[3] T. Bagby, *Rational interpolation with restricted poles*, J. Approx. Theory 7 (1973), 1–7.
[4] N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wiss. 190, Springer, New York, 1972.
[5] T. Ransford, *Potential Theory in the Complex Plane*, London Math. Soc. Student Texts 28, Cambridge Univ. Press, Cambridge, 1995.
[6] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.
[7] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Encyclopedia Math. Appl., Cambridge Univ. Press, Cambridge, 1992.
[8] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 4th ed., Amer. Math. Soc. Colloq. Publ. 20, Amer. Math. Soc., Providence, R.I., 1965.
[9] A. E. Wegert and L. N. Trefethen, *From the Buffon needle problem to the Kreiss matrix theorem*, Amer. Math. Monthly 101 (1994), 132–139.

Department of Mathematics
Umeå University
S-901 87 Umeå, Sweden
and
Department of Mathematics
Tbilisi University
Republic of Georgia
E-mail: amiran.ambroladze@math.lth.se

Department of Mathematics
Umeå University
S-901 87 Umeå, Sweden
E-mail: Hans.Wallin@mathdept.umu.se

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On the joint spectral radius of a nilpotent Lie algebra of matrices

by

ENRICO BOASSO (Buenos Aires)

Abstract. For a complex nilpotent finite-dimensional Lie algebra of matrices, and a Jordan–Hölder basis of it, we prove a spectral radius formula which extends a well-known result for commuting matrices.

1. Introduction. Let $T = (T_1, \dots, T_n)$ be an n-tuple of $d \times d$ complex matrices. A point $\lambda \in \mathbb{C}^n$ is in the *joint point spectrum* of T, $\sigma_{\mathrm{pt}}(T)$, if there exists a nonzero $x \in \mathbb{C}^d$ with $T_i(x) = \lambda_i x$, $1 \le i \le n$. Given p such that $1 \le p \le \infty$, R. Bhatia and T. Bhattacharyya [1] introduced the algebraic spectral radius of an n-tuple T, $\varrho_p(T)$, whose definition depends of the usual p-norm of $\mathbb{C}^d$, and proved that if T is an n-tuple of commuting matrices, then the algebraic spectral radius coincides with the geometric spectral radius, i.e., $\varrho_p(T) = r_p(T) = \max\{|\lambda|_p : \lambda \in \sigma_{\mathrm{pt}}(T)\}$ (see [1] or Section 2 for more details). This is a generalization of the well-known spectral radius formula for a single matrix; for $p = 2$, it was proved by M. Chō and T. Huruya [6].

M. Chō and M. Takaguchi [7] proved that if T is a commuting n-tuple of matrices, then $\sigma_{\mathrm{pt}}(T) = \mathrm{Sp}(T, \mathbb{C}^d)$, where $\mathrm{Sp}(T, \mathbb{C}^d)$ denotes the Taylor joint spectrum of T (see [12]). A. McIntosh, A. Pryde and W. Ricker [9], as a consequence of a more general result which also concerns infinite-dimensional spaces, extended the above identity to many other joint spectra including the commutant, the bicommutant and the Harte joint spectra.

On the other hand, in [4] we defined a joint spectrum, $\mathrm{Sp}(L, E)$, for complex solvable finite-dimensional Lie algebras L of operators acting on a Banach space E. We proved that $\mathrm{Sp}(L, E)$ is a compact nonempty subset of L^* satisfying the projection property for ideals. Moreover, when L is a commutative algebra, $\mathrm{Sp}(L, E)$ reduces to the Taylor joint spectrum in the following sense. If $\dim L = n$ and $\{T_i\}_{1 \le i \le n}$ is a basis of L, then $\{(f(T_1), \dots, f(T_n)) : f \in \mathrm{Sp}(L, E)\} = \mathrm{Sp}(T, E)$ for $T = (T_1, \dots, T_n)$, i.e.,

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$\mathrm{Sp}(L,E)$ in terms of the basis of L^* dual to $\{T_i\}_{1\le i\le n}$ coincides with the Taylor joint spectrum of the n-tuple T. Furthermore, in [2] we also extended to complex solvable finite-dimensional Lie algebras the Słodkowski joint spectra $\sigma_{\delta,k}$ and $\sigma_{\pi,k}$, and we proved the most important properties of a spectral theory, i.e., compactness, nonemptiness and the projection property for ideals.

For E finite-dimensional, in [3] we extended the characterization of [7], and partially that of [9], to complex nilpotent Lie algebras acting on E. Indeed, we proved that $\mathrm{Sp}(L,E)$; $\sigma_{\delta,k}(L,E)$ and $\sigma_{\pi,k}(L,E)$ coincide with the set of all weights of the Lie algebra L for the vector space E, which is the intrinsic algebraic description of the joint point spectrum, and showed that in suitable bases of L and L^* it reduces to the joint point spectrum of the basis of L (see [3] or Section 2 for details). This also extended the characterization of [7] and [9] to the Słodkowski joint spectra of an n-tuple of commuting matrices.

Thus, if L is a complex nilpotent finite-dimensional Lie algebra acting on a complex finite-dimensional vector space E, and if instead of considering the elements of $\mathrm{Sp}(L,E)$ as linear functionals on L we work with their coordinates in a basis of L^*, dual to a suitable basis of L, then, as in the former basis $\mathrm{Sp}(L,E)$ reduces to the joint point spectrum of the latter basis, it is possible to compute the geometric and algebraic spectral radii of L with respect to its basis, and to look for a generalization of the main result of [1]. In this article we extend the spectral radius formula of R. Bhatia and T. Bhattacharyya to the case under consideration. The argument is quite elementary, and it furnishes another proof of the formula for the commutative case.

The paper is organized as follows. In Section 2 we review several definitions and results which we need. In Section 3, we prove our main theorem and study some examples to show that in the solvable not nilpotent case the spectral radius formula fails.

2. Preliminaries. We briefly recall several definitions and results related to the spectrum of a Lie algebra (see [4]). Although in [4] we considered complex solvable finite-dimensional Lie algebras of linear bounded operators acting on a Banach space, for our present purpose we restrict ourselves to the case of complex finite-dimensional nilpotent Lie algebras of linear transformations defined on finite-dimensional vector spaces. Moreover, as in this case the Słodkowski joint spectra and the Taylor joint spectrum coincide, we concentrate on the latter; for more information about the Słodkowski joint spectra see [11] and [2].

From now on E denotes a complex finite-dimensional vector space, $\mathcal{L}(E)$ the algebra of all linear transformations defined on E, and L a complex nilpotent finite-dimensional Lie subalgebra of $\mathcal{L}(E)^{\mathrm{op}}$, the algebra $\mathcal{L}(E)$ with opposite product. Such an algebra is called a *nilpotent Lie algebra of linear transformations* in E. If $\dim L = n$ and f is a *character* of L, i.e., $f \in L^*$ and $f(L^2) = 0$, where $L^2 = \{[x,y] : x,y \in L\}$, consider the chain complex $(E \otimes \bigwedge L, d(f))$, where $\bigwedge L$ denotes the exterior algebra of L and

$$d_p(f) : E \otimes \textstyle\bigwedge^p L \to E \otimes \bigwedge^{p-1} L,$$

$$d_p(f)e\langle x_1 \wedge \ldots \wedge x_p\rangle = \sum_{k=1}^{p} (-1)^{k+1} e(x_k - f(x_k))\langle x_1 \wedge \ldots \wedge \widehat{x_k} \wedge \ldots \wedge x_p\rangle$$
$$+ \sum_{1\le k<l\le p} (-1)^{k+l} e([x_k,x_l]) \wedge x_1 \wedge \ldots \wedge \widehat{x_k} \wedge \ldots \wedge \widehat{x_l} \wedge \ldots \wedge x_p\rangle,$$

where $\widehat{\ }$ means deletion. If $p \le 0$ or $p \ge n+1$, we define $d_p(f) = 0$.

If we denote by $H_*(E \otimes \bigwedge L, d(f))$ the homology of the complex $(E \otimes \bigwedge L, d(f))$, we may state our first definition.

DEFINITION 1. With E, L and f as above, the set $\{f \in L^* : f(L^2) = 0, H_*(E \otimes \bigwedge L, d(f)) \ne 0\}$ is the *joint spectrum of L acting on E*, and it is denoted by $\mathrm{Sp}(L,E)$.

As already mentioned, in [4] we proved that $\mathrm{Sp}(L,E)$ is a compact nonempty subset of L^*, which reduces, in the sense explained in the introduction, to the Taylor joint spectrum when L is a commutative algebra. Moreover, if I is an ideal of L, and π denotes the projection map from L^* to I^*, then

$$\mathrm{Sp}(I,E) = \pi(\mathrm{Sp}(L,E)),$$

i.e., the projection property for ideals still holds. In this connection, we wish to mention the paper of C. Ott [10] who pointed out a gap in [4] in the proof of this result and gave another proof.

We recall the most important results of the theory of weight spaces, essentially Theorems 7 and 12, Chapter II of [8]. For a complete exposition see [8; II].

Let L and E be as above. A *weight* of L for E is a mapping $\alpha : L \to \mathbb{C}$ such that there exists a nonzero vector v in E with the following property: for each x in L there is $m_{v,x}$ in $\mathbb{N}$ such that $(x - \alpha(x))^{m_{v,x}}(v) = 0$. The set of vectors, zero included, which satisfy this condition is a subspace of E, denoted by E_α and called the *weight space* of E corresponding to the weight α.

Under our assumptions we have the following properties (see [8; II, 7,12]):

(i) the weights are linear functions on L which vanish on L^2, i.e., they are characters of L,

(ii) E has only a finite number of distinct weights; the weight spaces are submodules, and E is their direct sum,

(iii) for each weight α, the restriction of any $x \in L$ to E_α has only one characteristic root, $\alpha(x)$, with a certain multiplicity,

(iv) there is a basis of E such that for each weight α and each $x \in L$ the matrix of x_α, the restriction of x to E_α, is

$$x_\alpha = \begin{pmatrix} \alpha(x) & & * \\ & \ddots & \\ 0 & & \alpha(x) \end{pmatrix}.$$

We now recall the following theorem of [3] which will be crucial for our main result.

THEOREM A. *Let E be a complex finite-dimensional vector space, and L a complex finite-dimensional nilpotent Lie subalgebra of $\mathcal{L}(E)^{\mathrm{op}}$. Then*

$$\mathrm{Sp}(L, E) = \{\alpha \in L^* : \alpha \textit{ is a weight of } L \textit{ for } E\}.$$

We observe that the right hand side set is a generalization of the joint point spectrum. In fact, if we consider a commutative algebra L, and $T = (T_1, \ldots, T_n)$ is an n-tuple of matrices such that $\{T_i\}_{1 \le i \le n}$ is a basis of L, then the set of weights of L for E represented in terms of the basis of L^* dual to $\{T_i\}_{1 \le i \le n}$ coincides with $\sigma_{\mathrm{pt}}(T)$.

As we shall work with the coordinates of elements of $\mathrm{Sp}(L, E)$, we need to construct suitable bases for L and L^*.

By [5; IV,1], there is a Jordan–Hölder sequence of ideals, $(L_i)_{0 \le i \le n}$, such that:

(i) $\{0\} = L_0 \subseteq L_i \subseteq L_n = L$,
(ii) $\dim L_i = i$,
(iii) there is a k $(0 \le k \le n)$ such that $L_k = L^2$,
(iv) if $i < j$, then $[L_i, L_j] \subseteq L_{i-1}$.

As a consequence, if $\{x_j\}_{1 \le j \le n}$ is a basis of L such that $\{x_j\}_{1 \le j \le i}$ is a basis of L_i for each i, then

$$[x_j, x_i] = \sum_{h=1}^{i-1} c_{ij}^h x_h,$$

where $1 \le i < j \le n$. Such a basis will be called a *Jordan–Hölder basis*.

Now, given a Jordan–Hölder basis and an $\alpha \in \mathrm{Sp}(L, E)$, we may represent it by its coordinates in the dual basis of L^*, i.e., by $(\alpha(x_1), \ldots, \alpha(x_n))$. The set of all such n-tuples will be denoted by $\mathrm{Sp}((x_i)_{1 \le i \le n}, E)$, i.e.,

$$\mathrm{Sp}((x_i)_{1 \le i \le n}, E) = \{(\alpha(x_1), \ldots, \alpha(x_n)) : \alpha \in \mathrm{Sp}(L, E)\}.$$

We observe that $\mathrm{Sp}((x_i)_{1 \le i \le n}, E) = \sigma_{\mathrm{pt}}((x_i)_{1 \le i \le n})$.

Recall that when L is a solvable Lie algebra, by [5; IV,2] we may also construct a sequence $(L_i)_{0 \le i \le n}$ of ideals with properties (i), (ii) and (iii), and property (iv)′: if $i < j$, then $[L_i, L_j] \subseteq L_i$. Thus, if $\{x_j\}_{1 \le j \le n}$ is a basis of L such that $\{x_j\}_{1 \le j \le i}$ is a basis of L_i for each i, then

$$[x_j, x_i] = \sum_{h=1}^{i} c_{ij}^h x_h,$$

where $1 \le i < j \le n$. As in the nilpotent case, such a basis will be called a *Jordan–Hölder basis* of the solvable Lie algebra L.

We now recall the geometric and algebraic spectral radii, as defined in [1].

Let E_p, $1 \le p \le \infty$, denote the space $\mathbb{C}^d$ provided with the usual p-norm

$$\|x\|_p = \Big(\sum_{i=1}^{d} |x_i|^p\Big)^{1/p}, \quad 1 \le p < \infty,$$

$$\|x\|_\infty = \sup_{1 \le i \le d} |x_i|.$$

On the other hand, for $\lambda \in \mathbb{C}^n$, we denote its p-norm by $|\lambda|_p$.

If $T = (T_1, \ldots, T_n)$ is an n-tuple of $d \times d$ complex matrices, then the *geometric spectral radius* of T is defined as

$$r_p(T) = \max\{|\lambda|_p : \lambda \in \sigma_{\mathrm{pt}}(T)\}.$$

On the other hand, we may identify a matrix M with the associated linear transformation, which we also denote by M. Thus, the n-tuple T induces an operator from E_p to the direct sum of n copies of E_p, considered with its natural p-norm. The norm of this operator, also denoted by T, is

$$\|T\|_p = \sup_{\|x\|_p \le 1} \Big(\sum_{j=1}^{n} \|T_j(x)\|_p^p\Big)^{1/p}.$$

Now, given an $m \in \mathbb{N}$, we consider the n^m-tuple whose entries are all the products $T_{i_1} \ldots T_{i_m}$, where $1 \le i_1, \ldots, i_m \le n$, with repetitions allowed and the indices arranged lexicographically. We denote this n^m-tuple by T^m. Then the *algebraic spectral radius* of the n-tuple T is defined as

$$\varrho_p(T) = \inf \|T^m\|_p^{1/m}, \quad 1 \le p \le \infty.$$

As already mentioned, R. Bhatia and T. Bhattacharyya [1] proved that for a commuting n-tuple T of matrices the algebraic and geometric spectral radii coincide. Now, given a nilpotent Lie algebra L of matrices in $\mathcal{L}(\mathbb{C}^d)^{\mathrm{op}}$, and a Jordan–Hölder basis $\{x_i\}_{1 \le i \le n}$ of L, where n is $\dim L$, we may consider the n-tuple $(x_i)_{1 \le i \le n}$. In the next section we shall see that the geometric and algebraic spectral radii of $(x_i)_{1 \le i \le n}$ also coincide.

3. The main result. In this section we prove that the spectral radius formula proved in [1] extends to nilpotent Lie algebras of matrices.

Observe that if $T = (T_1, \dots, T_n)$ is a commuting n-tuple of matrices, and if U is an invertible matrix such that $UTU^{-1} = (UT_1U^{-1}, \dots, UT_nU^{-1})$ is an n-tuple of commuting upper triangular matrices, then $\sigma_{\mathrm{pt}}(T) = \sigma_{\mathrm{pt}}(UTU^{-1})$, $r_p(T) = r_p(UTU^{-1})$ and $\varrho_p(T) = \varrho_p(UTU^{-1})$; see [1] or Proposition 2 below. If we decompose the matrices UT_iU^{-1}, $1 \le i \le n$, into their diagonal and nilpotent parts, we have two new commuting n-tuples of matrices, N and D. The spectral radii of T coincide with the corresponding radii of D. This suggests that in the computations of the spectral radii the nilpotent parts of the matrices are not of great importance. In this section, in order to prove our main theorem, we consider an n-tuple T of upper triangular matrices with some additional properties, and we give an upper bound for the associated n-tuple N in order to show that the algebraic and geometric spectral radii of T coincide. As the nilpotent and commuting cases may be reduced to this situation, we obtain our main result as well as a new proof for the commuting case.

Let us begin with two propositions which simplify our work.

PROPOSITION 1. *Let T be an n-tuple of $d \times d$ matrices. Then*

$$r_\infty(T) = \lim_{p\to\infty} r_p(T), \qquad \varrho_\infty(T) = \lim_{p\to\infty} \varrho_p(T).$$

Proof. If $x \in \mathbb{C}^q$, $q \in \mathbb{N}$, then

$$\|x\|_\infty \le \|x\|_p \le q^{1/p}\|x\|_\infty.$$

In particular, by the definition of $r_p(T)$,

$$r_\infty(T) \le r_p(T) \le n^{1/p} r_\infty(T),$$

which gives us the first part of the proposition.

On the other hand, an easy calculation using the above inequality yields

$$\|T\|_p \le (dn)^{1/p}\|T\|_\infty, \qquad \|T\|_\infty \le \|T\|_p d^{1/p}.$$

Then, if $m \in \mathbb{N}$, we obtain

$$\|T^m\|_\infty \le \|T^m\|_p d^{1/p} \le (d^2 n^m)^{1/p}\|T^m\|_\infty,$$

which implies that

$$\|T^m\|_\infty^{1/m} \le \|T^m\|_p^{1/m} d^{1/(pm)} \le d^{2/(pm)} n^{1/p}\|T^m\|_\infty^{1/m}.$$

Thus,

$$\varrho_\infty(T) \le \varrho_p(T) \le n^{1/p}\|T\|_\infty,$$

which shows that $\lim_{p\to\infty} \varrho_p(T) = \varrho_\infty(T)$. ∎

Thus, we may restrict our proof to the case $1 \le p < \infty$.

PROPOSITION 2. *Let T be an n-tuple of matrices acting in E_p, and U an invertible matrix acting in E_p. Set $UTU^{-1} = (UT_1U^{-1}, \dots, UT_nU^{-1})$. Then*

$$r_p(T) = r_p(UTU^{-1}), \qquad \varrho_p(T) = \varrho_p(UTU^{-1}).$$

Proof. As $\sigma_{\mathrm{pt}}(UTU^{-1}) = \sigma_{\mathrm{pt}}(T)$, we have $r_p(UTU^{-1}) = r_p(T)$.

For the algebraic spectral radius, we first observe that $(UTU^{-1})^m = U(T^m)U^{-1}$ for all $m \in \mathbb{N}$. Thus, if k is such that $U^{-1}(B(0,k)) \subseteq B(0,1)$, then $\|(UTU^{-1})^m\|_p \le \|U\|_p\|T^m\|_p k^{-1}$, which gives

$$\varrho_p(UTU^{-1}) \le \varrho_p(T).$$

As $T = U^{-1}(UTU^{-1})U$, we obtain the desired equality. ∎

From now on we consider an n-tuple T of matrices such that $\mathbb{C}^d$ decomposes into a direct sum of linear subspaces, $\mathbb{C}^d = \bigoplus_{1\le j\le s} M_j$, such that for each i, $1 \le i \le n$, the linear transformation associated with T_i satisfies $T_i(M_j) \subseteq M_j$ for $1 \le j \le s$. In addition, we assume that for each j, there is a basis of M_j in which the matrix of $T_i|M_j$ has an upper triangular form for all i. Moreover, we also assume that all the diagonal entries of $T_i|M_j$ coincide. By Proposition 2 we may suppose that the above basis is the canonical one and that each M_i is generated by elements of the canonical basis of $\mathbb{C}^d$.

A straightforward calculation shows that for such T we have $\sigma_{\mathrm{pt}}(T) = \{(c_j^1, \dots, c_j^n) : 1 \le j \le s\}$ where c_j^i denotes the diagonal entries of $T_i|M_j$ in the above basis of M_j.

On the other hand, by the theory of weight spaces recalled in Section 2, if $\{x_i\}_{1\le i\le n}$ is a Jordan–Hölder basis of an n-dimensional nilpotent Lie algebra of linear transformations defined on a complex finite-dimensional vector space, then the n-tuple $(x_i)_{1\le i\le n}$ clearly satisfies the above conditions.

Moreover, if $T = (T_1, \dots, T_n)$ is a commuting n-tuple of $d \times d$ complex matrices, and if $\widetilde{L}$ is the $\mathbb{C}$-vector subspace of $\mathcal{L}(\mathbb{C}^d)$ generated by T_i, $1 \le i \le n$, then $\widetilde{L}$ is a commuting Lie subalgebra of $\mathcal{L}(\mathbb{C}^d)$ and $\mathcal{L}(\mathbb{C}^d)^{\mathrm{op}}$, in particular, $\widetilde{L}$ is a nilpotent Lie subalgebra of $\mathcal{L}(\mathbb{C}^d)^{\mathrm{op}}$. Thus, we may apply the weight space theory to $\widetilde{L}$ and $\mathbb{C}^d$ to obtain a decomposition and a basis of $\mathbb{C}^d$ in which the above conditions are satisfied by T. Indeed, as the properties (i)–(iv) of Section 2 are satisfied by each x in $\widetilde{L}$, they are, in particular, satisfied by T_i, $1 \le i \le n$. This approach gives a more precise description of the joint spectrum of an n-tuple of commuting matrices, refining those of [1], [7] and [9].

Now, denote by $D = (D_1, \dots, D_n)$, respectively $N = (N_1, \dots, N_n)$, the n-tuple of the diagonal, respectively nilpotent, parts of the matrices T_i, $1 \le i \le n$. As $\sigma_{\mathrm{pt}}(D) = \sigma_{\mathrm{pt}}(T)$, we have $r_p(D) = r_p(T)$. Furthermore, as D is an n-tuple of commuting matrices, by [1], Lemma 6, $\varrho_p(D) = r_p(D)$.

If $s = 1$ and $D_i = c^i \mathrm{Id}$, $1 \le i \le n$, then an easy calculation gives

$$r_p(D)^m = \|(c^1, \dots, c^n)\|_p^m = \Big(\sum_{(i_1,\dots,i_m)\in I_m} \prod_{k=1}^{m} |c^{i_k}|^p \Big)^{1/p} = \|D^m\|_p,$$

where $I_m = \{(i_1, \dots, i_m) : 1 \le i_k \le n \text{ for } 1 \le k \le m\}$.

We now start the proof of our main result: for an n-tuple T which satisfies the above conditions, $r_p(T) = \varrho_p(T)$.

PROPOSITION 3. *Let T, D and N be as above. Then*

$$r_p(T) \le \varrho_p(T).$$

Proof. Suppose that $D_i|M_j = c_j^i \mathrm{Id}_j$ for all $1 \le i \le n$ and $1 \le j \le s$, where Id_j denotes the identity on M_j. Then, as $r_p(T) = r_p(D)$, and as D_i is the diagonal part of T_i, if $(c_{j_0}^1, \dots, c_{j_0}^n)$ is such that $\|(c_{j_0}^1, \dots, c_{j_0}^n)\|_p = r_p(D)$, there is an element $x \in M_{j_0}$ such that $\|x\|_p = 1$ and, for all $(i_1, \dots, i_m) \in I_m$,

$$T_{i_1} \dots T_{i_m}(x) = D_{i_1} \dots D_{i_m}(x) = \prod_{1\le k\le m} c_{j_0}^{i_k} x.$$

Thus, by the previous observation,

$$r_p(T)^m = r_p(D)^m = \|(c_{j_0}^1, \dots, c_{j_0}^n)\|_p^m = \Big(\sum_{(i_1,\dots,i_m)\in I_m} \prod_{k=1}^{m} |c_{j_0}^{i_k}|^p \Big)^{1/p}$$

$$= \Big(\sum_{(i_1,\dots,i_m)\in I_m} \|T_{i_1} \dots T_{i_m}(x)\|_p^p \Big)^{1/p} \le \|T^m\|_p,$$

which implies the assertion. ∎

In order to prove that $\varrho_p(T) \le r_p(T)$ we need some preparation. We begin by studying the form of the product of m upper triangular matrices with constant diagonal entries.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of $b \times b$ upper triangular matrices whose diagonal entries coincide, i.e., for each i $(1 \le i \le n)$, there is a $c^i \in \mathbb{C}$ such that $(T_i)_{tt} = c^i$ for all $1 \le t \le b$. Let $m \in \mathbb{N}$ and $1 \le i_k \le n$ for $1 \le k \le m$. Then $(T_{i_1} \dots T_{i_m})_{st} = 0$ if $1 \le t < s \le b$, and $(T_{i_1} \dots T_{i_m})_{st} = \prod_{k=1}^m c^{i_k}$ if $s = t$. As T_i are upper triangular, a straightforward calculation shows that if $1 \le s < t \le b$ then

$$(T_{i_1} \dots T_{i_m})_{st} = \sum_{h_0,\dots,h_m\in J} \prod_{1\le k\le m} (T_{i_k})_{h_{k-1}h_k},$$

where $J = \{(h_0, \dots, h_m) : s = h_0 \le \dots \le h_m = t\}$. Decompose J as $\bigcup_{0\le q\le m-1} J_q$, where $J_q = \{(h_0, \dots, h_m) \in J :$ there are q-indices k with $h_k = h_{k+1}$, $0 \le k \le m-1\}$. As $t > s$, there cannot be m such k. Moreover, if $J_q \ne \emptyset$ there are $m-q$ pairs $(k, k+1)$, $0 \le k \le m-1$, such that $h_k \ne h_{k+1}$, which implies that $t - s \ge m - q$. Thus, $J = \bigcup_{m-t+s\le q\le m-1} J_q$.

For $1 \le q \le t - s$, we may reppresent J_{m-q} as

$$J_{m-q} = \{(s, \dots, s, s + l_1, \dots, s + l_1, s + l_2, \dots, s + l_2, \dots, s + l_{q-1}, \dots, s + l_{q-1}, t, \dots t) : 1 \le l_1 < \dots < l_{q-1} \le t - s - 1\},$$

where the jumps occur at the indices k_u, $1 \le u \le q$, and $1 \le k_1 < \dots < k_q \le m-1$. With this representation it is easy to see that J_{m-q} has $\binom{m-1}{q}\binom{t-s-1}{q-1}$ elements, and that

$$(T_{i_1} \dots T_{i_m})_{st} = \sum_{1\le q\le t-s} \sum_{K_q, L_q} \prod_{k=1, k\ne k_u}^{m} c^{i_k} \prod_{u=1}^{q} (T_{i_{k_u}})_{s+l_{u-1}s+l_u},$$

where $K_q = \{(k_1, \dots, k_q) : 1 \le k_1 < \dots < k_q \le m-1\}$ and $L_q = \{(l_1, \dots, l_{q-1}) : 1 \le l_1 < \dots < l_{q-1} \le t-s-1\}$, $1 \le q \le t-s$.

Now we prove the reverse inequality of Proposition 3 for the case under consideration.

If $\|y\|_p \le 1$ then

$$\|T_{i_1} \dots T_{i_m}(y)\|_p^p = \Big\| \Big(\sum_{v=1}^{b} (T_{i_1} \dots T_{i_m})_{wv} y_v \Big)_{1\le w\le b} \Big\|_p^p$$

$$\le \Big(\sum_{w=1}^{b} \Big(\sum_{v=w}^{b} |(T_{i_1} \dots T_{i_m})_{wv}| \Big)^p \Big) \|y\|_p^p.$$

If $w = v$, we have $|(T_{i_1} \dots T_{i_m})_{ww}| = \prod_{k=1}^m |c^{i_k}|$. On the other hand, if $1 \le w < v \le b$, as there is a constant $R_1 > 1$ such that $|(T_i)_{st}| \le R_1$ for all s, t and i with $1 \le s, t \le b$ and $1 \le i \le n$, we have

$$|(T_{i_1} \dots T_{i_m})_{wv}| \le R_1^{b-1}(b-1)! \sum_{1\le q\le v-w} \sum_{(k_1,\dots,k_q)\in K_q} \prod_{k=1, k\ne k_u}^{m} |c^{i_k}|.$$

Now, if w_0 and v_0 are such that $|T_{1_1} \dots T_{i_m}|_{wv} \le |T_{i_1} \dots T_{i_m}|_{w_0v_0}$ for all w, v with $1 \le w < v \le b$, we obtain

$$\|T_{i_1} \dots T_{i_m}(y)\|_p^p \le C\|y\|_p^p \Big(\prod_{k=1}^{m} |c^{i_k}|^p + \Big(\sum_{1\le q\le v_0-w_0} \sum_{(k_1\dots,k_q)\in K_q} \prod_{k=1, k\ne k_u}^{m} |c^{i_k}| \Big)^p \Big),$$

where $C = R_1^{p(b-1)} b b!^p$.

Observe that if there are l indices $i_1, \dots, i_l$ such that $c^{i_j} = 0$, and if $l > v_0 - w_0$, then $T_{i_1} \dots T_{i_m} = 0$.

Set $\tilde{I}_m = \bigcup_{0\le r\le \tilde{b}} I_m^r$, where $\tilde{b} = \min\{b-1, m\}$ and

$$I_m^r = \{(i_1, \dots, i_m) \in I_m : \text{there are } r \text{ indices } l \text{ such that } c^{i_l} = 0\}.$$

Then, if $(i_1,\dots,i_m)\in I_m^0$, we have

$$\|T_{i_1}\dots T_{i_m}\|_p^p \le C\|y\|_p^p\prod_{k=1}^m|c^{i_k}|^p\Big(1+\Big(\sum_{1\le q\le v_0-w_0}\ \sum_{(k_1\dots,k_q)\in K_q}\ \prod_{u=1}^q\frac{1}{|c^{i_{k_u}}|}\Big)^p\Big).$$

If $R_2\ge\max\{R_1,1/|c^i|:c^i\ne 0,1\le i\le n\}$ then

$$\|T_{i_1}\dots T_{i_m}(y)\|_p^p\le C_0\|y\|_p^p\prod_{k=1}^m|c^{i_k}|^p,$$

where $C_0=2^pC(b-1)^p(m-1)^{(b-1)p}R_2^{p^2}$.

Moreover, if $(i_1,\dots,i_m)\in I_m^r$ $(1\le r\le\widetilde{b})$, a similar calculation gives

$$\|T_{i_1}\dots T_{i_m}(y)\|_p^p\le C_r\|y\|_p^p\prod_{k=1,k\ne l}^m|c^{i_k}|^p,$$

where i_l $(1\le l\le r)$, are such that $c^{i_l}=0$, and $C_r=2^pC(b-r)^p\times(m-1)^{(b-1)p}R_2^{p^2}\le C_0$ for all $1\le r\le\widetilde{b}$.

Now, for $m\ge b$, an easy calculation shows that

$$\begin{aligned}\|T^m\|_p^p&=\sup_{\|y\|_p\le1}\Big(\sum_{(i_1,\dots,i_m)\in I_m}\|T_{i_1}\dots T_{i_m}(y)\|_p^p\Big)\\&=\sup_{\|y\|_p\le1}\Big(\sum_{0\le r\le b-1}\ \sum_{(i_1,\dots,i_m)\in I_m^r}\|T_{i_1}\dots T_{i_m}(y)\|_p^p\Big)\\&\le C_0\sum_{0\le r\le b-1}\ \sum_{(i_1,\dots,i_m)\in I_m^r}\ \prod_{k=1,\,c^{i_k}\ne0}^m|c^{i_k}|^p\\&=C_0\sum_{0\le r\le b-1}h^r\prod_{j=0}^{r-1}(m-j)\sum_{(i_1,\dots,i_{m-r})\in I_{m-r}}\ \prod_{k=1}^m|c^{i_k}|^p,\end{aligned}$$

where $h=\sharp\{l:c^l=0,\ 1\le l\le n\}$.

By the observation which precedes Proposition 3,

$$\|T^m\|_p^p\le C_0h^{b-1}m^br_p(D)^{p(m-b+1)}\Big(\sum_{j=0}^{b-1}r_p(D)^{pj}\Big).$$

Thus, for $m\ge b$ there is a constant $K>1$ such that

$$\|T^m\|_p^{1/m}\le K^{1/m}(m^{1/m})^{2b}r_p(D)^{(m-b+1)/m},$$

which implies that

$$\varrho_p(T)\le r_p(D)=r_p(T),$$

and by Proposition 3 we obtain

$$\varrho_p(T)=r_p(D)=r_p(T).$$

Let us now return to the general case. For an n-tuple T as specified after Proposition 2, and for $s=1$, we have just seen that $\varrho_p(T)=r_p(T)$. We now prove the general case. Let us begin by the following observation. By Proposition 2, as we may assume that the subspaces M_j $(1\le j\le s)$, are generated by elements of the canonical basis of $\mathbb{C}^d$, if $y=(y_j)_{1\le j\le s}\in\mathbb{C}^d$ and $\|y\|_p\le1$, then

$$\|T_{i_1}\dots T_{i_m}(y)\|_p^p=\Big\|\sum_{1\le j\le s}T_{i_1}\dots T_{i_m}(y_j)\Big\|_p^p=\sum_{1\le j\le s}\|T_{i_1}\dots T_{i_m}(y_j)\|_p^p.$$

Now, as $\|y_j\|_p\le\|y\|_p\le1$ for $1\le j\le s$, we may proceed as follows:

$$\begin{aligned}\|T^m\|_p&=\sup_{\|y\|_p\le1}\Big(\sum_{(i_1,\dots,i_m)\in I_m}\|T_{i_1}\dots T_{i_m}(y)\|_p^p\Big)^{1/p}\\&\le\sup_{\|y\|_p\le1}\Big(\sum_{(i_1,\dots,i_m)\in I_m}\ \sum_{1\le j\le s}\|T_{i_1}\dots T_{i_m}(y_j)\|_p^p\Big)^{1/p}\\&\le\sum_{1\le j\le s}\|(T^j)^m\|_p\le s\|(T^{j_0})^m\|_p,\end{aligned}$$

where T^j is the n-tuple defined by $T_i^j=T_i|M_j$ $(1\le i\le n)$, and j_0 is such that $\|(T^j)^m\|_p\le\|(T^{j_0})^m\|_p$ for all j.

Thus, if $d_{j_0}=\dim M_{j_0}$, then there is a constant K_0 such that

$$\begin{aligned}\|T^m\|_p^{1/m}&\le s^{1/m}K_0^{1/m}(m^{1/m})^{2d_{j_0}}r_p(D^{j_0})^{(m-d_j+1)/m}\\&\le s^{1/m}K_0^{1/m}(m^{1/m})^{2d_{j_0}}r_p(D)^{(m-d_j+1)/m},\end{aligned}$$

which implies that

$$\varrho_p(T)\le r_p(D)=r_p(T).$$

Thus, $\varrho_p(T)=r_p(T)$.

We have thus proved our main theorem.

THEOREM 1. *Let L be a complex nilpotent Lie algebra of linear transformations in $\mathcal{L}(\mathbb{C}^d)^{\mathrm{op}}$, and $\{x_i\}_{1\le i\le n}$ a Jordan–Hölder basis of L. Then*

$$\varrho_p((x_i)_{1\le i\le n})=r_p((x_i)_{1\le i\le n}).$$

Moreover, if $T=(T_1,\dots,T_n)$ is an n-tuple of commuting $d\times d$ matrices, then

$$\varrho_p(T)=r_p(T),$$

where $1\le p\le\infty$. ∎

Finally, we consider several examples to show that our result fails in the solvable not nilpotent case.

Consider a complex solvable finite-dimensional Lie algebra of linear transformations, L, acting on a complex finite-dimensional vector space E. By [5; IV,2], as in the nilpotent case, we may construct a Jordan–Hölder sequence of ideals and a Jordan–Hölder basis for L (see Section 2). However, in the solvable not nilpotent case, Theorem A is no longer true, i.e., if $\dim L = n$ and $\{x_i\}_{1\le i\le n}$ is a Jordan–Hölder basis of L, then $\mathrm{Sp}((x_i)_{1\le i\le n}, E) \neq \sigma_{\mathrm{pt}}((x_i)_{1\le i\le n})$. Thus, we may consider $\max\{|\lambda|_p : \lambda \in \mathrm{Sp}((x_i)_{1\le i\le n}, E)\}$ instead of $r_p((x_i)_{1\le i\le n})$, and try to see if $\varrho_p((x_i)_{1\le i\le n}) = \max\{|\lambda|_p : \lambda \in \mathrm{Sp}((x_i)_{1\le i\le n}, E)\}$. However, as we shall see, this equality also fails.

In the following examples we consider the space $E = \mathbb{C}^2$, and the solvable not nilpotent algebras we work with are representations in $\mathcal{L}(\mathbb{C}^2)^{\mathrm{op}}$ of the two-dimensional solvable algebra $L = \langle y\rangle \oplus \langle x\rangle$ with bracket $[x, y] = y$. Observe that (y, x) is a Jordan–Hölder basis of L.

As our first example consider the algebra $L_1 \subseteq \mathcal{L}(\mathbb{C}^2)^{\mathrm{op}}$ with Jordan–Hölder basis

$$y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Then

$$\sigma_{\mathrm{pt}}((y, x)) = \{(0, -1/2)\}, \quad \mathrm{Sp}((y, x), E) = \{(0, 1/2), (0, -3/2)\},$$

and

$$r_\infty((y, x)) = 1/2 = \varrho_\infty((y, x)), \quad \text{but} \quad \max\{|\lambda|_\infty : \lambda \in \mathrm{Sp}((y, x), E)\} = 3/2.$$

On the other hand, if $L_2 \subseteq \mathcal{L}(\mathbb{C}^2)^{\mathrm{op}}$ is the algebra with Jordan–Hölder basis,

$$y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix},$$

we obtain

$$\sigma_{\mathrm{pt}}((y, x)) = \{(0, 2)\}, \quad \mathrm{Sp}((y, x), E) = \{(0, 1), (0, 3)\},$$

and

$$r_\infty((y, x)) = 2, \quad \text{but} \quad \varrho_\infty((y, x)) = 3 = \max\{|\lambda|_\infty : \lambda \in \mathrm{Sp}((y, x), E)\}.$$

Our last example is the algebra $L_3 \subseteq \mathcal{L}(\mathbb{C}^2)^{\mathrm{op}}$ generated by

$$y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} -1/3 & 0 \\ 0 & 2/3 \end{pmatrix}.$$

Then

$$\sigma_{\mathrm{pt}}((y, x)) = \{(0, -1/3)\}, \quad \mathrm{Sp}((y, x), E) = \{(0, -4/3), (0, 2/3)\},$$

and

$$r_\infty((y, x)) = 1/3, \quad \varrho_\infty((y, x)) = 2/3,$$
$$\max\{|\lambda|_\infty : \lambda \in \mathrm{Sp}((y, x), E)\} = 4/3.$$

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References

[1] R. Bhatia and T. Bhattacharyya, *On the joint spectral radius of commuting matrices*, Studia Math. 114 (1995), 29–38.
[2] E. Boasso, *Dual properties and joint spectra for solvable Lie algebras of operators*, J. Operator Theory 33 (1995), 105–116.
[3] —, *Joint spectra and nilpotent Lie algebras of Linear transformations*, Linear Algebra Appl. 263 (1997), 49–62.
[4] E. Boasso and A. Larotonda, *A spectral theory for solvable Lie algebras of operators*, Pacific J. Math. 158 (1993), 15–22.
[5] N. Bourbaki, *Éléments de Mathématique, Groupes et Algèbres de Lie, Algèbres de Lie Fasc. XXVI*, Hermann, 1960.
[6] M. Chō and T. Huruya, *On the joint spectral radius*, Proc. Roy. Irish Acad. Sect. A 91 (1991), 39–44.
[7] M. Chō and M. Takaguchi, *Identity of Taylor's joint spectrum and Dash's joint spectrum*, Studia Math. 70 (1982), 225–229.
[8] N. Jacobson, *Lie Algebras*, Interscience Publ., 1962.
[9] A. McIntosh, A. Pryde and W. Ricker, *Comparison of joint spectra for certain classes of commuting opertors*, Studia Math. 88 (1988), 23–36.
[10] C. Ott, *A note on a paper of E. Boasso and A. Larotonda*, Pacific J. Math. 173 (1996), 173–179.
[11] Z. Słodkowski, *An infinite family of joint spectra*, Studia Math. 61 (1973), 239–235.
[12] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. 6 (1970), 172–191.

Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria, Pab. I
1428 Buenos Aires, República Argentina
E-mail: eboasso@dm.uba.ar

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