On inessential and improjective operators

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Abstract. We give several characterizations of the improjective operators, introduced by Tanabé, and we characterize the inessential operators among the improjective operators. It is an interesting problem whether both classes of operators coincide in general. A positive answer would provide, for example, an intrinsic characterization of the inessential operators. We give several equivalent formulations of this problem and we show that the inessential operators acting between certain pairs of Banach spaces coincide with the improjective operators.

1. Introduction. An important class which occurs in the perturbation theory of Fredholm operators is that of inessential operators, introduced by Kleinecke [7] as the inverse image in $L(X)$ by the quotient map

$$
\pi : L(X) \rightarrow L(X)/K(X)
$$

of the radical of the Calkin algebra $L(X)/K(X)$, where $X$ is a Banach space, $L(X)$ is the set of all (continuous linear) operators on $X$ and $K(X)$ is the subset of all compact operators.

Other authors [9, 10] have defined and studied inessential operators acting between different Banach spaces $X,Y$. Let $L(X,Y)$ be the set of all (continuous linear) operators acting from $X$ into $Y$. An operator $T \in L(X,Y)$ is Fredholm, in symbols $T \in \Phi(X,Y)$, if its kernel $\ker(T)$ is finite-dimensional and its range $R(T)$ is finite-codimensional. The inessential operators can be defined by

$$
\mathcal{I}n(X,Y) := \{T \in L(X,Y) : I_X - ST \in \Phi(X) \text{ for every } S \in L(Y,X)\},
$$

where $I_X$ is the identity operator in $X$ and $\Phi(X) = \Phi(X, X)$. Equivalently [2],

$$
\mathcal{I}n(X,Y) := \{T \in L(X,Y) : I_Y - TS \in \Phi(Y) \text{ for every } S \in L(Y,X)\}.
$$

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This class of operators can be characterized in more algebraic terms.

**Theorem 1.1** ([9], [2, Lemma 1.1 and Theorem 1.4]). For an operator \( T \in \mathcal{L}(X,Y) \) the following assertions are equivalent:

(a) \( T \in \mathcal{I}_n(X,Y) \);
(b) \( \ker(I_X - ST) \) is finite-dimensional for every \( S \in \mathcal{L}(Y,X) \);
(c) \( \ker(I_Y - TS) \) is finite-dimensional for every \( S \in \mathcal{L}(Y,X) \);
(d) \( R(I_Y - ST) \) is finite-dimensional for every \( S \in \mathcal{L}(Y,X) \);
(e) \( R(I_Y - TS) \) is finite-dimensional for every \( S \in \mathcal{L}(Y,X) \).

It is well known that \( \mathcal{I}_n(X,Y) \) is a closed subspace of \( \mathcal{L}(X,Y) \). Moreover, the class \( \mathcal{I}_n \) of all the inessential operators is an operator ideal, in the sense of Pietsch [10], that contains the operator ideals which occur in Fredholm theory, namely the compact, the strictly singular and the strictly cosingular operators.

The characterizations of \( T \in \mathcal{I}_n \) existing in the literature are expressed, like those in Theorem 1.1, in terms of the properties of the product of \( T \) by a large class of operators. It is a problem of certain interest to find an "intrinsic" characterization of the inessential operators, for instance in terms of their action on the complemented subspaces. This would be obtained, for example, if the class \( \mathcal{I}_n \) coincided with the class of the injective operators, introduced by Tarafdar in [11, 12]. As a consequence, we would obtain some structural information about the complemented subspaces of products of Banach spaces: if no infinite-dimensional complemented subspace of \( X \) is isomorphic to a complemented subspace of \( Y \), then every complemented subspace of \( X \times Y \) would be isomorphic to the product of a complemented subspace of \( X \) and a complemented subspace of \( Y \) (see [4]).

In fact, inessential operators are injective, but it is not known if these two classes coincide. Tarafdar [12] gave an affirmative answer in some special cases.

In this paper we give, in Theorem 2.3, a dual characterization of the injective operators. We apply this result to study these operators and to characterize the inessential operators among the injective operators, in Theorem 2.6.

In the third section we consider the question whether the classes of inessential operators and injective operators coincide. We give several formulations of this question and describe some related problems.

Finally, in the fourth section we describe some families of pairs of Banach spaces \( X, Y \) such that the injective operators in \( \mathcal{L}(X,Y) \) are inessential.

Along the paper, \( \mathbb{K} \) is the field \( \mathbb{R} \) of real numbers or the field \( \mathbb{C} \) of complex numbers. The results are valid in both cases if the field is not explicitly mentioned. Further, \( X, Y, Z \) and \( W \) are Banach spaces over \( \mathbb{K} \), and we denote by \( X^* \) the dual space of \( X \), and by \( \mathcal{L} \) the class of all (continuous linear) operators between Banach spaces. Given a subclass \( \mathcal{A} \) of \( \mathcal{L} \), the subsets

\[ A(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y) \]

are called the components of \( \mathcal{A} \). Moreover, \( A(X) := A(X,X) \).

We denote by \( T^* \) \( \in \mathcal{L}(Y^*, X^*) \) the conjugate operator of \( T \in \mathcal{L}(X,Y) \).

Subspaces of a Banach space are not necessarily closed. Given a closed subspace \( M \) of \( X \), we denote by \( J_M \) the inclusion of \( M \) into \( X \), and by \( Q_M \) the quotient map from \( X \) onto \( X/M \). A subspace \( M \) of \( X \) is complemented if there exists \( P \in \mathcal{L}(X) \) so that \( P^2 = P \) (i.e., \( P \) is a projection) and \( R(P) = M \). Of course, complemented subspaces are closed. Given subspaces \( M \) of \( X \) and \( U \) of \( X^* \), we denote by \( M^1 \subset X^* \) and \( U_1 \subset X \) their respective annihilators.

2. Characterizations of inessential operators and injective operators. Injective operators were introduced by Tarafdar in the following way.

**Definition 2.1** ([11, 12]). An operator \( T \in \mathcal{L}(X,Y) \) is said to be injective if there is no infinite-dimensional closed subspace \( M \) of \( X \) such that the restriction \( T|_M \) is an isomorphism and \( T(M) \) is a complemented subspace of \( Y \).

We denote by \( \mathcal{I}mp \) the class of injective operators. It was proved in [11, Theorem 3.6] that \( \mathcal{I}mp(X,Y) \) is a closed subset of \( \mathcal{L}(X,Y) \).

Next we give a lemma which will be useful in the study of these operators and their relation to other classes of operators. A result similar to the first part was proved before in [11, Lemma 1.1].

**Lemma 2.2.** Let \( T \in \mathcal{L}(X,Y) \).

(a) If \( M \) is a closed subspace of \( X \) such that \( T|_M \) is an isomorphism, \( T(M) \) is complemented in \( Y \) and \( N \) is a closed complement of \( T(M) \), then \( M \) is complemented in \( X \) and \( T^{-1}(N) \) is a closed complement of \( M \).

(b) If \( N \) is a closed subspace of \( Y \) such that \( Q_N T \) is surjective, \( T^{-1}(N) \) is complemented in \( X \) and \( M \) is a closed complement of \( T^{-1}(N) \), then \( N \) is complemented in \( Y \) and \( T(M) \) is a closed complement of \( N \).

**Proof.** (a) If \( N \) is a closed complement of \( T(M) \) in \( Y \) and \( T|_M \) is an isomorphism, then \( T^{-1}(N) \cap M = \{0\} \), both \( T^{-1}(N) \) and \( M \) are closed subspaces and \( X = T^{-1}(N) \oplus M \). Hence the result is a direct consequence of the closed graph theorem.

(b) If \( M \) is a closed complement of \( T^{-1}(N) \) in \( X \), then since \( \ker(T) \) is contained in \( T^{-1}(N) \), we have \( T(M) \cap N = \{0\} \). Moreover, since \( Q_N T \) is surjective, we obtain \( T(M) \oplus N = T(X) + N = Y \), and it follows from [13, Theorem IV.5.10] that \( T(M) \) is closed; hence \( N \) is complemented in \( Y \). ■
Improtective operators admit the following characterization in terms of quotient maps.

**Theorem 2.3.** An operator $T \in \mathcal{L}(X,Y)$ is improtective if and only if there is no infinite-codimensional closed subspace $N$ of $Y$ such that $Q_N T$ is surjective and $T^{-1}(N)$ is a complemented subspace of $X$.

**Proof.** Assume that $T \in \mathcal{L}(X,Y)$ is improtective and let $N$ be a closed subspace of $Y$ such that $Q_N T$ is surjective and $T^{-1}(N)$ is a complemented subspace of $X$. By Lemma 2.2, if $M$ is a closed complement of $T^{-1}(N)$ then $T(M)$ is a closed complement of $N$. Observe that the restriction of $T$ to $M$ is an isomorphism and $T$ is improtective. Therefore $T(M)$ is finite-dimensional; hence $N$ is finite-codimensional.

Conversely, assume that $T$ is not improtective and take an infinite-dimensional closed subspace $M$ of $X$ such that $T|_M$ is an isomorphism and $T(M)$ is complemented in $Y$. Given a closed complement $N$ of $T(M)$, we see that $N$ is infinite-codimensional and $Q_N T$ is surjective. Hence, by Lemma 2.2, we conclude that $M$ is a closed complement of $T^{-1}(N)$.

Tarafdar [12] proved, for operators in $\mathcal{L}(X)$, that inessential operators are improtective. Here we give an elementary proof for the general case.

**Proposition 2.4.** Inessential operators are improtective.

**Proof.** If $T \in \mathcal{L}(X,Y)$ is not improtective, then there exists an infinite-dimensional closed subspace $M$ of $X$ such that the restriction $T|_M$ is an isomorphism and $T(M)$ is a complemented subspace of $Y$. By Lemma 2.2, $M$ is also complemented in $X$ and

$$X = M \oplus T^{-1}(N) \quad \text{and} \quad Y = T(M) \oplus N,$$

where $N$ and $T^{-1}(N)$ are closed subspaces of $Y$ and $X$, respectively. So we can define an operator $S \in \mathcal{L}(Y,X)$ by

$$Sy := \begin{cases} (T|_M)^{-1}y & \text{if } y \in T(M), \\ 0 & \text{if } y \in N. \end{cases}$$

We have $\ker(I_Y - ST) = M$; hence by Theorem 1.1, the operator $T$ is not inessential.

The following lemma will be the key to characterizing the inessential operators among the improtective operators.

**Lemma 2.5.** Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,X)$.

(a) If the subspace $M := \ker(I_Y - ST)$ is complemented in $X$ with closed complement $U$, then both subspaces $T(M)$ and $S^{-1}(U)$ are complemented in $Y$; in fact, we have $Y = T(M) \oplus S^{-1}(U)$.

(b) If the subspace $N := R(I_Y - TS)$ is complemented in $Y$ with closed complement $V$, then both subspaces $T^{-1}(N)$ and $S(V)$ are complemented in $X$; in fact, we have $X = S(V) \oplus T^{-1}(N)$.

**Proof.** (a) Let $P$ denote the projection from $X$ onto $M$ along $U$; thus $R(P) = M$ and $\ker(P) = U$. Since $(I_X - ST)P = 0$, we have $P = STP$. Therefore, defining $Q := TPS$, we have $Q^2 = TP(STP)S = TP^2 S = Q$; i.e., $Q$ is a projection in $Y$.

From $P = STP$ we obtain $\ker(T) \cap R(P) = \{0\}$; thus $\ker(Q) = \ker(PS) = S^{-1}(U)$.

Moreover, $P = STP$ implies that $ST(M) = M$; hence $S^{-1}(U) \cap T(M) = \{0\}$. On the other hand, from $Q = TPS$, it follows that $R(Q) \subset T(M)$; thus we conclude that $R(Q) = T(M)$.

(b) Denote by $Q$ the projection from $Y$ onto $V$ along $N$. Then we have $(I_Y - TS) = 0$; hence $Q = QTS$. Therefore, as in the previous part, we have $P = STQ$, which defines a projection in $X$.

From $P = STQ$ we obtain $\ker(S) \cap R(Q) = \{0\}$; thus $\ker(P) = \ker(PS) = T^{-1}(N)$.

Moreover, $Q = QTS$ implies that $(TS)^{-1}(N) = N$; hence $S(V) \cap T^{-1}(N) = \{0\}$. On the other hand, from $P = STQ$, it follows that $R(P) \subset S(V)$; thus we conclude that $R(P) = S(V)$.

Finally, we give several characterizations of the inessential operators among the improtective operators in terms of the complementability of some subspaces.

**Theorem 2.6.** For an operator $T \in \mathcal{L}(X,Y)$ the following assertions are equivalent:

(a) $T \in \mathcal{I}(X,Y)$;

(b) $T \in \mathcal{I}(X,Y)$ and $\ker(I_X - ST)$ is complemented for every $S \in \mathcal{L}(Y,X)$;

(c) $T \in \mathcal{I}(X,Y)$ and $\ker(I_Y - TS)$ is complemented for every $S \in \mathcal{L}(Y,X)$;

(d) $T \in \mathcal{I}(X,Y)$ and $R(I_X - ST)$ is complemented for every $S \in \mathcal{L}(Y,X)$;

(e) $T \in \mathcal{I}(X,Y)$ and $R(I_Y - TS)$ is complemented for every $S \in \mathcal{L}(Y,X)$.

**Proof.** First we show that (a) implies the other assertions.

Assume that $T$ is inessential. By Proposition 2.4, $T$ is improtective. Moreover, by Theorem 1.1, for every $S \in \mathcal{L}(Y,X)$, $\ker(I_X - ST)$ and $\ker(I_Y - TS)$ are finite-dimensional, and $R(I_X - ST)$ and $R(I_Y - TS)$ are finite-codimensional; hence all of them are complemented.
(b)⇒(a). Assume that $T \in \text{Imp}(X,Y)$ and $M := \ker(I_X - ST)$ is complemented. Then $T$ is an isomorphism on $M$ and, by Theorem 2.5, $T(M)$ is complemented. Hence $M$ is finite-dimensional.

We have seen that (b) implies that $\ker(I_X - ST)$ is finite-dimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$.

(c)⇒(a). Assume that $T \in \text{Imp}(X,Y)$ and $N := \ker(I_Y - TS)$ is complemented. Then $S$ is an isomorphism on $N$ (so that $S(N)$ is closed), $T$ is an isomorphism on $S(N)$ and $T(S(N)) = N$ is complemented. Hence $N$ is finite-dimensional.

We have seen that (c) implies that $\ker(I_Y - TS)$ is finite-dimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$.

(d)⇒(a). Assume that $T \in \text{Imp}(X,Y)$ and $M := \overline{R(I_X - ST)}$ is complemented. Since $Q_M(I_X - ST) = 0$, we have $Q_MST = Q_M$; in particular, $R(ST) + M = X$. Then $R(T) + S^{-1}(M) = Y$; i.e., $Q_{S^{-1}(M)}T$ is surjective. Moreover,

$$T^{-1}S^{-1}(M) = (ST)^{-1}(M) = ((ST)^*(M^1))_1$$

$$= (T^*S^*(\ker(I_X^* - T^*S^*))_1$$

$$= (\ker(I_X^* - T^*S^*))_1 = M$$

is complemented. By Theorem 2.3 we deduce that $S^{-1}(M)$ is finite-codimensional; hence so is $M = T^{-1}S^{-1}(M)$.

Therefore, (d) implies that $\overline{R(I_X - ST)}$ is finite-codimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$.

(e)⇒(a). Assume that $T \in \text{Imp}(X,Y)$ and $N := \overline{R(I_Y - TS)}$ is complemented. Since $Q_N(I_Y - TS) = 0$, we have $Q_NT = Q_N$; in particular, $Q_NT$ is surjective and, by Lemma 2.5, $T^{-1}(N)$ is complemented. Applying Theorem 2.3 we conclude that $N$ is finite-codimensional.

Therefore, (e) implies that $\overline{R(I_Y - TS)}$ is finite-codimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$. ■

3. On the inclusion $\mathcal{I}n(X,Y) \subset \text{Imp}(X,Y)$. We begin this section by giving an alternative proof of the fact that the imprejective operators form an ideal with respect to the product, and introducing the concept of quasi-operator ideal, which will be useful in our discussions.

**Proposition 3.1** [11, Theorem 1.2]. Let $A \in \mathcal{L}(Y,Z)$, $K \in \text{Imp}(X,Y)$ and $B \in \mathcal{L}(W,X)$. Then $KB \in \text{Imp}(W,Y)$ and $AK \in \text{Imp}(X,Z)$.

**Proof.** Assume first that $AK$ is not imprejective. Then we can find an infinite-dimensional closed subspace $M$ of $X$ such that $AKJ_M$ is an isomorphism and $AK(M)$ is complemented in $Z$. Note that $KJ_M$ is also an isomorphism; hence $K(M)$ is closed.

Since $AJ(K(M))$ is an isomorphism, it follows from Lemma 2.2 that $K(M)$ is complemented in $Y$. Hence $K$ is not imprejective.

In the case in which $KB$ is not imprejective, Theorem 2.3 allows us to select an infinite-codimensional closed subspace $N$ of $Y$ such that $Q_NB$ is surjective and $(KB)^{-1}(N)$ is a complemented subspace of $W$.

Putting $M := K^{-1}(N)$ we deduce that $M$ is an infinite-codimensional closed subspace of $X$ such that $Q_MB$ is surjective and $B^{-1}(M)$ is complemented. It follows from Lemma 2.2 that $M$ is complemented in $X$. Moreover, $Q_NK$ is surjective; hence it follows from Theorem 2.3 that $K$ is not imprejective. ■

Let $\mathcal{F}$ denote the class of all operators with finite-dimensional range.

**Definition 3.2.** A subclass $\mathcal{A}$ of $\mathcal{L}$ is said to be a quasi-operator ideal if it satisfies

(a) $\mathcal{F} \subset \mathcal{A}$;
(b) $A \in \mathcal{L}(Y,Z)$, $K \in \mathcal{A}(X,Y)$, $B \in \mathcal{L}(W,X) \Rightarrow AKB \in \mathcal{A}(W,Z)$.

A quasi-operator ideal is an operator ideal [in the sense of Piestch [10]] if and only if $\mathcal{A}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$ for every pair $X,Y$ of Banach spaces.

The main question that remains open concerns the equality $\mathcal{I}n = \text{Imp}$.

**Question 1.** Is it true that $\mathcal{I}n(X,Y) = \text{Imp}(X,Y)$ for every pair $X,Y$ of Banach spaces?

We say that a quasi-operator ideal $\mathcal{A}$ is proper if $I_X \in \mathcal{A}$ for no infinite-dimensional space $X$.

As far as we know, it is an open problem whether $\mathcal{I}n$ is the greatest proper operator ideal. However, we have the following result, essentially due to Tůtařar [11, Corollary 3.3], which follows easily from Definition 2.1 and Proposition 3.1.

**Proposition 3.3.** The class of all imprejective operators is the greatest proper quasi-operator ideal.

Therefore, a positive answer to the following question would provide a description of the greatest proper operator ideal.

**Question 2.** Is it true that $\text{Imp}(X,Y)$ is a subspace for every pair $X,Y$?

Later we shall show that the equality $\mathcal{I}n(X,Y) = \text{Imp}(X,Y)$ holds in many cases. We think that Questions 1 and 2 are difficult. So a positive answer to the following one could be useful. For example, it would show that the previous two are equivalent.

**Question 3.** Fix a pair $X,Y$ of Banach spaces. Is it true that $\mathcal{I}n(X,Y) = \text{Imp}(X,Y)$ whenever $\text{Imp}(X,Y)$ is a subspace?
The next result shows a symmetry of the equality $\mathcal{I}n = \mathcal{I}mp$.

**Proposition 3.4.** Let $X$ and $Y$ be Banach spaces. Then $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ if and only if $\mathcal{I}n(Y,X) = \mathcal{I}mp(Y,X)$.

**Proof.** Suppose that $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ and let $T \in \mathcal{I}mp(Y,X)$. We only have to show that $T \in \mathcal{I}n$.

Given $S \in \mathcal{L}(X,Y)$, by Proposition 3.1 we have $STS \in \mathcal{I}mp(X,Y) = \mathcal{I}n(X,Y)$; hence $(TS)^2 \in \mathcal{I}n$. Therefore $I_X - (TS)^2 = (I_X + TS)(I_X - TS) \in \Phi(X)$, and hence $\ker(I_X - TS) \subset \ker(I_X - (TS)^2)$ is finite-dimensional. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$.

Let $X$ be a Banach space and let $T \in \mathcal{L}(X)$. Given an invariant subspace $M$ of $T$, i.e., $M$ closed and $T(M) \subset M$, we can define two associated operators

$$T_M : M \to M \quad \text{and} \quad T^M : X/M \to X/M$$

in the natural way: $T_M m := Tm$ and $T^M(x + M) := Tx + M$.

To give some sufficient conditions for $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$, we consider two classes of operators $\Omega_+$ and $\Omega_-$ introduced in [1]:

$$\Omega_+(X) := \{T \in \mathcal{L}(X) : T_M \text{ is an (into) isomorphism for no infinite-dimensional, invariant subspace } M \}$$

$$\Omega_-(X) := \{T \in \mathcal{L}(X) : T^M \text{ is surjective for no infinite-codimensional, invariant subspace } M \}$$

**Theorem 3.5.** Let $A$ be a quasi-operator ideal and let $Y$ be a Banach space.

(a) If $A(Y) \subset \Omega_+(Y)$, then $A(Y,Z) \subset \mathcal{I}n(Y,Z)$ for every Banach space $Z$.

(b) If $A(Y) \subset \Omega_-(Y)$, then $A(X,Y) \subset \mathcal{I}n(X,Y)$ for every Banach space $X$.

**Proof.** (a) Suppose that there exists $T \in A(Y,Z) \setminus \mathcal{I}n(Y,Z)$. By Theorem 1.1, we can find $S \in \mathcal{L}(Z,Y)$ so that $M := \ker(I_Y - ST)$ is infinite-dimensional. However, since $ST \in A(Y) \subset \Omega_+(Y)$ and $ST$ coincides with the identity on $M$, we find that $M$ is finite-dimensional; a contradiction.

(b) Suppose that there exists $T \in A(X,Y) \setminus \mathcal{I}n(X,Y)$. By Theorem 1.1, we can find $S \in \mathcal{L}(Y,X)$ so that $N := \mathcal{R}(I_Y - TS)$ is infinite-codimensional. However, since $TS \in A(Y) \subset \Omega_-(Y)$, $N$ is an invariant subspace of $TS$ and $(TS)^N$ is surjective, we deduce that $N$ is finite-codimensional; a contradiction.

In the case $A = \mathcal{I}mp$, we obtain further characterizations of pairs $X,Y$ satisfying $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$.

**Corollary 3.6.** For a Banach space $X$ the following statements are equivalent:

(a) $\mathcal{I}n(X) = \mathcal{I}mp(X)$

(b) $\mathcal{I}mp(X) \subset \Omega_+(X)$

(c) $\mathcal{I}mp(X) \subset \Omega_-(X)$

(d) $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ for every Banach space $Y$.

**Proof.** For the implications (a)$\Rightarrow$(b) and (a)$\Rightarrow$(c), it is enough to observe that $\mathcal{I}n(X) \subset \Omega_+(X) \cap \Omega_-(X)$. Indeed, if $T \in \mathcal{L}(X) \setminus \mathcal{I}n(X)$, then using Theorem 1.1 we can find an operator $S \in \mathcal{L}(X)$ so that $M := \ker(I_X - ST)$ is infinite-dimensional. Since the restriction of $T$ to $M$ is an isomorphism and $T(M) \subset M$, we conclude $T \notin \Omega_+$. Analogously, we can conclude $T \notin \Omega_-$.

(b)$\Rightarrow$(d) follows from Theorem 3.5 and $\mathcal{I}n(X,Y) \subset \mathcal{I}mp(X,Y)$.

(c)$\Rightarrow$(d) follows from Theorem 3.5, the inclusion $\mathcal{I}n(Y,X) \subset \mathcal{I}mp(Y,X)$ and the equivalence $\mathcal{I}n(X,Y) = \mathcal{I}mp(Y,X) \iff \mathcal{I}n(Y,X) = \mathcal{I}mp(Y,X)$, proved in Proposition 3.4.

(d)$\Rightarrow$(a) is trivial.

Corollary 3.6 shows that Question 1 is equivalent to the following one.

**Question 4.** Is it true that $\mathcal{I}n(X) = \mathcal{I}mp(X)$ for every Banach space $X$?

Let $X$ be a complex Banach space and $T \in \mathcal{L}(X)$. Recall [6, §48, page 203] that a subset $\sigma$ of the spectrum $\sigma(T)$ of $T$ is a spectral set if both $\sigma$ and $\sigma(T)$ are closed.

**Proposition 3.7.** Let $X$ be an infinite-dimensional complex Banach space and let $T \in \mathcal{I}mp(X)$. Then we have

(a) $0 \in \sigma(T)$;

(b) if $\sigma \subset \sigma(T)$ is a spectral set and $0 \notin \sigma$, then the spectral projection associated with $\sigma$ has finite-dimensional range.

**Proof.** (a) Clearly, an invertible operator in an infinite-dimensional space cannot be isomorphic.

(b) Let $P$ denote the spectral projection associated with $\sigma$ (see [6, §49]). Then taking $M := \mathcal{R}(P)$, we have $T(M) = M$ and the restriction $T|_M$ is an isomorphism. Since $T \in \mathcal{I}mp$ we deduce that $M$ is finite-dimensional.

Next we show that, for complex Banach spaces, Question 4 can be formulated in terms of the spectral properties of imprimitive operators.

**Proposition 3.8.** Let $X$ be a complex Banach space. Then $\mathcal{I}mp(X) = \mathcal{I}n(X)$ if and only if for every $T \in \mathcal{I}mp(X)$, the spectrum $\sigma(T)$ is either a finite set or a sequence which clusters at 0.
Proof. The direct implication follows from a well-known property of the spectrum of inessential operators [1]. For the converse, fix an operator $T \in \text{Imp}(X)$. For every $S \in \mathcal{L}(X)$, we have $ST \in \text{Imp}(X)$. Now, by the hypothesis and Proposition 3.7, either $I_X - ST$ is bijective, or 1 is an isolated point in $\sigma(ST)$ and the spectral projection associated with the spectral set $\{1\}$ has finite-dimensional range. In any case, $\ker(I_X - ST)$ is finite-dimensional, and applying Theorem 1.1, we conclude that $T$ is inessential. 

Proposition 3.8 shows that, for complex Banach spaces, Question 4 is equivalent to the first part of the following one.

**Question 5.** Let $X$ be a complex Banach space $X$.

(a) Is it true that $\sigma(T)$ is either a finite set or a sequence which clusters at 0, for every $T \in \text{Imp}(X)$?

(b) We remark that we do not know if the assertion of Proposition 3.8 is valid for a single operator, instead of the whole set $\text{Imp}(X)$. Therefore, we ask: Assume that the spectrum of $T \in \text{Imp}(X)$ is either a finite set or a sequence which clusters at 0. Is $T$ inessential?

4. Some examples. Here we present several examples of pairs $X,Y$ of Banach spaces for which we have the equality $\text{Inn}(X,Y) = \text{Imp}(X,Y)$. Note that, by Proposition 3.4, this is equivalent to $\text{Inn}(Y,X) = \text{Imp}(Y,X)$. Some of these examples correspond to the case in which one of the spaces admits many projections, like the subprojective spaces and the superprojective spaces, and some others to the opposite case in which one of the spaces admits only trivial projections. First we recall some results of [2, 4] which give examples of pairs $X,Y$ such that $\mathcal{L}(X,Y) = \text{Inn}(X,Y)$, or equivalently, $\mathcal{L}(Y,X) = \text{Inn}(Y,X)$ [4, Proposition 1]. We refer to [4] for the definitions of the concepts involved in the results.

**Theorem 4.1** ([2, Theorem 2.3]; [4, Theorem 1]). We have the equalities $\mathcal{L}(X,Y) = \text{Imp}(X,Y) = \text{Inn}(X,Y)$

in the following cases:

(a) $X$ is reflexive and $Y$ has the Dunford-Pettis property;
(b) $X$ has the reciprocal Dunford-Pettis property and $Y$ has the Schur property;
(c) $X$ contains no copies of $\ell_\infty$ and $Y = \ell_\infty$, $H^{\infty}$ or $\mathcal{C}(K)$ with $K$ $\sigma$-stonian;
(d) $X$ contains no copies of $c_0$ and $Y = \mathcal{C}(K)$;
(e) $X$ contains no complemented copies of $c_0$ and $Y = L[0,1]$
(f) $X$ contains no complemented copies of $\ell_1$ and $Y = L_1(\mu)$;

(g) $X$ contains no complemented copies of $\ell_p$ and $Y = \ell_p$, $1 \leq p < \infty$;
(h) $X$ contains no complemented copies of $\ell_p$ or $\ell_2$ and $Y = L_p[0,1]$, $1 < p < \infty$.

We observe that Theorem 4.1 includes and extends the list of examples of Tarafdar [11, 12] of pairs $X,Y$ for which $\mathcal{L}(X,Y) = \text{Imp}(X,Y)$.

It was proved by Tarafdar [12, Theorem 3.2] that $\mathcal{L}(X,Y) = \text{Imp}(X,Y)$ if and only if no infinite-dimensional complemented subspace of $X$ is isomorphic to a complemented subspace of $Y$. So, the following question is natural.

**Question 6.** Let $X$ and $Y$ be Banach spaces such that $\mathcal{L}(X,Y) = \text{Imp}(X,Y)$. Is it true that every complemented subspace of $X \times Y$ is isomorphic to the product of a complemented subspace of $X$ and a complemented subspace of $Y$?

The answer is affirmative when $\mathcal{L}(X,Y) = \text{Inn}(X,Y)$ [4, Theorem 3].

**Definition 4.2.** A Banach space $X$ is said to be subprojective if every infinite-dimensional closed subspace of $X$ contains an infinite-dimensional subspace which is complemented in $X$. The space $X$ is said to be superprojective if every infinite-codimensional closed subspace of $X$ is contained in an infinite-codimensional subspace which is complemented in $X$.

The spaces $\ell_p$ ($1 < p < \infty$) are subprojective and superprojective, and the spaces $L_p[0,1]$ are subprojective for $2 \leq p < \infty$ and superprojective for $1 < p \leq 2$. Moreover, $\ell_1$ and $c_0$ are subprojective, but $L_1[0,1]$ and $C[0,1]$ are neither subprojective nor superprojective [15]. For further information on subprojective and superprojective spaces, we refer to [2].

If one of the spaces is subprojective or superprojective, the improjective operators are inessential.

**Theorem 4.3.** Assume that one of the spaces $X,Y$ is subprojective or superprojective. Then $\text{Imp}(X,Y) = \text{Inn}(X,Y)$.

**Proof.** Assume first that $Y$ is superprojective. If $T \in \mathcal{L}(X,Y)$ is not inessential, then we can find an operator $S \in \mathcal{L}(Y,X)$ such that $M := R(I_Y - TS)$ is infinite-codimensional in $Y$. We take an infinite-codimensional, complemented subspace $N$ of $Y$ containing $M$, and we select a projection $P$ with $\text{ker}(P) = N$.

We see that $R(P)$ is infinite-dimensional. Moreover, since $P(I_Y - TS) = 0$, we find that $PTS$ restricted to $R(P)$ coincides with the identity operator. Then $PTS$ is not improjective, and by Proposition 3.1, $T$ is not improjective.

Now we consider the case in which $X$ is subprojective. If $T \in \mathcal{L}(X,Y)$ is not inessential, then we can find an operator $S \in \mathcal{L}(Y,X)$ such that $M := R(I_Y - TS)$ is infinite-codimensional in $Y$. We take an infinite-codimensional, complemented subspace $N$ of $Y$ containing $M$, and we select a projection $P$ with $\text{ker}(P) = N$.
ker(\(I_X - ST\)) is infinite-dimensional in \(X\). We take an infinite-dimensional, complemented subspace \(N\) of \(X\) contained in \(M\).

Since \(ST\) restricted to the subspace \(N\) coincides with the identity, we see that \(T(N)\) is closed, the restriction \(SJ_{T(N)}\) is an isomorphism and \(S(T(N))\) is complemented. Hence, by Lemma 2.2, \(T(N)\) is complemented. Since \(TJ_N\) is an isomorphism, we conclude that \(T\) is not imjective.

For the remaining cases it is enough to apply Proposition 3.4 and the previously proved cases. ■

Recall that an operator \(T \in \mathcal{L}(X,Y)\) is strictly singular if no restriction \(TJ_M\) of \(T\) to an infinite-dimensional closed subspace \(M\) of \(X\) is an isomorphism. The operator \(T\) is strictly cosingular if there is no infinite-codimensional closed subspace \(N\) of \(Y\) such that \(R(T) + N = Y\). We denote by \(SS\) and \(SC\) the classes of all strictly singular and strictly cosingular operators, respectively. These classes are closed operator ideals \([10, 1.9, 1.10, 4.2.7]\). Moreover, both \(SS\) and \(SC\) are contained in \(\mathcal{I}\) in \([10, 26.7.3]\). In particular, the inclusion
\[SS(\mathcal{X}, \mathcal{Y}) \cup SC(\mathcal{X}, \mathcal{Y}) \subset \mathcal{I}_{mp}(\mathcal{X}, \mathcal{Y})\]
holds for each pair \(\mathcal{X}, \mathcal{Y}\) of Banach spaces.

In the following result we show that, in some cases in which one of the spaces is subprojective or superprojective, the imjective operators coincide with the strictly singular or the strictly cosingular operators. We observe that part (a) was proved before in \([11, \text{Theorem 1.3}]\).

**Theorem 4.4.**
(a) If \(Y\) is subprojective, then \(\mathcal{I}_{mp}(\mathcal{X}, \mathcal{Y}) = SS(\mathcal{X}, \mathcal{Y})\).
(b) If \(X\) is superprojective, then \(\mathcal{I}_{mp}(\mathcal{X}, \mathcal{Y}) = SC(\mathcal{X}, \mathcal{Y})\).

**Proof.** (a) Assume that \(Y\) is subprojective and \(T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})\) is not strictly singular. Take an infinite-dimensional closed subspace \(M\) of \(X\) such that \(TJ_M\) is an isomorphism. Since \(Y\) is subprojective, we can take an infinite-dimensional subspace \(N\) of \(T(M)\) which is complemented in \(Y\). Then \(A := (T|_M)^{-1}(N)\) is an infinite-dimensional closed subspace of \(X\) contained in \(M\). In particular, \(TJ_A\) is an isomorphism. Moreover, \(T(A) = N\) is complemented in \(Y\). Hence \(T\) is not imjective.

(b) Assume that \(X\) is superprojective and \(T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})\) is not strictly cosingular. Take an infinite-codimensional closed subspace \(N\) of \(X\) such that \(Q_NT\) is surjective. Then \(ker(Q_NT) = T^{-1}(N)\) is closed and infinite-codimensional; indeed, if \(A\) is an algebraic complement of \(T^{-1}(N)\) in \(X\), then \(Y = N + T(A)\).

Since \(X\) is superprojective, we can take an infinite-codimensional complemented subspace \(L\) of \(X\) containing \(T^{-1}(N)\). Now, since \(Q_NT\) is surjective and \(ker(Q_NT) \subset L\), we see that \(Q_NT(L)\) is closed; hence \(B := Q_NT^{-1}(Q_NT(L)) = T(L) + N\) is closed as well.

Note that \(Q_NT\) is surjective and \(T^{-1}(B) = L\); in particular, \(B\) is infinite-codimensional. Therefore, by Theorem 2.3 we conclude that \(T\) is not imjective.

The following examples show that we cannot change the order of the spaces \(\mathcal{X}, \mathcal{Y}\) in Theorem 4.4. In (a), if \(X\) is subprojective, then \(\text{Imp}(\mathcal{X}, \mathcal{Y}) = SS(\mathcal{X}, \mathcal{Y})\) is not true in general, and analogously in (b).

**Example 4.5.**
(a) The space \(\ell_2\) is subprojective and, by Theorem 4.1(c), we have
\[\mathcal{L}(\ell_2, \ell_\infty) = \text{Imp}(\ell_2, \ell_\infty) = \mathcal{I}_{n}(\ell_2, \ell_\infty);\]
however, \(\mathcal{L}(\ell_2, \ell_\infty) \neq SS(\ell_2, \ell_\infty)\), because \(\ell_\infty\) contains a closed subspace isomorphic to \(\ell_2\) \([3, \text{Theorem IV.II.2}]\).

Another example may be derived from the fact that the natural inclusion of \(L_2[0,1]\) in \(L_1[0,1]\) is not strictly singular, because it is an isomorphism on the closed subspace generated by the Rademacher functions \([3, \text{Proposition VI.1.1}]\). However, by Theorem 4.1(f), we have \(\mathcal{L}(L_2[0,1], L_1[0,1]) = \text{Imp}(L_2[0,1], L_1[0,1])\).

(b) The space \(\ell_2\) is superprojective and, by \([3, \text{Proposition IV.II.2}]\), every operator \(T \in \mathcal{L}(\ell_1, \ell_2)\) is strictly singular. Therefore, we have
\[\mathcal{L}(\ell_1, \ell_2) = \text{Imp}(\ell_1, \ell_2) = \mathcal{I}_{n}(\ell_1, \ell_2).\]

However, \(\mathcal{L}(\ell_1, \ell_2) \neq SC(\ell_1, \ell_2)\), because \(\ell_1\) has a quotient isomorphic to \(\ell_2\) \([3, \text{Theorem IV.II.1}]\).

Next we consider some examples \(X, Y\) for which \(\mathcal{I}_{n}(X, Y) = \text{Imp}(X, Y)\) because one of the spaces has very few projections.

**Definition 4.6.** A Banach space \(X\) is said to be indecomposable if it does not contain pairs of infinite-dimensional closed subspaces \(M, N\) so that \(X = M \oplus N\).

Recall that an operator \(T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})\) is upper semi-Fredholm, in symbols \(T \in \Phi_+(\mathcal{X}, \mathcal{Y})\), if its range \(R(T)\) is closed and its kernel \(ker(T)\) is finite-dimensional; \(T\) is lower semi-Fredholm, in symbols \(T \in \Phi_-(\mathcal{X}, \mathcal{Y})\), if \(R(T)\) is finite-codimensional (hence closed \([13, \text{Theorem IV.5.10}]\)).

The following result, essentially proved by Weis ([14, Corollary 2.3] contains the hard implications), characterizes the class of Banach spaces such that any operator either is semi-Fredholm or belongs to one of the operator ideals \(SS\) or \(SC\), in terms of the decomposability of their closed subspaces or quotients. We include a proof for the convenience of the reader and for future reference to some of the steps.
Theorem 4.7 [14]. Let $Y$ be a Banach space.

(a) We have $L(Y, Z) = SS(Y, Z) \cup \Phi_+(Y, Z)$ for every Banach space $Z$ if and only if all the closed subspaces of $Y$ are indecomposable.

(b) We have $L(X, Y) = SC(X, Y) \cup \Phi_-(X, Y)$ for every Banach space $X$ if and only if all the quotients of $Y$ are indecomposable.

Proof. (a) Assume that $T \in L(Y, Z)$, but $T \notin SS \cup \Phi_+$. Since $T$ is not strictly singular, we can find a number $\delta > 0$ and an infinite-dimensional closed subspace $M$ of $Y$ such that $\|Tx\| \geq \delta \|x\|$ for every $x \in M$. Now, since $T$ is not upper semi-Fredholm, there exists an infinite-dimensional closed subspace $N$ of $Y$ so that $|Tz| < (\delta/2)\|z\|$ for every $z \in N$. It is immediate to check that $M \cap N = \{0\}$ and $M + N$ is closed; hence the closed subspace $M \oplus N$ of $Y$ is not decomposable.

Conversely, assume that not all the closed subspaces of $Y$ are indecomposable. Then we can find two closed infinite-dimensional subspaces $M, N$ of $Y$ such that $M \cap N = \{0\}$ and $M + N$ is closed. Thus, the quotient map $Q_\| \in L(Y, Y/M)$ does not belong to $SS \cup \Phi_+$, because $Q_\| \in \Phi_+$ is an isomorphism on $M$ and $N$ is infinite-dimensional.

(b) Assume that $T \in L(X, Y)$, but $T \notin SC \cup \Phi_-$. Since $T$ is not strictly singular, we can find an infinite-dimensional closed subspace $M$ of $Y$ such that $Q_\| T \in T$. By the open mapping theorem, there exists a number $\delta > 0$ so that $\delta B_{Y/M} \subset Q_\| T(B_{X})$; hence $\|T^{*} f\| \geq \|f\|$ for every $f \in M^\perp$.

On the other hand, since $T$ is not lower semi-Fredholm, there exists an infinite-codimension closed subspace $N$ of $Y$ so that $\|Q_\| T x\| < (\delta/2)\|x\|$ for every $x \in X$; hence $\|T^{*} f\| < (\delta/2)\|f\|$ for every $f \in N^\perp$. We have $M^\perp \cap N^\perp = \{0\}$ and $M^\perp + N^\perp$ is closed. Then $M + N = Y$ and $M \cap N$ is infinite-codimensional in $M$ and in $N$, hence the quotient $Y/(M \cap N) = M/(M \cap N) \oplus N/(M \cap N)$ is decomposable. Therefore $Y$ is not quotient hereditarily indecomposable.

Conversely, if not all the quotients of $Y$ are indecomposable, then we can find closed subspaces $U, M$ and $N$ of $Y$ such that $U = M \cap N$, $M + N = Y$ and both $M/U$ and $N/U$ are infinite-dimensional, i.e., $Y/U = M/U \oplus N/U$ is not indecomposable. Then the natural injection $J_M \in L(M, Y)$ does not belong to $SC \cup \Phi_-$, because $R(J_M) + N = Y$ and both $M$ and $N$ are infinite-codimensional.

We observe that, at the time Weis proved this result, the existence of Banach spaces satisfying the hypothesis of Theorem 4.7 was an open problem. However, Gowers and Maurey [5] have recently constructed examples of spaces satisfying these conditions.

Example 4.8 [5]. There exists a reflexive Banach space $X_{OM}$ such that all its closed subspaces are indecomposable. Hence, all the quotients of $X_{OM}$ are indecomposable.

Now we can show further examples of Banach spaces $X, Y$ for which $\text{Im}(h_{X, Y}) = \text{Im}(h_{X, Y})$. Given a complex Banach space and an operator $T \in L(X, Y)$, we define $\sigma_+(T) := \{z \in \mathbb{C} : zI_X - T \notin \Phi_+\}$, $\sigma_-(T) := \{z \in \mathbb{C} : zI_X - T \notin \Phi_-\}$.

We need the following fact.

Lemma 4.9. Let $X$ be an infinite-dimensional complex Banach space. Then for every operator $T \in L(X)$, the sets $\sigma_+(T)$ and $\sigma_-(T)$ are compact and non-empty.

Proof. It is well known that $\sigma_+(T) := \{z \in \mathbb{C} : zI_X - T \notin \Phi_+\}$ is a non-empty compact set, because it coincides with the spectrum of the image of $T$ in the Calkin algebra $L(X)/\mathcal{K}(X)$ [6, §3]. Moreover, by the stability of the index of a semi-Fredholm operator under small perturbations [8, Proposition 2.2.9], the boundary of $\sigma_+(T)$ is contained both in $\sigma_+(T)$ and in $\sigma_-(T)$.

Part (a) of the following result was proved by Gowers and Maurey [5]. However, their proof is quite long. Here we present a much shorter proof, based on Lemma 4.9 and some ideas in [5].

Proposition 4.10. Let $X$ be an infinite-dimensional Banach space.

(a) If every closed subspace of $X$ is indecomposable, then $L(X) = \{sI_X \} \oplus SS(X)$.

(b) If every quotient of $X$ is indecomposable, then $L(X) = \{sI_X \} \oplus SC(X)$.

Proof. (a) First we assume that $K = \mathbb{C}$, the complex field. Let $T \in L(X)$. By Lemma 4.9, there exists $\eta_0 \in \mathbb{C}$ such that $\eta_0 I_X - T \notin \Phi_+$. Since all the closed subspaces of $X$ are indecomposable, by Theorem 4.7, $\eta_0 I_X - T$ is strictly singular, and the result is clear.

In the case $K = \mathbb{R}$, denote by $S$ the natural extension of $T$ to the complexification $X_\mathbb{C}$ of $X$. Observe that $X_\mathbb{C}$ can be represented by a product $X \times X$. We refer to [8, Proof of Theorem 2.2.13] for a brief description. Using this representation, it is easy to check that $(zI_\mathbb{C} - S)(x, y) = (zI_\mathbb{C} - S)(x, z)$, $(x, y) \in X \times X$.

where $\bar{z}$ is the complex conjugate of $z$. In particular, $zI_\mathbb{C} - S \notin \Phi_+$ if and only if $\bar{z}I_\mathbb{C} - S \notin \Phi_+$. Moreover, $S \in SS$ if and only if $T \in SS$. Hence, by Theorem 4.7, the set $\sigma_+(S)$ contains at most one real number.
Assume that there exist numbers $z, w \in \sigma_+(S)$ such that $z \neq w \neq \overline{z}$, and define $T_w := T^2 - 2 \text{Re} w T + |w|^2 I_X$. Then we have

$$(S - \overline{w} I_X) (S - w I_X) (x, y) = (T_w y, T_w y).$$

Therefore, $T_z$ and $T_w$ are not upper semi-Fredholm, hence they are strictly singular, and we get $T_z - T_w = \alpha T + b I_X \in \mathcal{S}$, for some $\alpha, b \in \mathbb{R}$, not both 0.

Note that $\alpha \neq 0$. Otherwise, we would have $I_X \in \mathcal{S}$, hence $X$ finite-dimensional, in contradiction with the hypothesis. Therefore, there is a real number $t$ so that $t I_X - T \in \mathcal{S}$, and the result is proved.

(b) The proof is very similar.

**Proposition 4.11.** Let $X, Y$ be Banach spaces. Assume that each closed subspace of $X$ is indecomposable, or each quotient of $X$ is indecomposable. Then $\text{Imp}(X, Y) = \text{In}(X, Y)$.

**Proof.** By Corollary 3.6 it is enough to show that $\text{Imp}(X) = \text{In}(X)$. So we fix $T \in \mathcal{L}(X) \setminus \mathcal{I}$. By Proposition 4.10, in both cases we can write $T = \lambda I_X + S$, where $S \in \mathcal{L}(X)$ is inessential. Hence $\lambda \neq 0$ and $T$ is a Fredholm operator. Thus $T \notin \text{Imp}$.

**References**


