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Ergodic theorems for subadditive superstationary families of random sets with values in Banach spaces

by

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Abstract. Under different compactness assumptions pointwise and mean ergodic theorems for subadditive superstationary families of random sets whose values are weakly (or strongly) compact convex subsets of a separable Banach space are presented. The results generalize those of [14], where random sets in $\mathbb{R}^d$ are considered. Techniques used here are inspired by [3].

1. Introduction. A generalization of Birkhoff's pointwise ergodic theorem for superstationary families of random variables was given by Krengel [13]. In [12] Kingman proved Birkhoff's ergodic theorem for stationary subadditive processes. Abid [1] generalized previous results and showed an ergodic theorem for subadditive superstationary families of $\mathbb{R}$-valued random variables. Using Abid's result Schürger [14] proved pointwise and mean ergodic theorems for subadditive superstationary families of convex compact random sets in $\mathbb{R}^d$. Here the results of [14] are generalized.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. Subadditive superstationary families $(F_{s,t})$ of multivalued functions on Ω with values being weakly (respectively strongly) compact convex subsets of a separable Banach space are considered. Certain compactness conditions are imposed upon $(F_{s,t})$. Namely, it is assumed that $\frac{1}{t}F_{0,t}(\omega)$ are a.s. contained in some, dependent on ω, ball-compact set for all $t \in \mathbb{N}$, or, in the case when the Banach space and its dual both have the Radon–Nikodym property, $\operatorname{cl}\operatorname{co}\bigcup_{t=1}^{\infty}\frac{1}{t}\int_A F_{0,t}$ is supposed to be w-compact for all subsets A of an underlying σ-algebra. It is noteworthy that in the finite-dimensional case there is no distinction between weak and strong topology and that the conditions mentioned above are automatically satisfied. Later, under some additional conditions, the convergence of subadditive superstationary families of subsets of a Banach space to a constant limit is proved.

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The main idea used in the proofs is to scalarize elements of subadditive superstationary families using support functions, then use Abid's one-dimensional results and prove the existence of multivalued infinite-dimensional limits using techniques from [3] and [10].

2. Preliminaries. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let X be a separable Banach space with norm $\|\cdot\|$. We will denote by X^* the dual of X and $\langle\cdot,\cdot\rangle$ will stand for the usual duality. The strong and weak topology on X will be denoted by s and w respectively. Let $\mathcal{P}_{\mathrm{wk}}(X)$ (respectively $\mathcal{P}_{\mathrm{sk}}(X)$) denote a family of w-compact (s-compact) subsets of X. We will write $\mathcal{P}_{\mathrm{wkc}}(X)$ and $\mathcal{P}_{\mathrm{skc}}(X)$ for the families of w-compact convex and s-compact convex subsets of X. A subset of X will be called *w-ball-compact* (*s-ball-compact*) if its intersection with any closed ball with center at the origin is w-compact (s-compact). Denote the family of w-ball-compact (resp. s-ball-compact) sets by $\mathcal{R}_{\mathrm{w}}$ (resp. $\mathcal{R}_{\mathrm{s}}$).

A *multifunction* is any mapping $F : \Omega \to \mathcal{P}_{\mathrm{wkc}}(X)$. A multifunction F is said to be (*Effrös*) *measurable* if the *preimage* $F^{-}U := \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ belongs to $\mathcal{A}$ for any s-open set $U \subset X$. Measurable multivalued functions will be considered and the adjective will often be omitted. Also, the term *random set* (*r.s.*) will be used to denote measurable multivalued functions.

The *support function* and the *radius* of the set $C \in \mathcal{P}_{\mathrm{wkc}}(X)$ will be defined in the following way:

$$s(x^*, C) := \sup_{x\in C}\langle x, x^*\rangle, \qquad \|C\| := \sup_{x\in C}\|x\|.$$

A sequence $(C_n) \subset \mathcal{P}_{\mathrm{wkc}}(X)$ *converges scalarly* to a $C \in \mathcal{P}_{\mathrm{wkc}}(X)$ if

$$\lim s(x^*, C_n) = s(x^*, C) \quad \text{for all } x^* \in X^*.$$

The topology of scalar convergence will be denoted by $\mathcal{T}_{\mathrm{scalar}}$.

Given any topology η on X, the *sequential* η-*Kuratowski limes superior* (η-$\mathrm{Ls}\, C_n$) of a sequence $(C_n) \subset X$ is the set of all η-limits of subsequences (x_{n_j}) such that $x_{n_j} \in C_{n_j}$ for all n_j. In the sequel η will be the weak or the strong topology.

Denote by D a countable subset of the unit ball B^* in X^* dense with respect to the Mackey topology τ. Let H be the set of all rational linear combinations of vectors in D. For details see [5, III.32]. Notice also that, by [5, III.34], for any $C \in \mathcal{P}_{\mathrm{wkc}}(X)$,

$$C = \bigcap_{x^*\in H}\{x \in X : \langle x, x^*\rangle \le s(x^*, C)\}. \tag{1}$$

Denote $\mathbb{N}\cup\{0\}$ by $\mathbb{N}_0$. Let $\Delta := \{(s,t) \in (\mathbb{N}_0)^2 : s < t\}$. Families $(F_{s,t})_{(s,t)\in\Delta}$ in $\mathcal{P}_{\mathrm{wkc}}(X)$ (or $\mathcal{P}_{\mathrm{skc}}(X)$) will be considered. They will be denoted simply by $(F_{s,t})$. The concepts of subadditivity and superstationarity are defined in the following way.

DEFINITION 2.1. A family $(C_{s,t})_{(s,t)\in\Delta} \subset \mathcal{P}_{\mathrm{wkc}}(X)$ (or $\mathcal{P}_{\mathrm{skc}}(X)$) is called *subadditive* if

$$C_{s,t} \subset C_{s,u} + C_{u,t} \quad \text{for all } s < u < t.$$

Let $\mathcal{D}$ be a space of families $F = (F_{s,t})$ in $\mathcal{P}_{\mathrm{wkc}}(X)^{\Delta}$ (or $\mathcal{P}_{\mathrm{skc}}(X)^{\Delta}$):

$$F = \begin{bmatrix} F_{0,1} & F_{1,2} & F_{2,3} & \cdots \\ F_{0,2} & F_{1,3} & F_{2,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ F_{0,n} & F_{1,n+1} & F_{2,n+2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

endowed with the product topology. Let $(i,j) \in \mathbb{N}^2$ and $\pi_{i,j} : \mathcal{D} \to \mathcal{P}_{\mathrm{wkc}}(X)$ (or $\mathcal{P}_{\mathrm{skc}}(X)$) be the projection that gives the (i,j)th coordinate of $F \in \mathcal{D}$, i.e. $\pi_{i,j}(F) = F_{j-1,i+j-1}$.

Define the shift $T : \mathcal{D} \to \mathcal{D}$ such that $\pi_{i,j}(T(F)) = F_{j,i+j}$ for all $i,j \in \mathbb{N}$. Let $\mathcal{M}(\mathcal{D})$ be the family of probability measures defined on $\mathcal{B}(\mathcal{D})$, the Borel σ-algebra of subsets of $\mathcal{D}$.

A probability measure $P_1 \in \mathcal{M}(\mathcal{D})$ is *stochastically smaller* than $P_2 \in \mathcal{M}(\mathcal{D})$ (notation: $P_1 \prec P_2$) if

$$\int_{\mathcal{D}} f\,dP_1 \le \int_{\mathcal{D}} f\,dP_2$$

for all bounded measurable functions $f : \mathcal{D} \to \mathbb{R}$ which are increasing, i.e. $C \lesssim D$ implies that $f(C) \le f(D)$, where $C \lesssim D$ means that $C_{i,j} \subseteq D_{i,j}$ for all $(i,j) \in \Delta$.

DEFINITION 2.2. Let $F = (F_{s,t})$ be a family of $\mathcal{P}_{\mathrm{wkc}}(X)$-valued random sets defined on $(\Omega, \mathcal{A}, P)$. Let Q_i denote the probability distribution of T^iF, i.e. $Q_i(A) = P(T^iF \in A)$, $A \in \mathcal{B}(\mathcal{D})$. The family F is called *superstationary* if $Q_1 \prec Q_0$.

Note also that if F is a superstationary process then $Q_{i+1} \prec Q_i$ for all $i \in \mathbb{N}$.

A nontrivial example is the following.

EXAMPLE 2.1. Consider the positive half of the real line. Let (p_n) be a non-increasing sequence of numbers in $(0,1)$. For any $n \in \mathbb{N}$ define random variables

$$x_n := \begin{cases} 1 & \text{with probability } p_n, \\ 0 & \text{with probability } 1 - p_n. \end{cases}$$

The binomial distributions on $\mathbb{Z}_+$ define a probability measure on the space $\Omega := \{0,1\}^{\mathbb{Z}_+}$. For any $\omega \in \Omega$ and $i \in \mathbb{Z}_+$ set $\omega_i := \omega(i)$. Define a family of random intervals $F_{m,n}(\omega) := (0, \sum_{i=m}^{n-1} x(\omega_i))$ where $x(\omega_i) := x_i$. Then $(F_{m,n})$ is an additive and superstationary process.

3. Results. Theorems 3.1–3.3 give different conditions under which Theorem (4.1) of [14] can be generalized for r.s. with values in a Banach space.

We begin with pointwise ergodic theorems.

THEOREM 3.1. *Let $F = (F_{s,t})$ be a subadditive superstationary family in $\mathcal{P}_{\mathrm{wkc}}(X)$ satisfying the following assumptions:*

(i) *$(F_{s,s+1})$ is $\mathcal{L}^1_{\mathcal{P}_{\mathrm{wkc}}(X)}$-bounded, i.e. there exists a constant $\widetilde{k}$ such that $\int_\Omega \|F_{s,s+1}\| \le \widetilde{k}$ for all $s \in \mathbb{N}_0$,*

(ii) *for almost all $\omega \in \Omega$, $\bigcup_{t=1}^{\infty} \frac{1}{t} F_{0,t}(\omega)$ is contained in some, dependent on ω, element of $\mathcal{R}_{\mathrm{w}}$.*

Then there exists $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{\mathrm{wkc}}(X)}$ such that

(a) $\frac{1}{t} F_{0,t} \xrightarrow{\mathcal{T}_{\mathrm{scalar}}} F_\infty$,

(b) $F_\infty(\omega) \subset \mathrm{cl\,co}\, w\text{-}\mathrm{Ls} \frac{1}{t} F_{0,t}(\omega)$ *a.s. in* Ω.

Note that if the space X is reflexive then balls in X are w-compact, thus condition (ii) of Theorem 3.1 is trivially satisfied. Namely, X itself is w-ball compact.

The next theorem is a version of the previous one for $\mathcal{P}_{\mathrm{skc}}(X)$-valued functions.

THEOREM 3.2. *Under the assumptions of Theorem 3.1, where $\mathcal{P}_{\mathrm{wkc}}(X)$ and $\mathcal{R}_{\mathrm{w}}$ are replaced by $\mathcal{P}_{\mathrm{skc}}(X)$ and $\mathcal{R}_{\mathrm{s}}$ respectively, there exists an $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{\mathrm{skc}}(X)}$ such that*

(a) $\lim_{t\to\infty} \varrho_{\mathrm{H}}(\frac{1}{t} F_{0,t}, F_\infty) = 0$, *where ϱ_{H} denotes the Hausdorff distance,*

(b) $F_\infty(\omega) \subset \mathrm{cl\,co}\, s\text{-}\mathrm{Ls} \frac{1}{t} F_{0,t}(\omega)$ *a.s. in* Ω.

Another generalization of Theorem (4.1) in [14] is possible. The assumptions of the next theorem are inspired by [3]. In this result both X and X^* are required to have the *Radon–Nikodym property* (RNP). Recall that X has the RNP with respect to $(\Omega, \mathcal{A}, P)$ if any P-absolutely continuous measure Q with bounded variation has a density $f \in \mathcal{L}^1_X$ with respect to P, that is, $Q(A) = \int_A f\,dP$ for all $A \in \mathcal{A}$.

THEOREM 3.3. *Suppose that the Banach space X and its dual X^* (with the dual norm) both have the RNP. Let $F = (F_{s,t})$ be a subadditive superstationary family in $\mathcal{P}_{\mathrm{wkc}}(X)$ satisfying the following assumptions:*

(i) *$(F_{s,s+1})$ is $\mathcal{L}^1_{\mathcal{P}_{\mathrm{wkc}}(X)}$-bounded, i.e. there exists a constant $\widetilde{k}$ such that $\int_\Omega \|F_{s,s+1}\| \le \widetilde{k}$ for all $s \in \mathbb{N}_0$,*

(ii) *the set $\mathrm{cl\,co} \bigcup_{t=1}^{\infty} \frac{1}{t} \int_A F_{0,t}$ is w-compact for all $A \in \mathcal{A}$.*

Then there exists $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{\mathrm{wkc}}(X)}$ such that

(a) $\frac{1}{t} F_{0,t} \xrightarrow{\mathcal{T}_{\mathrm{scalar}}} F_\infty$,

(b) $F_\infty(\omega) \subset \bigcap_m \mathrm{cl\,co} \bigcup_{t>m} \frac{1}{t} F_{0,t}(\omega)$ *a.s. in* Ω.

Observe that here condition (ii) is of more global nature than in Theorem 3.1. Note also that for a separable Banach space X, X^* has the RNP if and only if X^* is separable for the dual norm (see [8, Stegall's theorem, p. 195]).

The following theorems give conditions under which limits that occur in Theorems 3.1, 3.2, 3.3 are constant a.s. These conditions appeared in [9], [14].

THEOREM 3.4. *Let $F = (F_{s,t})$ be a family of random sets satisfying the assumptions of Theorem 3.1. Suppose also that*

(i) $\int_\Omega s(x^*, F_{0,t})^2 < \infty$, *for all $t \in \mathbb{N}$, $x^* \in H$,*

(ii) $\lim_{t\to\infty} \frac{1}{t^2} \int_\Omega (s(x^*, F_{0,t}))^2 = \alpha_{x^*}$ *exists and is finite for all $x^* \in H$,*

(iii) *there is a constant $0 \le \delta_{x^*} < 1$ depending on $x^* \in H$ such that for all $x^* \in H$,*

$$\mathrm{Var}(s(x^*, F_{0,2t})) + \Big(\int_\Omega s(x^*, F_{0,2t})\Big)^2 \le 2(1+\delta_{x^*})\,\mathrm{Var}(s(x^*, F_{0,t})) + 4\Big(\int_\Omega s(x^*, F_{0,t})\Big)^2,$$

where Var *denotes the variance.*

Then there exists a set $C \in \mathcal{P}_{\mathrm{wkc}}(X)$ such that

(a) $\frac{1}{t} F_{0,t} \xrightarrow{\mathcal{T}_{\mathrm{scalar}}} C$,

(b) $\lim_{t\to\infty} \frac{1}{t} s(x^*, F_{0,t}) = s(x^*, C)$ *in $\mathcal{L}^2_{\mathbb{R}}$, for all $x^* \in X^*$.*

We have an immediate analogue of Theorem 3.4 for r.s. with values in $\mathcal{P}_{\mathrm{skc}}(X)$.

THEOREM 3.5. *Let $F = (F_{s,t})$ be a family of random sets satisfying the assumptions of Theorem 3.2 and (i), (ii), (iii) of Theorem 3.4. Then there exists a set $C \in \mathcal{P}_{\mathrm{skc}}(X)$ such that*

(a) $\lim_{t\to\infty} \varrho_{\mathrm{H}}(\frac{1}{t} F_{0,t}, C) = 0$,

(b) $\lim_{t\to\infty} \frac{1}{t} s(x^*, F_{0,t}) = s(x^*, C)$ *in $\mathcal{L}^2_{\mathbb{R}}$, for all $x^* \in X^*$.*

THEOREM 3.6. *Suppose both X and X^* have the RNP. Let $F = (F_{s,t})$ be a family of random sets satisfying the assumptions of Theorem 3.3 and (i), (ii), (iii) of Theorem 3.4. Then there exists a set $C \in \mathcal{P}_{\mathrm{wkc}}(X)$ such that (a), (b) of Theorem 3.4 hold.*

REMARK 3.1. No analogue of Theorem 3.6 exists for $\mathcal{P}_{\rm skc}(X)$-valued r.s. The Radon–Nikodym theorem for multimeasures, which is used in the proof of Theorem 3.6, provides only the existence of a derivative multifunction with w-compact values.

We now present mean ergodic theorems.

PROPOSITION 3.1. *Let (F_n) be a sequence of $\mathcal{P}_{\rm wkc}(X)$-valued random sets such that for almost all $\omega\in\Omega$, $\bigcup_{n\in\mathbb{N}}F_n(\omega)$ is a subset of some w-ball-compact set dependent on ω. Suppose that for some $m\in\mathbb{N}$, $\lim_{n\to\infty}s(x^*,F_n)$ exists in $\mathcal{L}^m_{\mathbb{R}}$ for all $x^*\in H$ and that $(\|F_n\|)$ converges in $\mathcal{L}^m_{\mathbb{R}}$. Then there exists an $F_\infty\in\mathcal{L}^m_{\mathcal{P}_{\rm wkc}(X)}$ such that*

$$\int_\Omega |s(x^*,F_n)-s(x^*,F_\infty)|^m\to 0\quad \textit{for all } x^*\in H.$$

THEOREM 3.7. *Let $F=(F_{s,t})$ be a family of random sets in $\mathcal{P}_{\rm wkc}(X)$ satisfying the conditions of Theorem 3.1. Then there exists a random set $F_\infty\in\mathcal{L}^1_{\mathcal{P}_{\rm wkc}(X)}$ such that*

$$\lim_{t\to\infty}\int_\Omega\left|s\left(x^*,\frac{1}{t}F_{0,t}\right)-s(x^*,F_\infty)\right|=0\quad\textit{for all } x^*\in X^*.$$

REMARK 3.2. Suppose that both X and X^* have the RNP. It is possible to derive analogues of Theorem 3.7 (and also Proposition 3.1, which is used in the proof of Theorem 3.7). In that case the condition that $\bigcup_{t\in\mathbb{N}}\frac{1}{t}F_{0,t}(\omega)$ is a.s. contained in a w-ball-compact set, which depends on ω, is replaced by the following one: $\operatorname{cl}\operatorname{co}\bigcup_{t\in\mathbb{N}}\int_A\frac{1}{t}F_{0,t}$ is w-compact for all $A\in\mathcal{A}$.

Much more interesting results can be obtained for r.s. whose values are in $\mathcal{P}_{\rm skc}(X)$. Let us first state a $\mathcal{P}_{\rm skc}(X)$-valued version of Proposition 3.1.

PROPOSITION 3.2. *Let (F_n) be a sequence of random sets in $\mathcal{P}_{\rm skc}(X)$ such that for almost all $\omega\in\Omega$, $\bigcup_{n\in\mathbb{N}}F_n(\omega)$ is a subset of some, dependent on ω, s-ball-compact set. Suppose that for some $m\in\mathbb{N}$, $\lim s(x^*,F_n)$ exists in $\mathcal{L}^m_{\mathbb{R}}$ for all $x^*\in H$ and that $(\|F_n(\omega)\|)$ converges in $\mathcal{L}^m_{\mathbb{R}}$. Then there exists $F_\infty\in\mathcal{L}^1_{\mathcal{P}_{\rm skc}(X)}$ such that*

$$\int_\Omega |s(x^*,F_n)-s(x^*,F_\infty)|^m\to 0\quad \textit{for all } x^*\in H.$$

Now we have an analogue of Theorem 3.7.

THEOREM 3.8. *Let $F=(F_{s,t})$ be a family of random sets in $\mathcal{P}_{\rm skc}(X)$ satisfying the assumptions of Theorem 3.2. Then there exists an $\mathcal{L}^1_{\mathcal{P}_{\rm skc}(X)}$-bounded random set F_∞ satisfying*

$$\lim_{t\to\infty}\int_\Omega\left|s\left(x^*,\frac{1}{t}F_{0,t}\right)-s(x^*,F_\infty)\right|=0\quad\textit{for all } x^*\in X^*.$$

Under additional assumptions (namely, those that appear in Theorem 3.4) it is possible to derive mean-square convergence results (see Theorems 3.4(b) and 3.5(b)).

REMARK 3.3. Artstein and Hansen showed in [2] that the strong law of large numbers for convex-valued random sets can be extended to the non-convex case by applying the smart observation that given a sequence (K_i) of s-compact sets in a Banach space X and an s-compact convex set K_0 such that $\varrho_{\rm H}\big(\frac{1}{n}\sum_{i=1}^n\operatorname{co}K_i,K_0\big)\to 0$ as $n\to\infty$, also $\varrho_{\rm H}\big(\frac{1}{n}\sum_{i=1}^n K_i,K_0\big)\to 0$ as $n\to\infty$. In the case when compact subsets of $\mathbb{R}^d$ are considered, the well-known result of Shapley–Folkmann–Starr can be applied. However, this argument does not apply in the case of subadditive, superstationary processes. A counterexample (already in $\mathbb{R}^d$) is given in [14].

4. Applications. Recall that if a space X is reflexive then closed balls in X are w-compact. Keeping in mind this remark it is easy to see that the condition which appears in the results of Section 3, namely that $\bigcup_{t=1}^\infty\frac{1}{t}F_{0,t}(\omega)$ is a subset of some element in $\mathcal{R}_{\rm w}$, is automatically satisfied in reflexive spaces. (X itself is then w-ball-compact.) Thus Schürger's results for random sets in $\mathbb{R}^d$ with compact, convex values follow from the results presented in Section 3. Notice also that the scalar convergence topology is equivalent to the topology generated by the Hausdorff distance in that case.

Let $\mathcal{P}_{\rm kc}(\mathbb{R}^d)$ denote the family of all convex compact subsets of $\mathbb{R}^d$. We have an easy corollary from any of Theorems 3.1–3.3.

COROLLARY 4.1 (Theorem (4.1) of [14]). *Let $F:=(F_{s,t})$ be a subadditive superstationary family of $\mathcal{P}_{\rm kc}(\mathbb{R}^d)$-valued random sets defined on a common probability space $(\Omega,\mathcal{A},P)$. Assume that there exists a constant $\widetilde{K}>0$ such that $\int_\Omega\|F_{s,s+1}\|\le\widetilde{K}$ for $s\in\mathbb{N}_0$. Then $\lim_{t\to\infty}\frac{1}{t}F_{0,t}$ exists a.s. in $(\mathcal{P}_{\rm kc}(\mathbb{R}^d),\varrho_{\rm H})$.*

Theorem 3.4 (or 3.5, 3.6) implies:

COROLLARY 4.2 (Theorem (4.16) of [14]). *Let $F:=(F_{s,t})$ be a subadditive superstationary family of $\mathcal{P}_{\rm kc}(\mathbb{R}^d)$-valued random sets satisfying the conditions of Corollary 4.1 and conditions* (i), (ii), (iii) *of Theorem 3.4. Then there exists a set $C\in\mathcal{P}_{\rm kc}(\mathbb{R}^d)$ such that $\lim_{t\to\infty}\frac{1}{t}F_{0,t}=C$ a.s., $\alpha_{x^*}=s^2(x^*,C)$ and for all x^*, $\lim_{t\to\infty}\frac{1}{t}s(x^*,F_{0,t})=s(x^*,C)$ in $\mathcal{L}^2_{\mathbb{R}}$.*

Theorems 3.8 and 3.5(b) yield respectively:

COROLLARY 4.3 (Theorem (4.32) of [14]). *Let $F:=(F_{s,t})$ be a subadditive superstationary family of $\mathcal{P}_{\rm kc}(\mathbb{R}^d)$-valued random sets satisfying the assumptions of Corollary 4.1. Then there exists a $\mathcal{P}_{\rm kc}(\mathbb{R}^d)$-valued random* *set* G *with* $\int_\Omega \|G\|\,dP<\infty$ *and*

$$\lim_{t\to\infty}\int_\Omega \varrho_{\mathrm H}\Big(\frac{1}{t}F_{0,t},G\Big)\,dP=0.$$

COROLLARY 4.4 (Theorem (4.35) of [14]). *Let* $F:=(F_{s,t})$ *be a subadditive superstationary family of* $\mathcal P_{\mathrm{kc}}(\mathbb R^d)$*-valued random sets satisfying the assumptions of Corollary* 4.1. *Then there exists a* $\mathcal P_{\mathrm{kc}}(\mathbb R^d)$*-valued random set* G *such that* $\lim_{t\to\infty}\int_\Omega \varrho^2_{\mathrm H}(\frac1t F_{0,t},G)\,dP=0$.

5. Proofs of the results. The following result, due to Abid ([1]), plays a key role in the proofs of the results of this paper. Let us recall, after [1], the definition of a real-valued subadditive superstationary process. We consider a family $x=(x_{s,t})\subset\mathbb R^{\Delta}$ where $x_{s,t}$ are real-valued random variables. $\mathbb R^{\Delta}$ is equipped with the product topology. The shift T is defined analogously to the case of $\mathcal P_{\mathrm{wkc}}(X)$-valued families.

DEFINITION 5.1. We say that a real-valued process $x:=(x_{s,t})$ is *subadditive* and *superstationary* if

(i) $x_{s,t}\le x_{s,u}+x_{u,t}$, for any triple $s,u,t\in\mathbb N_0$ such that $s<u<t$,

(ii) the distribution of x is stochastically smaller than the distribution of Tx,

(iii) $\int_\Omega x_{0,t}<\infty$ for any $t\in\mathbb N$,

(iv) there exists an $M>0$ such that $\inf_{s\ge0}\int_\Omega x_{s,s+t}\ge -Mt$, for any $t\in\mathbb N$.

LEMMA 5.1 ([1]). *For any subadditive superstationary family* $x:=(x_{s,t})$, $\lim_{t\to\infty}\frac1t x_{0,t}$ *exists a.s. in* $\mathcal L^1_{\mathbb R}$.

Another important tool is an analogue of the Blaschke type lemma ([15]) for multivalued functions with values in $\mathcal P_{\mathrm{wkc}}(X)$ or $\mathcal P_{\mathrm{skc}}(X)$.

LEMMA 5.2 (Lemma 3.2 of [3]). (a) *For every* $K\in\mathcal P_{\mathrm{wkc}}(X)$ *the subset* $\mathcal K:=\{C\in\mathcal P_{\mathrm{wkc}}(X):C\subset K\}$ *of* $\mathcal P_{\mathrm{wkc}}(X)$ *is metrizable and compact for the scalar convergence topology.*

(b) *For every* $K\in\mathcal P_{\mathrm{skc}}(X)$ *the subset* $\mathcal K:=\{C\in\mathcal P_{\mathrm{skc}}(X):C\subset K\}$ *of* $\mathcal P_{\mathrm{skc}}(X)$ *is compact for the Hausdorff metric* $\varrho_{\mathrm H}$.

Consider the following lemma, which will be often used in the sequel.

LEMMA 5.3. *Let* $(C_n)\subset\mathcal P_{\mathrm{wkc}}(X)$. *Suppose there exists an* $R\in\mathcal R_{\mathrm w}$ *such that* $C_n\subset R$ *for all* $n\in\mathbb N$. *Suppose also that* $(\|C_n\|)$ *is bounded and that for all* $x^*\in H$ *the sequence* $s(x^*,C_n)$ *converges. Then there exists a* $C_\infty\in\mathcal P_{\mathrm{wkc}}(X)$ *such that* $C_n\xrightarrow{\mathcal T_{\mathrm{scalar}}}C_\infty$.

Proof. By the assumption, $s(x^*,C_n)$ converges for all $x^*\in H$. Define

$$\alpha_{x^*}:=\lim_{n\to\infty}s(x^*,C_n),\tag{2}$$

$$r:=\sup_n\|C_n\|.\tag{3}$$

Notice that $C_n\subset K:=R\cap B(0,r)$. Since $R\in\mathcal R_{\mathrm w}$, K is compact and by application of part (a) of Lemma 5.2 there exists a subsequence $(C_{n_i})\subset(C_n)$ which scalarly converges to some $C_\infty\in\mathcal P_{\mathrm{wkc}}(X)$. Obviously, $s(x^*,C_\infty)=\alpha_{x^*}$ for all $x^*\in H$. By (2), (1) and the fact that H is dense in X^*, C_∞ is the unique cluster point of the sequence (C_n) in the topology of scalar convergence. Therefore $C_n\xrightarrow{\mathcal T_{\mathrm{scalar}}}C_\infty$. ■

Proof of Theorem 3.1. It will be shown that for any $x^*\in H$, $s(x^*,F):=(s(x^*,F_{s,t}))$ and $\|F\|:=(\|F_{s,t}\|)$ are subadditive superstationary real-valued processes. By subadditivity of F, for any $x^*\in H$, $s(x^*,F_{s,t})\le s(x^*,F_{s,u})+s(x^*,F_{u,t})$ for all $s<u<t$. By superstationarity of F, for all $x^*\in H$, $u\in\mathbb R$, $(s,t)\in\Delta$ we have

$$P[s(x^*,F_{s,t})>u]\ge P[s(x^*,F_{s+1,t+1})>u].$$

Assumption (i) and subadditivity yield

(i′) $\int_\Omega s(x^*,F_{0,t})<\infty$, for all $x^*\in H$, $t\in\mathbb N$,

(i″) $\inf_{s\ge0}\int_\Omega s(x^*,F_{s,s+t})\ge -\widetilde k t\|x^*\|_*$, for all $x^*\in H$, $t\in\mathbb N$.

Indeed,

$$\int_\Omega s(x^*,F_{0,t})\le\sum_{u=0}^{t-1}\int_\Omega s(x^*,F_{u,u+1})\le\|x^*\|_*\sum_{u=0}^{t-1}\int_\Omega\|F_{u,u+1}\|\le\|x^*\|_*\widetilde k t<\infty$$

and

$$\frac{1}{\|x^*\|_*}\int_\Omega s(x^*,F_{s,s+t})\ge-\int_\Omega\|F_{s,s+t}\|\ge\sum_{u=0}^{t-1}\int_\Omega\|F_{u,u+1}\|\ge-\widetilde k t,$$

for all $s\in\mathbb N_0$, $t\in\mathbb N$, $x^*\in X^*$. Assertion (i″) follows by taking the infimum over all $s\in\mathbb N_0$. Now, for all $x^*\in H$, $s(x^*,F)$ is a real-valued subadditive superstationary process in the sense of [1] (see Definition 5.1). Analogously it can be shown that $(\|F_{s,t}\|)$ is also a subadditive superstationary process. Abid's pointwise ergodic theorem implies that for all $x^*\in H$ there exist null sets N_{x^*} and M and functions $\varphi_{x^*},\psi:\Omega\to\mathbb R$ such that

$$\lim_{t\to\infty}\frac1t s(x^*,F_{0,t}(\omega))=\varphi_{x^*}(\omega)\quad\text{for all }\omega\in N^{\mathrm c}_{x^*},\tag{4}$$

$$\lim_{t\to\infty}\frac1t\|F_{0,t}(\omega)\|=\psi_{x^*}(\omega)\quad\text{for all }\omega\in M^{\mathrm c}.\tag{5}$$

Define the null set $N:=\bigcup_{x^*\in H}N_{x^*}\cup M$. Fix any $\omega\in N^{\mathrm c}$. Lemma 5.3, applied to $C_t:=\frac1t F_{0,t}(\omega)$, yields the existence of a set C^ω_∞ such that

$\frac{1}{t}F_{0,t}(\omega) \xrightarrow{\mathcal{T}_{\text{scalar}}} C^\omega_\infty$. Define

$$F_\infty(\omega) := \begin{cases} C^\omega_\infty, & \omega \in N^c, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Obviously, $\frac{1}{t}F_{0,t}(\omega) \xrightarrow{\mathcal{T}_{\text{scalar}}} F_\infty(\omega)$ for all $\omega \in N^c$. Since for all $x \in X$,

$$d(x, F_\infty(\omega)) = \sup_{x^*\in D} [\langle x, x^*\rangle - s(x^*, F_\infty(\omega))]$$

is measurable in ω (for $s(x^*, F_\infty(\cdot))$ is a limit of measurable functions), F_∞ is measurable (see [5], Theorem III.9). Moreover, $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{\text{wkc}}(X)}$. Indeed,

$$\int_\Omega \|F_\infty\| \le \int_\Omega \liminf_{t\to\infty} \left\|\frac{1}{t}F_{0,t}\right\| \le \liminf_{t\to\infty} \int_\Omega \left\|\frac{1}{t}F_{0,t}\right\| = \liminf_{t\to\infty} \frac{1}{t}\int_\Omega \|F_{0,t}\| \le \frac{1}{t}\cdot t\widetilde{k} < \infty.$$

The proof of part (b) is an adaptation of that of part (b) of Theorem 2.1 of [3]. ∎

Proof of Theorem 3.2. Denote by $R : \Omega \to \mathcal{R}_s$ a function such that $\bigcup_{t=1}^\infty \frac{1}{t}F_{0,t}(\omega) \subset R(\omega)$ a.s. As in the proof of Theorem 3.1 one can show that there exists a function $F^w_\infty \in \mathcal{L}^1_{\mathcal{P}_{\text{wkc}}(X)}$ such that

$$\frac{1}{t}F_{0,t} \xrightarrow{\mathcal{T}_{\text{scalar}}} F^w_\infty. \tag{6}$$

Also, again as in the proof of Theorem 3.1, $\|\frac{1}{t}F_{0,t}\|$ converges to some $\psi \in \mathcal{L}^1_{\mathbb{R}}$. Thus for sufficiently large t,

$$\frac{1}{t}F_{0,t}(\omega) \subset K(\omega) := R(\omega) \cap B(0, \psi(\omega)+1),$$

where $K(\omega) \in \mathcal{P}_{\text{skc}}(X)$ (by the s-ball-compactness of $R(\omega)$). Part (b) of Lemma 5.2 implies that there is a subsequence $(t_i) \subset (t)$ and $F_\infty \in \mathcal{P}_{\text{skc}}(X)$ such that

$$\varrho_h\left(\frac{1}{t_i}F_{0,t_i}, F_\infty\right) \to 0 \quad \text{a.s.} \tag{7}$$

Now (6) and (7) imply that $F_\infty(\omega) = F^w_\infty(\omega)$ (a.s.) is a unique ϱ_{H}-cluster point of the sequence $(\frac{1}{t}F_{0,t}(\omega))$. Thus (a) is proved. To prove part (b) one can adapt the proof Theorem 2.1(b) in [3]. ∎

Proof of Theorem 3.3. As in the proof of Theorem 3.1, it can be shown that $(s(x^*, F_{s,t}))$ and $(\|F_{s,t}\|)$ are subadditive, superstationary families in

Abid's sense. Thus there exist $\varphi_{x^*}, \psi \in \mathcal{L}^1_{\mathbb{R}}$ such that for all $x^* \in H$,

$$\begin{aligned} \varphi_{x^*}(\omega) &= \lim_{t\to\infty} \frac{1}{t} s(x^*, F_{0,t}(\omega)), \\ \psi(\omega) &= \lim_{t\to\infty} \frac{1}{t}\|F_{0,t}(\omega)\|. \end{aligned} \tag{8}$$

The sets $G_t(\omega) := \frac{1}{t}F_{0,t}(\omega)$ are uniformly bounded by $\psi(\omega)+1$ (for sufficiently large t). Hence $(\frac{1}{t}s(x^*, F_{0,t}(\omega)))$ is equicontinuous on X^*, thus (8) holds for all $x^* \in X^*$. Put $R_A := \operatorname{cl}\operatorname{co}\bigcup_{t=1}^\infty \frac{1}{t}\int_A F_{0,t}$. Define $\psi_A : X^* \to \mathbb{R}$ by $\psi_A(x^*) = \int_A \varphi_{x^*}$. The function $\psi_A(x^*)$ is subadditive on X^* and

$$\psi_A(x^*) \le s(x^*, R_A), \quad \text{for all } x^* \in X^*, \tag{9}$$

therefore ψ_A is τ-continuous on X^* and is also w*-lower semicontinuous. Theorem II.16 of [5] implies that there exists a nonempty closed convex subset $M(A) \subset X$ such that $\psi_A(\cdot) = s(\cdot, M(A))$. Obviously, $s(x^*, M(A)) = \int_A \varphi_{x^*} \le \int_A \psi$, thus

$$\|M(A)\| = \sup_{x^*\in D} s(x^*, M(A)) \le \int_A \psi < \infty.$$

By (9), $M(A) \subset R_A$, thus it is w-compact. Now, as was done in the proof of Theorem 2.5 in [3], it can be shown that $M : \mathcal{A} \to \mathcal{P}_{\text{wkc}}(X)$ is additive, absolutely continuous with respect to P, has bounded variation and $s(x^*, M(\cdot))$ is σ-additive for all $x^* \in X^*$. Thus the multivalued Radon–Nikodym theorem (see [6, Théorème 3] or [7, Théorème 8, p. III.31]) can be applied. It implies the existence of a multifunction $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{\text{wkc}}(X)}$ such that

$$M(A) = \int_A F_\infty\, dP, \quad \text{for all } A \in \mathcal{A}. \tag{10}$$

The multifunction F_∞ is defined uniquely up to a null set. Recalling ([11]) that for multivalued functions with $s(x^*, \int_A F) = \int_A s(x^*, F)$ for all $A \in \mathcal{A}$ part (a) follows. The proof of part (b) is the same as of Theorem 2.5(b) in [3]. ∎

Proof of Theorem 3.4. By Theorem 3.1, $\frac{1}{t}s(x^*, F_{0,t})$ converges in $\mathcal{L}^1_{\mathbb{R}}$ (for all $x^* \in X^*$), therefore there exists the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_\Omega s(x^*, F_{0,t}) =: \widetilde{\alpha}_{x^*} < \infty, \quad \text{for all } x^* \in X^*. \tag{11}$$

(Note that $\widetilde{\alpha}_{x^*}$ does not depend on ω.) Following the proof in [9], p. 675, assumptions (i)–(iii) and (11) imply that

$$\lim_{n\to\infty} \frac{1}{2^n m} s(x^*, F_{0,2^n m}(\omega)) = \widetilde{\alpha}_{x^*} \quad \text{a.s.} \quad \text{for all } x^* \in H,\ m \in \mathbb{N} \tag{12}$$

and

$$\lim_{t\to\infty} \frac{1}{t} s(x^*, F_{0,t}) = \widetilde{\alpha}_{x^*} \quad \text{in } \mathcal{L}^2_{\mathbb{R}} \quad \text{for all } x^* \in H. \tag{13}$$

As in the proof of Theorem 3.1 it can be shown that $(\frac{1}{t}\|F_{0,t}(\omega)\|)$ is a.s. bounded. Thus by (12) and Lemma 5.3 there exists a set $C \in \mathcal{P}_{\mathrm{wkc}}(X)$ such that

$$\frac{1}{2^n m} F_{0,2^n m} \xrightarrow{\mathcal{T}_{\mathrm{scalar}}} C$$

(for any m as $n \to \infty$). In view of Theorem 3.1, $\frac{1}{t}F_{0,t}(\omega) \xrightarrow{\mathcal{T}_{\mathrm{scalar}}} C$ a.s. in Ω. Recalling that $s(x^*, C) = \widetilde{\alpha}_{x^*}$ for all $x^* \in X^*$ and taking into account (13) we get (b). ■

Proof of Theorem 3.5. Part (b) holds by the argument used in the proof of Theorem 3.4. Let us prove (a). Theorem 3.4 implies that there exists $C^w \in \mathcal{P}_{\mathrm{wkc}}(X)$ such that

$$\lim_{t\to\infty} \frac{1}{t} s(x^*, F_{0,t}(\omega)) = s(x^*, C^w) \quad \text{a.s.} \quad \text{for all } x^* \in X^*.$$

By Theorem 3.2, there exists a $\mathcal{P}_{\mathrm{skc}}(X)$-valued random set F_∞ satisfying

$$\lim_{t\to\infty} \varrho_{\mathrm{H}}\Big(\frac{1}{t} F_{0,t}(\omega), F_\infty(\omega)\Big) = 0 \quad \text{a.s.}$$

Thus also $\lim_{t\to\infty} \frac{1}{t} s(x^*, F_{0,t}(\omega)) = s(x^*, F_\infty(\omega))$ a.s. for all $x^* \in X^*$. Therefore (a) holds. ■

Proof of Theorem 3.6. The proof of Theorem 3.4 can be adapted. It is enough to use Theorem 3.3 where Theorem 3.1 is used. ■

Proof of Proposition 3.1. The sequence $(\|F_n\|)$ is $\mathcal{L}^m$-convergent, thus it has an a.s. convergent subsequence $(\|F_{n_k}\|)$. Let, for each $x^* \in H$, φ_{x^*} be the $\mathcal{L}^m$-limit of $(s(x^*, F_n))$. Take an arbitrary subsequence of (n_k) and denote it still by (n_k). Each sequence $(s(x^*, F_{n_k}))$ has an a.s. convergent subsequence. By the diagonal method we extract a subsequence of (n_k) (still denoted by (n_k)) and find a negligible set $N \subset \Omega$ such that for all $\omega \in N^c$,

$$\begin{aligned} \varphi_{x^*}(\omega) &:= \lim_{k\to\infty} s(x^*, F_{n_k}(\omega)) \quad \text{for all } x^* \in H, \\ \psi(\omega) &:= \lim_{k\to\infty} \|F_{n_k}(\omega)\|. \end{aligned} \tag{14}$$

By (14), for each $\omega \in N^c$ there is $k_\omega \in \mathbb{N}$ such that for all $k \ge k_\omega$, $(\|F_{n_k}(\omega)\|)$ is bounded. Lemma 5.3 yields the existence of a $\mathcal{P}_{\mathrm{wkc}}(X)$-valued multifunction F_∞ such that

$$F_{n_k}(\omega) \xrightarrow{\mathcal{T}_{\mathrm{scalar}}} F_\infty(\omega) \quad \text{a.s.} \tag{15}$$

Recall that $s(x^*, F_{n_k}(\omega)) \to \varphi_{x^*}(\omega)$ a.s. for each $x^* \in H$. Thus $s(x^*, F_\infty(\omega)) = \varphi_{x^*}(\omega)$ a.s. for all $x^* \in H$. Now we conclude that, for each $x^* \in H$,

$$\int_\Omega |s(x^*, F_n(\omega)) - s(x^*, F_\infty(\omega))|^m \, dP(\omega) \to 0.$$

To show that $F_\infty \in \mathcal{L}^m_{\mathcal{P}_{\mathrm{wkc}}(X)}$ recall that if $C_n \xrightarrow{\mathcal{T}_{\mathrm{scalar}}} C_\infty$ then $\|C_\infty\| \le \liminf_n \|C_n\|$. Therefore $\|F_\infty\|_m^m \le \liminf \|F_n\|_m^m < \infty$. ■

Proof of Theorem 3.7. Theorem 3.1 implies that there exists $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{\mathrm{wkc}}(X)}$ such that $\frac{1}{t}s(x^*, F_{0,t})$ converges to $s(x^*, F_\infty)$ for any $x^* \in X^*$. Also, as follows from the proof of Theorem 3.1, there exists $\psi \in \mathcal{L}^1_{\mathbb{R}}$ such that $\mathcal{L}^1\text{-}\lim_{t\to\infty} \|F_{0,t}(\omega)\| = \psi(\omega)$. Now the result follows from Proposition 3.1. ■

Proof of Proposition 3.2. Analogous to the proof of Proposition 3.1. ■

Proof of Theorem 3.8. By Theorem 3.2 there exists an $F_\infty \in \mathcal{L}^1_{\mathcal{P}_{\mathrm{skc}}(X)}$ such that a.s. $\varrho_{\mathrm{H}}(\frac{1}{t}F_{0,t}(\omega), F_\infty(\omega)) \to 0$, thus $(\frac{1}{t}s(x^*, F_{0,t}))$, $(\frac{1}{t}\|F_{0,t}\|)$ converge in $\mathcal{L}^1_{\mathbb{R}}$. The result now follows from Proposition 3.2. ■

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