphi \in \Delta(\mathcal{A})$ we define Λ_φ^α to be the union of all sets $\sigma_{w^*}(\varphi_a)$ where $a \in \mathcal{A}$, and Λ_α to be the closure of the union of all sets Λ_φ^α where $\varphi \in \Delta(\mathcal{A})$, and call Λ_α the unitary spectrum of α.

Starting by showing that the closure of Λ_φ^α (for fixed $\varphi \in \Delta(\mathcal{A})$) is a subsemigroup of $\widehat{G}$ we characterize the structure properties of the group representation α such as norm continuity, growth and existence of non-trivial invariant subspaces through its unitary spectrum Λ_α.

For an automorphism T of a semisimple commutative Banach algebra $\mathcal{A}$ we consider the group representation $\mathbf{T} : \mathbb{Z} \to \mathrm{Aut}(\mathcal{A})$ defined by $\mathbf{T}_n := T^n$ for all $n \in \mathbb{Z}$. It is shown that $\Lambda_{\mathbf{T}} = \sigma(T) \cap \mathbb{T}$, where $\sigma(T)$ is the spectrum of T and $\mathbb{T}$ is the unit circle. From this fact we give an easy proof of the Kamowitz–Scheinberg theorem which asserts that the spectrum $\sigma(T)$ either contains $\mathbb{T}$ or is a finite union of finite subgroups of $\mathbb{T}$.

Introduction. Let $\mathcal{A}$ be a commutative Banach algebra with Gelfand space $\Delta(\mathcal{A})$ (i.e., the space of regular maximal ideals of $\mathcal{A}$). Denote by $\mathrm{Aut}(\mathcal{A})$ the group of all continuous automorphisms of $\mathcal{A}$. For an automorphism T on $\mathcal{A}$ we consider the group representation $\mathbf{T} : \mathbb{Z} \to \mathrm{Aut}(\mathcal{A})$ given by $\mathbf{T}_n := T^n$ for all $n \in \mathbb{Z}$. For each $a \in \mathcal{A}$ and $\varphi \in \Delta(\mathcal{A})$ the function $\varphi_a(n) := \varphi(T^n a)$ ($n \in \mathbb{Z}$) belongs to the space $C(\mathbb{Z})$ of all continuous and bounded functions on the group $\mathbb{Z}$. The weak-star spectrum $\sigma_{w^*}(\varphi_a)$ of φ_a, as a closed subset of the unit circle $\mathbb{T}$, is defined in the classical way (see [14] or [20]). Note that $\mathbb{T}$ is the dual group of $\mathbb{Z}$.

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