Contents of Volume 132, Number 1

STUDIA MATHEMATICA

Executive Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: studia@impan.gov.pl

Subscription information (1999): Vols. 132–137 (18 issues); $33.50 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997
E-mail: publ@impan.gov.pl

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset using TEX at the Institute
Printed and bound by

drukarnia
herman & herman

PRINTED IN POLAND

ISSN 0039-3223

Rational interpolants with preassigned poles, theoretical aspects

by

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Abstract. Let f be an analytic function on a compact subset K of the complex plane $\mathbb{C}$, and let $r_n(z)$ denote the rational function of degree n with poles at the points $\{b_{ni}\}_{i=1}^n$ and interpolating f at the points $\{a_{ni}\}_{i=0}^n$. We investigate how these points should be chosen to guarantee the convergence of r_n to f as $n \to \infty$ for all functions f analytic on K. When K has no "holes" (see [8] and [3]), it is possible to choose the poles $\{b_{ni}\}_{i,n}$ without limit points on K. In this paper we study the case of general compact sets K, when such a separation is not always possible. This fact causes changes both in the results and in the methods of proofs. We consider also the case of functions analytic in open domains. It turns out that in our general setting there is no longer a "duality" ([8], Section 8.3, Corollary 2) between the poles and the interpolation points.

1. Introduction. Let f be an analytic function on a set $A \subset \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and let, for each $n \geq 1$, $A_n = \{a_{ni}\}_{i=0}^n \subset A$ and $B_n = \{b_{ni}\}_{i=1}^n \subset \overline{\mathbb{C}}$, $A_n \cap B_n = \emptyset$, be two sets of points. Then (see [8], §8.1) there exists a unique rational function r_n of degree at most n (i.e. the degrees of the denominator and the numerator are at most n) with poles at the points of B_n (counting multiplicities) interpolating f at the points of A_n, i.e. the points of A_n are zeros of $R_n := f - r_n$ in $\overline{\mathbb{C}}$ (counting multiplicities).

Our general problem here is how to choose A_n and B_n, $n = 1, 2, \ldots$, to guarantee the convergence of r_n to f on A as $n \to \infty$ for all analytic functions f on A. We treat the case when A is open or closed. When A is open (Theorem 1) we assume that $\bigcup_{n\geq 1} B_n$ has no limit point in A, and when A is closed (Theorem 3) we assume that $B_n \subset \mathbb{C} \setminus A$ for all n. This paper is an independent continuation of [2].

We start with some general observations when $A = K$ is a closed subset of $\overline{\mathbb{C}}$, $K \neq \overline{\mathbb{C}}$, and f is analytic on K. If K has no "holes", that is, if $\overline{\mathbb{C}} \setminus K$ is connected, then it is possible to get convergence when we choose the set $\bigcup_{n\geq 1} B_n$ separated from K (see [8], [3] and [1]). Moreover, in this case f

1991 *Mathematics Subject Classification*: 30E10, 41A21, 41A05.

can be approximated by polynomials in z if $\infty \notin K$, and in $1/(z-b)$ for some $b \in \overline{\mathbb{C}} \setminus K$ otherwise.

The situation when rational functions arise in a natural way is when we allow K to have holes. Then each hole should contain points from B_n for all sufficiently large n in order to be able to get convergence when we interpolate, for example, functions of the type $1/(z-b), b \in \overline{\mathbb{C}} \setminus K$. Consequently, if the number of "holes" is infinite, then the set $\bigcup_{n\geq 1} B_n$ has a limit point on K. The case of compact sets with a finite number of "holes" is studied in [2].

This discussion shows that for general compact sets K we cannot any longer separate the poles $\{b_{ni}\}$ from K. We also note that even for compact sets with connected complements (K has no "holes") such a separation is unnecessarily restrictive as is shown by the following example.

EXAMPLE 1. Let $\Delta = \{|z| \leq 1\}$ and $A_n = \{a_{ni}\}_{i=0}^n \subset \{|z| < 1\}, n \geq 1$. Suppose, for example, that $a_{ni} = 0$ when $i = 0$, and choose $B_n = \{b_{ni} = 1/\overline{a}_{ni}\}_{i=1}^n$, i.e. b_{ni} and a_{ni} are symmetric with respect to the unit circle. Suppose also that $\bigcup_{n\geq 1} A_n$ is separated from the boundary $\partial\Delta$ of Δ in the sense that $\bigcup_{n\geq 1} A_n$ has no limit point on $\partial\Delta$. Consequently, the set $\bigcup_{n\geq 1} B_n$ of poles is also separated from $\partial\Delta$. Then (see Example 3 and Theorem 1 in [2]) for any analytic function f on Δ the corresponding rational approximants r_n converge to f in Δ. It turns out (see Theorem 3 below) that the separation condition is unnecessary in this example—for the convergence to hold it is enough that the poles and interpolation points are symmetric.

Now we discuss the convergence problem for functions analytic in domains and compare it with the case of closed sets. If in Example 1 we take the open unit disk instead of the closed disk and consider functions f analytic in the open disk, then the symmetry of A_n and B_n no longer guarantees the convergence of r_n to f in the open disk. For the convergence to hold we need the separation condition. Moreover, we show below (see Theorem 2) that separation of $\bigcup_{n\geq 1} A_n$ from $\{|z| = 1\}$ is a necessary condition not only in general, but also for any particular choice of $A_n, n \geq 1$.

It might seem that the case when A is an open disk should be easier than that of a closed disk. For the open disk we investigate convergence on compact subsets of the disk and any such compact subset is separated from the poles $\{b_{ni}\}$ if they have no limit point in A (Theorem 1). We do not assume this separation for the closed disk (Theorem 3). Moreover, as in the latter case, a function f analytic in an open disk is bounded on compact subsets. However, the crucial point here is that f can be unbounded on the whole open disk and we may get "wrong" information by interpolating f at points close to the boundary.

The fact that for general sets A, closed or open, we allow the points $\{a_{ni}\}$ and $\{b_{ni}\}$ to come close to the boundary of A (no separation) causes some difficulties which are illustrated by condition (2) in Theorem 1 and the corresponding condition in Theorem 3. In the case of separation (see [3] and [2]) a necessary and sufficient condition for convergence of the rational interpolants is that the limit distributions of the points in A_n and B_n, as $n \to \infty$, are equal after sweeping out onto the boundary of A. This condition is no longer sufficient in the general case. If, for example, f is an analytic function on a compact set $K \subset \mathbb{C}$, and if we choose one of the poles $\{b_{ni}\}_{i=1}^n$ sufficiently close to K (after having chosen all the other poles and all the interpolation points $\{a_{ni}\}_{i=0}^n$), then it is easy to check that the corresponding rational interpolant r_n can be arbitrarily large on K. This simple argument does not work for the case of functions analytic in domains; we discuss this case separately in Example 2 at the end of Section 4.

There are also technical difficulties in the case of non-separated poles and interpolation points. Firstly, in this case the union of $\bigcup_{n\geq 1} A_n$ and $\bigcup_{n\geq 1} B_n$ can be everywhere dense in $\overline{\mathbb{C}}$. This prevents us from applying the standard technique of Möbius transformations (see, for example, [3] or [2]) mapping $\bigcup A_n$ and $\bigcup B_n$ to disjoint compact sets in $\mathbb{C}$, and from introducing logarithmic potentials of the counting measures (see below) of A_n and B_n. Secondly, and this is more important, the integration contour Γ_n in the error formula (10) below becomes dependent on n since all the interpolation points in A_n should be inside Γ_n. This means that we have to control the length of Γ_n. For this purpose we choose Γ_n to be a level curve of some rational function and prove a lemma (see Lemma 3 below) about the length of the level curves of rational functions. The lemma has, in our opinion, an independent interest; it is similar in spirit to Spijker's lemma (see [9]), but, in contrast to it, Lemma 3 estimates the length of the preimages of circles on the plane (that is, the length of level curves) under rational functions, rather than of their images as in Spijker's lemma. (We do not even know how sharp the estimate in Lemma 3 is and which rational function is extremal for this estimate; evidently it is not z^n as in the case of Spijker's lemma.) We thank the referee for the reference to Spijker's lemma.

2. Definitions and notation. We use the following notation:

$\overline{\mathbb{C}}$:	The extended complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
D:	A regular domain (open, connected set) in $\overline{\mathbb{C}}$.
K:	A closed set in $\overline{\mathbb{C}}, K \neq \overline{\mathbb{C}}$.
f:	A function analytic on D or K.
∂D:	The boundary of D.
$\mathring{K}$:	The interior of K.

$\overline{D}$: The closure of D, $\overline{D} = D \cup \partial D$.

A_n, B_n: Sets of points in $\overline{\mathbb{C}}$, $A_n = \{a_{ni}\}_{i=0}^n$, $B_n = \{b_{ni}\}_{i=1}^n$.

δ_z: The Dirac measure: $\delta_z(\overline{\mathbb{C}}) = \delta_z(\{z\}) = 1$.

α_n, β_n: The normalized counting measures of the sets A_n and B_n; for example $\alpha_n = (\sum_{i=0}^n \delta_{a_{ni}})/(n+1)$.

$\alpha_n \to \alpha$: Weak-star convergence of measures: $\int \varphi\, d\mu_n \to \int \varphi\, d\mu$ for every continuous function φ on $\overline{\mathbb{C}}$.

$\operatorname{supp}(\alpha)$: The support of a measure α.

Q_n: $Q_n(z) = \prod_{i=1}^n (z - b_{ni})$ (the factors corresponding to $b_{ni} = \infty$ are omitted; if $b_{ni} = \infty$ for all $i = 1, \ldots, n$, then $Q_n = 1$).

H_n: $H_n(z) = \prod_{i=0}^n (z - a_{ni})$ (with the same remark as for $Q_n(z)$).

$\sigma(P)$: The zero-counting measure (not normalized) of a polynomial P: $\sigma(P) = \sum_a \delta_a$, where a ranges over the set of zeros of P (counting multiplicity).

$r_n(z)$: The rational function of degree at most n, with poles at the points of B_n, interpolating a given function f at the points of A_n. (See Section 1 for more details.)

$U_\alpha(z)$: The logarithmic potential of a measure α,

$$U_\alpha(z) = -\int \log|z-t|\, d\alpha(t).$$

Sweeping out. We define sweeping out by using the solution to Dirichlet's problem and the Riesz representation theorem. Let Ω be an open regular set in $\overline{\mathbb{C}}$; that is, each connected component of Ω is regular (for the definition of regular domains, see for example [5], p. 88). Then (see [5], Cor. 4.1.8) any continuous function g on $\partial\Omega$ has a unique harmonic continuation u_g inside Ω. Let μ be a finite positive measure with $\operatorname{supp}(\mu) \subset \overline{\Omega}$. We define a linear functional L by

$$g \mapsto Lg = \int u_g\, d\mu$$

on the space $C(\partial\Omega)$ of continuous functions g on $\partial\Omega$. By using the maximum principle for harmonic functions we can easily check that L is bounded. Then, by the Riesz representation theorem ([6], Chapter 2), there exists a unique measure μ' with $\operatorname{supp}(\mu') \subset \partial\Omega$ such that

$$Lg = \int g d\mu' \quad \text{for all } g \in C(\partial\Omega).$$

DEFINITION. μ' is the *sweeping out* (balayage) of μ onto $\partial\Omega$. (If μ is not supported in $\overline{\Omega}$, then by μ' we understand the sum of the measures obtained by sweeping out μ onto $\partial\Omega$ from all the connected components of $\overline{\mathbb{C}} \setminus \partial\Omega$.)

When $\partial\Omega$ and $\operatorname{supp}(\mu)$ are compact subsets of $\mathbb{C}$ the sweeping out process has an interpretation with potentials (see for instance [4], Chapter IV, or [7], Appendix VII):

$$U_{\mu'}(z) \le U_\mu(z) + c(\mu) \quad \text{for all } z \in \mathbb{C}$$

where equality holds for all $z \notin \Omega$ and $c(\mu)$ is a non-negative constant: $c(\mu) = \int_G g(z;\infty)\, d\mu(z)$, where $g(z;\infty)$ is the Green function of the connected component G of Ω containing the point at infinity. If in particular Ω is a bounded set in $\mathbb{C}$, then $c(\mu) = 0$.

Now, let a sequence of measures μ_n converge in the weak-star sense to a measure μ, as $n \to \infty$. Let L_n denote the corresponding linear functionals. Then, for all $g \in C(\partial\Omega)$, by the definition of weak-star convergence, $L_n g \to Lg$. Consequently, $L_n g = \int g\, d\mu_n' \to \int g\, d\mu'$ where μ_n' is the sweeping out of μ_n. By weak-star convergence the last limit relation means that $\mu_n' \to \mu'$.

3. Results. We start with the case of functions analytic in a domain D. We suppose that ∂D is bounded in $\mathbb{C}$ (otherwise we would apply a Möbius transformation of the form $1/(z-a)$, $a \in D$).

THEOREM 1. *Let $D \subset \overline{\mathbb{C}}$ be a regular domain with bounded boundary and let, for each $n \ge 1$, $A_n = \{a_{ni}\}_{i=0}^n \subset D$ and $B_n = \{b_{ni}\}_{i=1}^n$ be such that $\bigcup_{n\ge1} B_n$ has no limit point in D. Denote by α_n and β_n the normalized counting measures of these sets* (see Section 2) *and by α_n' and β_n' their sweeping out measures on ∂D. Assume that for any measure α which is a weak-star limit point of the set $\{\alpha_n\}$, we have*

$$\alpha(D) > 0. \tag{1}$$

Furthermore, assume that

$$\lim_{n\to\infty} [\sup_{z\in\partial D} (U_{\alpha_n'}(z) - U_{\beta_n'}(z))] = 0. \tag{2}$$

Then, for any bounded analytic function f in D the corresponding rational interpolant r_n with poles at $\{b_{ni}\}_{i=1}^n$, interpolating f at $\{a_{ni}\}_{i=0}^n$, converges to f in D:

$$r_n \to f \quad \text{as } n \to \infty$$

uniformly, with geometric degree of convergence, on compact subsets of D.

REMARK 1. Roughly speaking, condition (2) means that the points a_{ni} do not converge to ∂D too fast. This condition also implies (see Lemma 1 below) that if for some subsequence $\{n_i\}$ the measures α_{n_i} and β_{n_i} have weak-star limits α and β, then their sweeping out measures on ∂D coincide: $\alpha' = \beta'$. On the other hand, as remarked in the introduction, this last condition cannot substitute (2).

Condition (1) is needed to guarantee a geometric degree of convergence.

The next result shows that it is necessary in Theorem 1 to assume that f is bounded when we allow the points $\{a_{ni}\}$ to converge to the boundary of D; no restriction on the rate of this convergence can replace the boundedness condition on f.

THEOREM 2. *Let* $D = \{|z| < 1\}$ *and let, for each* $n \geq 1$, $A_n = \{a_{ni}\}_{i=0}^{n} \subset D$ *and* $B_n = \{b_{ni}\}_{i=1}^{n} \subset \overline{\mathbb{C}}$ *be two disjoint sets such that* $\bigcup_{n\geq 1} A_n$ *has a limit point on* $\{|z| = 1\}$. *Then, for every* $z_0 \in D \setminus \bigcup_{n\geq 1} A_n$, *there exists an analytic function* f *in* D *such that, for the corresponding rational interpolants* r_n, *we have*

$$\limsup_{n\to\infty} |f(z_0) - r_n(z_0)| = \infty. \tag{5}$$

Coming back to Theorem 1 and Remark 1, we show in Example 2 (see Section 4) that the condition $\alpha' = \beta'$ alone is not enough for the convergence to hold. This is different from the case when the interpolation points $\{a_{ni}\}$ are separated from ∂D (see [2]).

Now we turn to the case of functions analytic on a closed set $K \subset \overline{\mathbb{C}}$, $K \neq \overline{\mathbb{C}}$. Without loss of generality we may assume that K is a compact subset of $\mathbb{C}$ (otherwise we would apply a suitable Möbius transformation). We also assume that K is connected; this is for convenience rather than for necessity and it corresponds to the fact that in Theorem 1 above we consider functions analytic in domains, not in open sets. We assume that the interior of K, $\mathring{K}$, is empty or a regular set for Dirichlet's problem. This is in contrast to the analogous theorems in [8] and [3], where connectedness and regularity of $\overline{\mathbb{C}} \setminus K$ is assumed. In those papers $\mathring{K}$ is automatically regular, and in our case connectedness of K implies the regularity of $\overline{\mathbb{C}}\setminus K$ (which means the regularity of each connected component). In the general setting for functions analytic on a set A, closed or open, we should demand that all connected components of the complement of the boundary of A are regular, or, equivalently, that the boundary of A is thin at none of its points (see, for example, [4]).

THEOREM 3. *Let* $K \subset \overline{\mathbb{C}}$, $K \neq \overline{\mathbb{C}}$, *be a connected closed set whose interior is either empty or regular. For each* $n \geq 1$, *let* $A_n = \{a_{ni}\}_{i=0}^{n} \subset K$ *and* $B_n = \{b_{ni}\}_{i=1}^{n} \subset \overline{\mathbb{C}} \setminus K$. *Denote by* α_n *and* β_n *the normalized counting measures of these sets, and by* α_n' *and* β_n' *their sweeping out measures on* ∂K. *Assume that, for any measure* β *which is a weak-star limit point of the set* $\{\beta_n\}$, *we have*

$$\beta(B) > 0$$

for each connected component B *of* $\overline{\mathbb{C}} \setminus K$. *Furthermore, assume that*

$$\lim_{n\to\infty} [\inf_{z\in\partial K} (U_{\alpha_n'}(z) - U_{\beta_n'}(z))] = 0.$$

Then, for any analytic function f *on* K, *the corresponding rational interpolants* r_n *with poles at* $\{b_{ni}\}_{i=1}^{n}$ *interpolating* f *at* $\{a_{ni}\}_{i=0}^{n}$ *converge to* f *uniformly on* K *with geometric degree of convergence.*

4. Proofs. In this section we prove Theorems 1 and 2. The proof of Theorem 3 is analogous to that of Theorem 1.

Proof of Theorem 1. We prove the theorem for an unbounded domain D (then we have $\infty \in D$) which is technically the most complicated case. In this case $\bigcup_{n\geq 1} B_n$ is bounded. We first assume that the set $\bigcup_{n\geq 1} A_n$ is also bounded; later we give necessary comments on the general case. Without loss of generality we may assume that the corresponding counting measures α_n and β_n have weak-star limits, α and β, respectively.

LEMMA 1. *Under the conditions of Theorem* 1, *assume that* $\alpha_n \to \alpha$ *and* $\beta_n \to \beta$ *in the weak-star sense. Then the sweeping out measures of* α *and* β *on* ∂D *are equal.*

Proof. Let μ be any measure (finite, positive) on ∂D with continuous logarithmic potential. From (2) it follows that

$$\limsup_{n\to\infty} \int (U_{\alpha_n'}(z) - U_{\beta_n'}(z))\, d\mu(z) \leq 0.$$

By Fubini's theorem

$$\limsup_{n\to\infty} \int U_\mu(z)\, d(\alpha_n' - \beta_n')(z) \leq 0.$$

The condition $\alpha_n \to \alpha$ and $\beta_n \to \beta$ yields that $\alpha_n' \to \alpha'$ and $\beta_n' \to \beta'$, where α' and β' denote the sweeping out measures on ∂D of α and β, respectively (see Section 2).

From the last inequality, the definition of weak-star convergence (see Section 2) and continuity of $U_\mu(z)$ we get

$$\int U_\mu(z)\, d(\alpha' - \beta')(z) \leq 0.$$

Now we want to extend the last inequality from potentials U_μ to all positive continuous functions on ∂D. This will give $\alpha' - \beta' \leq 0$ (see [4], Theorem 0.2′), and together with $\alpha'(\partial D) = \beta'(\partial D) = 1$ we will get $\alpha' = \beta'$.

We prove that such an extension is possible using a lemma about uniform approximation of positive continuous functions on ∂D by absolute values of polynomials (see [7], Lemma 3.2.4). From that lemma we deduce that if g is a positive continuous function on ∂D then we can approximate e^{-g} on ∂D by $|P|$, where P is a polynomial, that is, g can be approximated on ∂D by $-\ln|P|$. We can assume that $P \neq 0$ on ∂D, i.e. $-\ln|P|$ is continuous on ∂D. The function $-\ln|P|$ is the potential of the zero-counting measure $\sigma = \sigma(P)$ of P. Denote by σ' the sweeping out measure of σ on ∂D. By the regularity of the domain D, and the regularity of $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ as complement of a connected set, the potential $U_{\sigma'}$ differs from U_σ by a constant on ∂D (see Section 2). In particular, $U_{\sigma'}$ is continuous. Hence, any positive continuous function on ∂D can be uniformly approximated on ∂D by potentials of measures supported by ∂D, plus constants. But the constants cancel out in the inequality $\int U_\mu d(\alpha' - \beta') \leq 0$ since $\alpha'(\partial D) = \beta'(\partial D)$.

Lemma 1 is proved.

From Lemma 1 it follows that (see Section 2)

$$U_\beta(z) - U_\alpha(z) \equiv c, \quad z \in \partial D, \tag{6}$$

where c is a constant which is positive since $\alpha(D) > 0$ by (1) (see [7], Appendix VII). Together with (6) we will use later the fact that, by the maximum principle,

$$U_\beta(z) - U_\alpha(z) < c, \quad z \in D, \tag{7}$$

and

$$U_\beta(z) - U_\alpha(z) \geq c, \quad z \in \overline{\mathbb{C}} \setminus \overline{D}. \tag{8}$$

(7) and (8) follow from sub- and superharmonicity of $U_\beta(z) - U_\alpha(z)$ in D and $\overline{\mathbb{C}} \setminus \overline{D}$, respectively. Furthermore, for all $z \in \partial D$, $U_{\beta_n}(z) - U_{\alpha_n}(z) = U_{\beta'_n}(z) - U_{\alpha'_n}(z) + c_n$, where $c_n \to c$ as $n \to \infty$ (see [7], Appendix VII). From this and (2) we conclude that

$$\lim_{n\to\infty} \big(\inf_{z \in \partial D} (U_{\beta_n}(z) - U_{\alpha_n}(z)) \big) = c. \tag{9}$$

Now we fix a compact (in $\overline{\mathbb{C}}$) set $F \subset D$. Let D_n be any open set with $\overline{D}_n \subset D$ containing the set F and the points A_n, with rectifiable boundary $\Gamma_n = \partial D_n$. We may easily generalize to this case (by using the deformation invariance theorem for contours) the standard error formula (see [8], p. 186),

$$R_n(z) := f(z) - r_n(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{H_n(z) \cdot Q_n(t)}{H_n(t) \cdot Q_n(z)} \cdot \frac{f(t)\,dt}{t - z}, \qquad z \in F, \tag{10}$$

where $H_n(z) = \prod_{i=0}^n (z - a_{ni})$, $Q_n(z) = \prod_{i=1}^n (z - b_{ni})$, and $\Gamma_n = \partial D_n$ has positive orientation with respect to the domain D.

Since there exists a constant M such that $|f(z)| < M$ for $z \in D$, from (10) we get, for all $z \in F$,

$$|2\pi R_n(z)|^{1/n} \leq \left| \frac{H_n(z)}{Q_n(z)} \right|^{1/n} \cdot \sup_{t \in \Gamma_n} \left| \frac{Q_n(t)}{H_n(t)} \right|^{1/n} \cdot \left(\frac{M}{\operatorname{dist}(F, \Gamma_n)} \right)^{1/n} \cdot (\ell(\Gamma_n))^{1/n},$$

where $\ell(\Gamma_n)$ denotes the length of Γ_n, and $\operatorname{dist}(F, \Gamma_n)$ the distance between F and Γ_n. By taking the logarithm of both sides we get, for $z \in F$,

$$\ln |2\pi R_n(z)|^{1/n} \leq (U_{\beta_n}(z) - U_{\alpha_n}(z)) + \sup_{t \in \Gamma_n} (U_{\alpha_n}(t) - U_{\beta_n}(t)) + \frac{1}{n} \ln M - \frac{1}{n} \ln(\operatorname{dist}(K, \Gamma_n)) + \frac{1}{n} \ln(\ell(\Gamma_n)). \tag{11}$$

Now we need:

Lemma 2. *For any compact (in $\overline{\mathbb{C}}$) set $F \subset D$, there exists an $\varepsilon = \varepsilon(F) > 0$ such that*

$$U_{\beta_n}(z) - U_{\alpha_n}(z) \leq c - \varepsilon, \qquad z \in F, \tag{12}$$

for all sufficiently large n, where c is the constant in (6).

Proof. Consider a domain V with $\overline{V} \subset D$ and $F \subset V$. By the upper semicontinuity of subharmonic functions, (7) holds uniformly on closed subsets of D. In particular, for some $\varepsilon > 0$ we have $U_\beta(z) - U_\alpha(z) \leq c - 2\varepsilon$ for $z \in V$. By the principle of descent (see [7], Appendix III) we get

$$\limsup_{\substack{n\to\infty \\ z_n \to z}} (U_{\beta_n}(z_n) - U_{\alpha_n}(z_n)) \leq c - 2\varepsilon$$

for all $z \in V$. But this locally uniform convergence in V is equivalent (see [7], Section 1.1) to the uniform convergence on compact subsets of V. This proves Lemma 2.

Let ε be defined by Lemma 2 and fix δ, $0 < \delta < \varepsilon$. Now we choose D_n to be $D_n = \{z : U_{\beta_n}(z) - U_{\alpha_n}(z) < c - \delta\}$. From (9) and the minimum principle for the superharmonic function $U_{\beta_n}(z) - U_{\alpha_n}(z)$ in $\overline{\mathbb{C}} \setminus \overline{D}$ it follows that $D_n \subset D$ for all sufficiently large n. Furthermore, since the measures β_n and α_n have finite supports, the function $U_{\beta_n}(z) - U_{\alpha_n}(z)$ is continuous on the Riemann sphere. Consequently, D_n is an open set and

$$U_{\beta_n}(z) - U_{\alpha_n}(z) = c - \delta \tag{13}$$

on the boundary Γ_n of D_n. From (12) we see that $F \subset D_n$ and $F \cap \Gamma_n = \emptyset$ for all sufficiently large n.

Since F is arbitrary, we may suppose that everything in the discussion above is also true for another compact (in $\overline{\mathbb{C}}$) set $F_1 \subset D$ containing F and the point at infinity in its interior. In particular, we assume that $\infty \in F_1 \subset D_n$ for large n. This implies that the diameter of Γ_n is bounded as $n \to \infty$, which in turn implies

$$\limsup_{n\to\infty} [\operatorname{diam}(\Gamma_n)]^{1/n} \leq 1. \tag{14}$$

In addition we have

$$\limsup_{n\to\infty} [\operatorname{dist}(F, \Gamma_n)]^{1/n} = 1. \tag{15}$$

We may assume that $A_n \cap B_n = \emptyset$ (otherwise we cancel common terms). Consequently, $U_{\beta_n}(a_{ni}) - U_{\alpha_n}(a_{ni}) = -\infty$, $i = 0, 1, \ldots, n$. This means that $A_n \subset D_n$. Hence, D_n and its boundary Γ_n satisfy all conditions in the error formula (10).

From (13) we can see that Γ_n is a level curve of the rational function $H_n(z)/Q_n(z)$: $\Gamma_n = \{z : |H_n(z)/Q_n(z)| = e^{n(c-\delta)}\}$.

Now we need the following lemma.

Lemma 3. *Let $R(z) = A(z)/B(z)$ be a rational function of order n, $\max\{\deg A, \deg B\} = n$. Let $\varrho \geq 0$ and $\gamma_\varrho = \{z : |R(z)| = \varrho\}$ be a level* *curve. Denote by $d(\gamma_\varrho)$ and $\ell(\gamma_\varrho)$ the diameter and total length of γ_ϱ, respectively. Then*

$$\ell(\gamma_\varrho) \le 4nd(\gamma_\varrho).$$

Proof. The points of γ_ϱ satisfy the equation $|A(z)|^2 - \varrho^2|B(z)|^2 = 0$, or $A(z)\overline{A(z)} - \varrho^2 B(z)\overline{B(z)} = 0$. If we put $z = x+iy$ this reduces to $P(x,y) = 0$, where $P(x,y)$ is a polynomial in x and y with real coefficients of degree at most $2n$.

Suppose $d(\gamma_\varrho) < \infty$ (otherwise Lemma 3 is trivially satisfied). Then γ_ϱ is contained in a square with sides of length $d(\gamma_\varrho)$ and parallel to the coordinate axes.

Fix $y = y_0$ and consider the equation $P(x, y_0) = 0$ which has at most $2n$ solutions. This means that any horizontal (and, similarly, vertical) line intersects γ_ϱ at $2n$ points at most.

Let us introduce a natural arc length parametrization s on γ_ϱ. Then

$$\ell(\gamma_\varrho) = \int_{\gamma_\varrho} ds \le \int_{\gamma_\varrho} |dx(s)| + \int_{\gamma_\varrho} |dy(s)|.$$

But $x(s)$, and similarly $y(s)$, takes each value at most $2n$ times and has range in an interval of length $d(\gamma_\varrho)$. Consequently,

$$\int_{\gamma_\varrho} |dx(s)| \le 2n \cdot d(\gamma_\varrho) \quad \text{and} \quad \int_{\gamma_\varrho} |dy(s)| \le 2n \cdot d(\gamma_\varrho),$$

which proves Lemma 3.

Lemma 3 and (14) give

$$\limsup_{n\to\infty} [\ell(\Gamma_n)]^{1/n} \le 1. \tag{16}$$

Now, finally, from (11) together with (12), (13), (15) and (16) it follows that

$$\limsup_{n\to\infty} \ln |R_n(z)|^{1/n} \le c - \varepsilon + \delta - c = \delta - \varepsilon < 0,$$

uniformly for $z \in F$. This proves Theorem 1 under the assumption that $\bigcup_{n\ge1} A_n$ is bounded.

Now we consider the general case. The main idea is to move the "remote" points $\{a_{ni}\}$ to a fixed circle.

We suppose again that $\alpha_n \to \alpha$. As above, we fix a compact (in $\overline{\mathbb{C}}$) set $F \subset D$. We may assume that the point at infinity is an interior point of F. In particular, this means that ∂F is a bounded set in $\mathbb{C}$.

Set $D_R = \{|z| > R\}$, where R is sufficiently large to guarantee that $D_R \subset D$. In addition, suppose that $\alpha(\partial D_R) = 0$. Denote by α_R the part of the measure α concentrated in D_R, and let α_R' be its sweeping out onto ∂D_R. Note that if we now sweep out α_R' onto ∂D, we get the measure which is the

sweeping out of α_R onto ∂D (this follows from the definition of sweeping out in Section 2).

Let $\{a_{ni}\}_{i=0}^{m(n)} = A_n \cap D_R$. Then the measures $\frac{1}{n}\sum_{i=0}^{m(n)} \delta_{a_{ni}}$ converge weakly to α_R as $n \to \infty$. We may choose $\{a'_{ni}\}_{i=0}^{m(n)} \subset \partial D_R$ so that the measures $\frac{1}{n}\sum_{i=0}^{m(n)} \delta_{a'_{ni}}$ converge weakly to α'_R, the sweeping out of α_R onto D_R. Consider a new set $A'_n = \{a'_{ni}\}_{i=0}^{n}$ of interpolation points, where $a'_{ni} = a_{ni}$ for $i > m(n)$. We may check (by changing the order of sweeping out and limit procedures; see Section 2) that the interpolation points A'_n and poles B_n still satisfy all the conditions of Theorem 1. But for this case we have proved above that the corresponding rational interpolants, which we denote by r'_n, converge to f on F with geometric rate.

In particular, we have this convergence on ∂F. Since ∂F is bounded in $\mathbb{C}$ and Γ_n are (uniformly in n) bounded in $\mathbb{C}$, it follows from the error formula for $f - r'_n$, analogous to (10), that if we replace a'_{ni} by a_{ni} in that formula, the change of the nth root of the integrand for $z \in \partial F$ will be arbitrarily small if R is sufficiently large. Consequently, $f - r_n$ also converges to zero on ∂F and, by the maximum principle, on the whole of F as well.

Theorem 1 is proved.

Proof of Theorem 2. Without loss of generality we may suppose that $z_0 = 0$. We shall construct the function f in the form $f(z) = zg(z)$, where $g(z)$ is analytic in D. Let Γ_n be a positively oriented circle with center at the origin and with A_n inside. We rewrite the error formula (10) for this case:

$$R_n(0) = f(0) - r_n(0) = \frac{1}{2\pi i}\int_{\Gamma_n} \frac{H_n(0)Q_n(t)}{H_n(t)Q_n(0)} \cdot \frac{t\cdot g(t)}{t-0}\,dt \tag{17}$$
$$= \frac{1}{2\pi i}\frac{H_n(0)}{Q_n(0)}\int_{\Gamma_n} \frac{Q_n(t)}{H_n(t)} g(t)\,dt.$$

For the sake of simplicity we assume that, for fixed n, the interpolation points $A_n = \{a_{ni}\}_{i=0}^{n}$ are pairwise different; otherwise we would split points which coincide observing that $R_n(z)$ depends continuously on the location of the points $\{a_{ni}\}_{i=0}^{n}$.

By Cauchy's residue theorem, from (17) we get

$$R_n(0) = \frac{H_n(0)}{Q_n(0)}\sum_{i=0}^{n} \operatorname{Res}\left(\frac{Q_n g}{H_n}, a_{ni}\right) = \frac{H_n(0)}{Q_n(0)}\sum_{i=0}^{n} \frac{Q_n(a_{ni})}{H'_n(a_{ni})} g(a_{ni}).$$

Hence,

$$R_n(0) = \sum_{i=0}^{n} c_{ni} g(a_{ni}), \tag{18}$$

where $c_{ni} \neq 0$ depend only on the interpolation scheme $\{A_n, B_n\}$ and not on the function g.

By using (18) we shall try to find a function g for which $R_n(0) \to \infty$ as $n \to \infty$. We cannot immediately apply Weierstrass' theorem about the existence of an analytic function in a domain taking given values at given points provided that the points have no limit point in the domain. Instead we use an analogous construction to find the function g as a convergent sum of polynomials g_n, $g = \sum_{n=1}^{\infty} g_n$.

We assume that $|a_{nn}| = \max\{|a_{ni}| : 0 \le i \le n\}$ and set $d_n = |a_{nn}|$. By the assumptions of Theorem 2, $d_n \to 1$ as $n \to \infty$. We also assume that $d_1 < d_2 < \ldots$ (otherwise we could consider a subsequence with this property). Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n \le 1$. Choose $g_1 \equiv 0$. Having chosen $g_1, \ldots, g_{n-1}$ we choose a polynomial g_n (not necessarily of degree n) with the following properties:

(i) $|g_n(a_{nn})| - \sum_{i=1}^{n-1} |g_i(a_{nn})| = 1/\varepsilon_n$.
(ii) $g_n(z) = 0$ for all $z \in \bigcup_{i=1}^{n} A_i, z \neq a_{nn}$.
(iii) $|g_n(z)| \le \varepsilon_n$ on the disk $\{|z| \le d_{n-1}\}$.

Such a choice is possible. In fact, we may choose a polynomial satisfying conditions (i) and (ii), and then, by multiplying it by $(z/d_n)^m$ with some sufficiently large $m = m(n)$, we get (iii).

Condition (iii) guarantees that the sum $g = \sum_{n=1}^{\infty} g_n$ converges uniformly on any disk $\{|z| < d_n\}$ and, consequently, it converges in the whole unit disk since $d_n = |a_{nn}| \to 1$ (this is where we use this condition). The function g is analytic in D.

From (i) and (iii) we obtain

$$(19)\quad |g(a_{nn})| = \Big|\sum_{i=1}^{\infty} g_i(a_{nn})\Big| \ge |g_n(a_{nn})| - \sum_{i=1}^{n-1} |g_i(a_{nn})| - \sum_{i>n} |g_i(a_{nn})| \ge \frac{1}{\varepsilon_n} - \sum_{i>n} \varepsilon_i \ge \frac{1}{\varepsilon_n} - 1.$$

Property (ii) gives, for $i < n$,

$$g(a_{ni}) = g_1(a_{ni}) + g_2(a_{ni}) + \ldots + g_{n-1}(a_{ni}),$$

and

$$g(a_{nn}) = g_1(a_{nn}) + g_2(a_{nn}) + \ldots + g_n(a_{nn}).$$

From the last two equalities we conclude that in the sum (18) all terms depend only on the polynomials $g_1, \ldots, g_{n-1}$, except the term $c_{nn} g_n(a_{nn})$. Now, having determined the functions $g_1, \ldots, g_{n-1}$ and the numbers $\varepsilon_1, \ldots, \varepsilon_{n-1}$, we take ε_n so small that $g(a_{nn})$ (see (19)) is arbitrarily large. Consequently, $R_n(0)$ will also be arbitrarily large.

Theorem 2 is proved.

EXAMPLE 2. We show that if $\alpha_n \to \alpha$, $\beta_n \to \beta$, and α', β' denote the corresponding sweeping out measures of α and β on ∂D, then the condition $\alpha' = \beta'$ is not enough for the convergence result of Theorem 1 to hold.

Let $D = \{|z| < 1\}$, and let, for each $n \ge 1$, all $b_{ni} = \infty$ for all $i = 1, \ldots, n$, and $a_{ni} = 0$ for $i = 0, \ldots, n-2$, and $a_{n,n-1} = a_{n,n} = a_n$. We assume that $|a_n| > 3/4$ and $|a_n| \to 1$ as $n \to \infty$. Denote by Γ_n a positively oriented circle with center at the origin and with a_n inside. Let γ denote the positively oriented circle of radius $1/2$ with center at the origin, and C_n a positively oriented circle with center at a_n, contained in D. Note that γ and C_n are disjoint.

Let $f(z) = \sum_{n=1}^{\infty} z^n/n^2$. This function is bounded and analytic in D, but its derivative f' is unbounded. Fix a point $z_0 \neq 0$ with $|z_0| < 1/2$. By using the error formula (10) and the deformation invariance theorem for contours, we get, for large n,

$$R_n(z_0) = \frac{1}{2\pi i} H_n(z_0) \int_{\Gamma_n} \frac{f(t)\,dt}{t^{n-1}(t-a_n)^2(t-z_0)}$$
$$= \frac{1}{2\pi i} H_n(z_0) \int_{\gamma} \frac{f(t)\,dt}{t^{n-1}(t-a_n)^2(t-z_0)} + \frac{1}{2\pi i} H_n(z_0) \int_{C_n} \frac{f(t)\,dt}{t^{n-1}(t-a_n)^2(t-z_0)}.$$

The first term on the right-hand side is, for a fixed n, bounded when a_n varies in the domain $3/4 < |z| < 1$. For large n the second term equals, by the residue theorem,

$$H_n(z_0)\mathrm{Res}\Big(\frac{f(t)}{t^{n-1}(t-a_n)^2(t-z_0)}, a_n\Big) = H_n(z_0) \frac{d}{dt}\Big(\frac{f(t)}{t^{n-1}(t-z_0)}\Big)(a_n)$$
$$= H_n(z_0) \frac{f'(a_n) a_n^{n-1}(a_n - z_0) - f(a_n)[n a_n^{n-1} - z_0(n-1) a_n^{n-2}]}{a_n^{2n-2}(a_n - z_0)^2}.$$

We can see that the last expression will be arbitrarily large if we can choose a_n so that $f'(a_n)$ is sufficiently large. But such a choice of a_n is possible since f' is unbounded in D.

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Received November 21, 1996 (3785)
Revised version February 24, 1998 and June 4, 1998

STUDIA MATHEMATICA 132 (1) (1999)

On the joint spectral radius of a nilpotent Lie algebra of matrices

by

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Abstract. For a complex nilpotent finite-dimensional Lie algebra of matrices, and a Jordan–Hölder basis of it, we prove a spectral radius formula which extends a well-known result for commuting matrices.

1. Introduction. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of $d \times d$ complex matrices. A point $\lambda \in \mathbb{C}^n$ is in the *joint point spectrum* of T, $\sigma_{\mathrm{pt}}(T)$, if there exists a nonzero $x \in \mathbb{C}^d$ with $T_i(x) = \lambda_i x$, $1 \le i \le n$. Given p such that $1 \le p \le \infty$, R. Bhatia and T. Bhattacharyya [1] introduced the algebraic spectral radius of an n-tuple T, $\varrho_p(T)$, whose definition depends of the usual p-norm of $\mathbb{C}^d$, and proved that if T is an n-tuple of commuting matrices, then the algebraic spectral radius coincides with the geometric spectral radius, i.e., $\varrho_p(T) = r_p(T) = \max\{|\lambda|_p : \lambda \in \sigma_{\mathrm{pt}}(T)\}$ (see [1] or Section 2 for more details). This is a generalization of the well-known spectral radius formula for a single matrix; for $p = 2$, it was proved by M. Chō and T. Huruya [6].

M. Chō and M. Takaguchi [7] proved that if T is a commuting n-tuple of matrices, then $\sigma_{\mathrm{pt}}(T) = \mathrm{Sp}(T, \mathbb{C}^d)$, where $\mathrm{Sp}(T, \mathbb{C}^d)$ denotes the Taylor joint spectrum of T (see [12]). A. McIntosh, A. Pryde and W. Ricker [9], as a consequence of a more general result which also concerns infinite-dimensional spaces, extended the above identity to many other joint spectra including the commutant, the bicommutant and the Harte joint spectra.

On the other hand, in [4] we defined a joint spectrum, $\mathrm{Sp}(L, E)$, for complex solvable finite-dimensional Lie algebras L of operators acting on a Banach space E. We proved that $\mathrm{Sp}(L, E)$ is a compact nonempty subset of L^* satisfying the projection property for ideals. Moreover, when L is a commutative algebra, $\mathrm{Sp}(L, E)$ reduces to the Taylor joint spectrum in the following sense. If $\dim L = n$ and $\{T_i\}_{1 \le i \le n}$ is a basis of L, then $\{(f(T_1), \ldots, f(T_n)) : f \in \mathrm{Sp}(L, E)\} = \mathrm{Sp}(T, E)$ for $T = (T_1, \ldots, T_n)$, i.e.,

1991 *Mathematics Subject Classification*: Primary 47A13.
Key words and phrases: Taylor spectrum, joint spectral radius, nilpotent Lie algebras.
Research supported by UBACYT.