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Department of Numerical Analysis
Eötvös L. University
Múzeum krt. 6-8
H-1088 Budapest, Hungary
E-mail: weisz@ludens.elte.hu

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On inessential and improjective operators

by

PIETRO AIENA (Palermo) and MANUEL GONZÁLEZ (Santander)

Abstract. We give several characterizations of the improjective operators, introduced by Tarafdar, and we characterize the inessential operators among the improjective operators. It is an interesting problem whether both classes of operators coincide in general. A positive answer would provide, for example, an intrinsic characterization of the inessential operators. We give several equivalent formulations of this problem and we show that the inessential operators acting between certain pairs of Banach spaces coincide with the improjective operators.

1. Introduction. An important class which occurs in the perturbation theory of Fredholm operators is that of *inessential operators*, introduced by Kleinecke [7] as the inverse image in $\mathcal{L}(X)$ by the quotient map

$$\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$$

of the radical of the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where X is a Banach space, $\mathcal{L}(X)$ is the set of all (continuous linear) operators on X and $\mathcal{K}(X)$ is the subset of all compact operators.

Other authors [9, 10] have defined and studied inessential operators acting between different Banach spaces X, Y. Let $\mathcal{L}(X,Y)$ be the set of all (continuous linear) operators acting from X into Y. An operator $T \in \mathcal{L}(X,Y)$ is *Fredholm*, in symbols $T \in \Phi(X,Y)$, if its kernel $\ker(T)$ is finite-dimensional and its range $R(T)$ is finite-codimensional. The *inessential operators* can be defined by

$$\mathcal{I}n(X,Y) := \{T \in \mathcal{L}(X,Y) : I_X - ST \in \Phi(X) \text{ for every } S \in \mathcal{L}(Y,X)\},$$

where I_X is the identity operator in X and $\Phi(X) = \Phi(X,X)$. Equivalently [2],

$$\mathcal{I}n(X,Y) := \{T \in \mathcal{L}(X,Y) : I_Y - TS \in \Phi(Y) \text{ for every } S \in \mathcal{L}(Y,X)\}.$$

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This class of operators can be characterized in more algebraic terms.

THEOREM 1.1 ([9], [2, Lemma 1.1 and Theorem 1.4]). *For an operator* $T \in \mathcal{L}(X,Y)$ *the following assertions are equivalent*:

(a) $T \in \mathcal{I}n(X,Y)$;
(b) $\ker(I_X - ST)$ *is finite-dimensional for every* $S \in \mathcal{L}(Y,X)$;
(c) $\ker(I_Y - TS)$ *is finite-dimensional for every* $S \in \mathcal{L}(Y,X)$;
(d) $\overline{R(I_X - ST)}$ *is finite-codimensional for every* $S \in \mathcal{L}(Y,X)$;
(e) $\overline{R(I_Y - TS)}$ *is finite-codimensional for every* $S \in \mathcal{L}(Y,X)$.

It is well known that $\mathcal{I}n(X,Y)$ is a closed subspace of $\mathcal{L}(X,Y)$. Moreover, the class $\mathcal{I}n$ of all the inessential operators is an operator ideal, in the sense of Pietsch [10], that contains the operator ideals which occur in Fredholm theory, namely the compact, the strictly singular and the strictly cosingular operators.

The characterizations of $T \in \mathcal{I}n$ existing in the literature are expressed, like those in Theorem 1.1, in terms of the properties of the product of T by a large class of operators. It is a problem of certain interest to find an "intrinsic" characterization of the inessential operators, for instance in terms of their action on the complemented subspaces. This would be obtained, for example, if the class $\mathcal{I}n$ coincided with the class of the improjective operators, introduced by Tarafdar in [11, 12]. As a consequence, we would obtain some structural information about the complemented subspaces of products of Banach spaces: if no infinite-dimensional complemented subspace of X is isomorphic to a complemented subspace of Y, then every complemented subspace of $X \times Y$ would be isomorphic to the product of a complemented subspace of X and a complemented subspace of Y (see [4]).

In fact, inessential operators are improjective, but it is not known if these two classes coincide. Tarafdar [12] gave an affirmative answer in some special cases.

In this paper we give, in Theorem 2.3, a dual characterization of the improjective operators. We apply this result to study these operators and to characterize the inessential operators among the improjective operators, in Theorem 2.6.

In the third section we consider the question whether the classes of inessential operators and improjective operators coincide. We give several formulations of this question and describe some related problems.

Finally, in the fourth section we describe some families of pairs of Banach spaces X, Y such that the improjective operators in $\mathcal{L}(X,Y)$ are inessential.

Along the paper, $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. The results are valid in both cases if the field is not explicitly mentioned. Further, X, Y, Z and W are Banach spaces over $\mathbb{K}$, and we denote by X^* the *dual space* of X, and by $\mathcal{L}$ the class of all (continuous linear) operators between Banach spaces. Given a subclass $\mathcal{A}$ of $\mathcal{L}$, the subsets

$$\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y)$$

are called the *components* of $\mathcal{A}$. Moreover, $\mathcal{A}(X) := \mathcal{A}(X,X)$.

We denote by $T^* \in \mathcal{L}(Y^*, X^*)$ the *conjugate operator* of $T \in \mathcal{L}(X,Y)$.

Subspaces of a Banach space are not necessarily closed. Given a closed subspace M of X, we denote by J_M the inclusion of M into X, and by Q_M the quotient map from X onto X/M. A subspace M of X is *complemented* if there exists $P \in \mathcal{L}(X)$ so that $P^2 = P$ (i.e., P is a *projection*) and $R(P) = M$. Of course, complemented subspaces are closed. Given subspaces M of X and U of X^*, we denote by $M^\perp \subset X^*$ and $U_\perp \subset X$ their respective annihilators.

2. Characterizations of inessential operators and improjective operators. Improjective operators were introduced by Tarafdar in the following way.

DEFINITION 2.1 [11, 12]. An operator $T \in \mathcal{L}(X,Y)$ is said to be *improjective* if there is no infinite-dimensional closed subspace M of X such that the restriction TJ_M is an isomorphism and $T(M)$ is a complemented subspace of Y.

We denote by $\mathcal{I}mp$ the class of improjective operators. It was proved in [11, Theorem 3.6] that $\mathcal{I}mp(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$.

Next we give a lemma which will be useful in the study of these operators and their relation to other classes of operators. A result similar to the first part was proved before in [11, Lemma 1.1].

LEMMA 2.2. *Let* $T \in \mathcal{L}(X,Y)$.

(a) *If* M *is a closed subspace of* X *such that* TJ_M *is an isomorphism,* $T(M)$ *is complemented in* Y *and* N *is a closed complement of* $T(M)$, *then* M *is complemented in* X *and* $T^{-1}(N)$ *is a closed complement of* M.

(b) *If* N *is a closed subspace of* Y *such that* $Q_N T$ *is surjective,* $T^{-1}(N)$ *is complemented in* X *and* M *is a closed complement of* $T^{-1}(N)$, *then* N *is complemented in* Y *and* $T(M)$ *is a closed complement of* N.

Proof. (a) If N is a closed complement of $T(M)$ in Y and TJ_M is an isomorphism, then $T^{-1}(N) \cap M = \{0\}$, both $T^{-1}(N)$ and M are closed subspaces and $X = T^{-1}(N) \oplus M$. Hence the result is a direct consequence of the closed graph theorem.

(b) If M is a closed complement of $T^{-1}(N)$ in X, then since $\ker(T)$ is contained in $T^{-1}(N)$, we have $T(M) \cap N = \{0\}$. Moreover, since $Q_N T$ is surjective, we obtain $T(M) \oplus N = T(X) + N = Y$, and it follows from [13, Theorem IV.5.10] that $T(M)$ is closed; hence N is complemented in Y. ■

Improjective operators admit the following characterization in terms of quotient maps.

THEOREM 2.3. *An operator $T \in \mathcal{L}(X,Y)$ is improjective if and only if there is no infinite-codimensional closed subspace N of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is a complemented subspace of X.*

Proof. Assume that $T \in \mathcal{L}(X,Y)$ is improjective and let N be a closed subspace of Y such that $Q_N T$ is surjective and $T^{-1}(N)$ is a complemented subspace of X. By Lemma 2.2, if M is a closed complement of $T^{-1}(N)$ then $T(M)$ is a closed complement of N. Observe that the restriction of T to M is an isomorphism and T is improjective. Therefore $T(M)$ is finite-dimensional; hence N is finite-codimensional.

Conversely, assume that T is not improjective and take an infinite-dimensional closed subspace M of X such that TJ_M is an isomorphism and $T(M)$ is complemented in Y. Given a closed complement N of $T(M)$, we see that N is infinite-codimensional and $Q_N T$ is surjective. Hence, by Lemma 2.2, we conclude that M is a closed complement of $T^{-1}(N)$. ■

Tarafdar [12] proved, for operators in $\mathcal{L}(X)$, that inessential operators are improjective. Here we give an elementary proof for the general case.

PROPOSITION 2.4. *Inessential operators are improjective.*

Proof. If $T \in \mathcal{L}(X,Y)$ is not improjective, then there exists an infinite-dimensional closed subspace M of X such that the restriction TJ_M is an isomorphism and $T(M)$ is a complemented subspace of Y. By Lemma 2.2, M is also complemented in X and

$$X = M \oplus T^{-1}(N) \quad \text{and} \quad Y = T(M) \oplus N,$$

where N and $T^{-1}(N)$ are closed subspaces of Y and X, respectively. So we can define an operator $S \in \mathcal{L}(Y,X)$ by

$$Sy := \begin{cases} (T|_M)^{-1}y & \text{if } y \in T(M), \\ 0 & \text{if } y \in N. \end{cases}$$

We have $\ker(I_X - ST) = M$; hence by Theorem 1.1, the operator T is not inessential. ■

The following lemma will be the key to characterizing the inessential operators among the improjective operators.

LEMMA 2.5. *Let $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y,X)$.*

(a) *If the subspace $M := \ker(I_X - ST)$ is complemented in X with closed complement U, then both subspaces $T(M)$ and $S^{-1}(U)$ are complemented in Y; in fact, we have $Y = T(M) \oplus S^{-1}(U)$.*

(b) *If the subspace $N := \overline{R(I_Y - TS)}$ is complemented in Y with closed complement V, then both subspaces $T^{-1}(N)$ and $S(V)$ are complemented in X; in fact, we have $X = S(V) \oplus T^{-1}(N)$.*

Proof. (a) Let P denote the projection from X onto M along U; thus $R(P) = M$ and $\ker(P) = U$. Since $(I_X - ST)P = 0$, we have $P = STP$. Therefore, defining $Q := TPS$, we have $Q^2 = TP(STP)S = TP^2S = Q$; i.e., Q is a projection in Y.

From $P = STP$ we obtain $\ker(T) \cap R(P) = \{0\}$; thus

$$\ker(Q) = \ker(PS) = S^{-1}(U).$$

Moreover, $P = STP$ implies that $ST(M) = M$; hence $S^{-1}(U) \cap T(M) = \{0\}$. On the other hand, from $Q = TPS$, it follows that $R(Q) \subset T(M)$; thus we conclude that $R(Q) = T(M)$.

(b) Denote by Q the projection from Y onto V along N. Then we have $Q(I_Y - TS) = 0$; hence $Q = QTS$. Therefore, as in the previous part, $P := SQT$ defines a projection in X.

From $Q = QTS$ we obtain $\ker(S) \cap R(Q) = \{0\}$; thus

$$\ker(P) = \ker(QT) = T^{-1}(N).$$

Moreover, $Q = QTS$ implies that $(TS)^{-1}(N) = N$; hence $S(V) \cap T^{-1}(N) = \{0\}$. On the other hand, from $P = SQT$, it follows that $R(P) \subset S(V)$; thus we conclude that $R(P) = S(V)$. ■

Finally, we give several characterizations of the inessential operators among the improjective operators in terms of the complementability of some subspaces.

THEOREM 2.6. *For an operator $T \in \mathcal{L}(X,Y)$ the following assertions are equivalent:*

(a) $T \in \mathcal{I}n(X,Y)$;
(b) $T \in \mathcal{I}mp(X,Y)$ *and* $\ker(I_X - ST)$ *is complemented for every* $S \in \mathcal{L}(Y,X)$;
(c) $T \in \mathcal{I}mp(X,Y)$ *and* $\ker(I_Y - TS)$ *is complemented for every* $S \in \mathcal{L}(Y,X)$;
(d) $T \in \mathcal{I}mp(X,Y)$ *and* $\overline{R(I_X - ST)}$ *is complemented for every* $S \in \mathcal{L}(Y,X)$;
(e) $T \in \mathcal{I}mp(X,Y)$ *and* $\overline{R(I_Y - TS)}$ *is complemented for every* $S \in \mathcal{L}(Y,X)$.

Proof. First we show that (a) implies the other assertions.

Assume that T is inessential. By Proposition 2.4, T is improjective. Moreover, by Theorem 1.1, for every $S \in \mathcal{L}(Y,X)$, $\ker(I_X - ST)$ and $\ker(I_Y - TS)$ are finite-dimensional, and $\overline{R(I_X - ST)}$ and $\overline{R(I_Y - TS)}$ are finite-codimensional; hence all of them are complemented.

(b)⇒(a). Assume that $T \in \mathcal{I}mp(X,Y)$ and $M := \ker(I_X - ST)$ is complemented. Then T is an isomorphism on M and, by Lemma 2.5, $T(M)$ is complemented. Hence M is finite-dimensional.

We have seen that (b) implies that $\ker(I_X - ST)$ is finite-dimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$.

(c)⇒(a). Assume that $T \in \mathcal{I}mp(X,Y)$ and $N := \ker(I_Y - TS)$ is complemented. Then S is an isomorphism on N (so that $S(N)$ is closed), T is an isomorphism on $S(N)$ and $T(S(N)) = N$ is complemented. Hence N is finite-dimensional.

We have seen that (c) implies that $\ker(I_Y - TS)$ is finite-dimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$.

(d)⇒(a). Assume that $T \in \mathcal{I}mp(X,Y)$ and $M := \overline{R(I_X - ST)}$ is complemented. Since $Q_M(I_X - ST) = 0$, we have $Q_M ST = Q_M$; in particular, $R(ST) + M = X$. Then $R(T) + S^{-1}(M) = Y$; i.e., $Q_{S^{-1}(M)}T$ is surjective. Moreover,

$$\begin{aligned} T^{-1}S^{-1}(M) &= (ST)^{-1}(M) = ((ST)^*(M^\perp))_\perp \\ &= (T^*S^*(\ker(I_{X^*} - T^*S^*))_\perp \\ &= (\ker(I_{X^*} - T^*S^*))_\perp = M \end{aligned}$$

is complemented. By Theorem 2.3 we deduce that $S^{-1}(M)$ is finite-codimensional; hence so is $M = T^{-1}S^{-1}(M)$.

Therefore, (d) implies that $\overline{R(I_X - ST)}$ is finite-codimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$.

(e)⇒(a). Assume that $T \in \mathcal{I}mp(X,Y)$ and $N := \overline{R(I_Y - TS)}$ is complemented. Since $Q_N(I_Y - TS) = 0$, we have $Q_N TS = Q_N$; in particular, $Q_N T$ is surjective and, by Lemma 2.5, $T^{-1}(N)$ is complemented. Applying Theorem 2.3 we conclude that N is finite-codimensional.

Therefore, (e) implies that $\overline{R(I_Y - TS)}$ is finite-codimensional for every $S \in \mathcal{L}(Y,X)$. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$. ■

3. On the inclusion $\mathcal{I}n(X,Y) \subset \mathcal{I}mp(X,Y)$. We begin this section by giving an alternative proof of the fact that the improjective operators form an ideal with respect to the product, and introducing the concept of quasi-operator ideal, which will be useful in our discussions.

PROPOSITION 3.1 [11, Theorem 1.2]. *Let $A \in \mathcal{L}(Y,Z)$, $K \in \mathcal{I}mp(X,Y)$ and $B \in \mathcal{L}(W,X)$. Then $KB \in \mathcal{I}mp(W,Y)$ and $AK \in \mathcal{I}mp(X,Z)$.*

Proof. Assume first that AK is not improjective. Then we can find an infinite-dimensional closed subspace M of X such that AKJ_M is an isomorphism and $AK(M)$ is complemented in Z. Note that KJ_M is also an isomorphism; hence $K(M)$ is closed.

Since $AJ_{K(M)}$ is an isomorphism, it follows from Lemma 2.2 that $K(M)$ is complemented in Y. Hence K is not improjective.

In the case in which KB is not improjective, Theorem 2.3 allows us to select an infinite-codimensional closed subspace N of Y such that $Q_N KB$ is surjective and $(KB)^{-1}(N)$ is a complemented subspace of W.

Putting $M := K^{-1}(N)$ we deduce that M is an infinite-codimensional closed subspace of X such that $Q_M B$ is surjective and $B^{-1}(M)$ is complemented. It follows from Lemma 2.2 that M is complemented in X. Moreover, $Q_N K$ is surjective; hence it follows from Theorem 2.3 that K is not improjective. ■

Let $\mathcal{F}$ denote the class of all operators with finite-dimensional range.

DEFINITION 3.2. A subclass $\mathcal{A}$ of $\mathcal{L}$ is said to be a *quasi-operator ideal* if it satisfies

(a) $\mathcal{F} \subset \mathcal{A}$;
(b) $A \in \mathcal{L}(Y,Z)$, $K \in \mathcal{A}(X,Y)$, $B \in \mathcal{L}(W,X) \Rightarrow AKB \in \mathcal{A}(W,Z)$.

A quasi-operator ideal is an *operator ideal* (in the sense of Pietsch [10]) if and only if $\mathcal{A}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$ for every pair X, Y of Banach spaces.

The main question that remains open concerns the equality $\mathcal{I}n = \mathcal{I}mp$.

QUESTION 1. Is it true that $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ for every pair X, Y of Banach spaces?

We say that a quasi-operator ideal $\mathcal{A}$ is *proper* if $I_X \in \mathcal{A}$ for no infinite-dimensional space X.

As far as we know, it is an open problem whether $\mathcal{I}n$ is the greatest proper operator ideal. However, we have the following result, essentially due to Tarafdar [11, Corollary 3.3], which follows easily from Definition 2.1 and Proposition 3.1.

PROPOSITION 3.3. *The class $\mathcal{I}mp$ of all improjective operators is the greatest proper quasi-operator ideal.*

Therefore, a positive answer to the following question would provide a description of the greatest proper operator ideal.

QUESTION 2. Is it true that $\mathcal{I}mp(X,Y)$ is a subspace for every pair X, Y?

Later we shall show that the equality $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ holds in many cases. We think that Questions 1 and 2 are difficult. So a positive answer to the following one could be useful. For example, it would show that the previous two are equivalent.

QUESTION 3. Fix a pair X, Y of Banach spaces. Is it true that $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ whenever $\mathcal{I}mp(X,Y)$ is a subspace?

The next result shows a symmetry of the equality $\mathcal{I}n = \mathcal{I}mp$.

PROPOSITION 3.4. *Let X and Y be Banach spaces. Then $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ if and only if $\mathcal{I}n(Y,X) = \mathcal{I}mp(Y,X)$.*

Proof. Suppose that $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ and let $T \in \mathcal{I}mp(Y,X)$. We only have to show that $T \in \mathcal{I}n$.

Given $S \in \mathcal{L}(X,Y)$, by Proposition 3.1 we have $STS \in \mathcal{I}mp(X,Y) = \mathcal{I}n(X,Y)$; hence $(TS)^2 \in \mathcal{I}n$. Therefore $I_X - (TS)^2 = (I_X + TS)(I_X - TS) \in \Phi(X)$, and hence $\ker(I_X - TS) \subset \ker(I_X - (TS)^2)$ is finite-dimensional. By Theorem 1.1 we conclude that $T \in \mathcal{I}n$. ∎

Let X be a Banach space and let $T \in \mathcal{L}(X)$. Given an *invariant subspace* M of T, i.e., M closed and $T(M) \subset M$, we can define two associated operators

$$T_M : M \to M \quad \text{and} \quad T^M : X/M \to X/M$$

in the natural way: $T_M m := Tm$ and $T^M(x+M) := Tx + M$.

To give some sufficient conditions for $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$, we consider two classes of operators Ω_+ and Ω_- introduced in [1]:

$$\Omega_+(X) := \{T \in \mathcal{L}(X) : T_M \text{ is an (into) isomorphism for no infinite-dimensional, invariant subspace } M \text{ of } T\}$$

$$\Omega_-(X) := \{T \in \mathcal{L}(X) : T^M \text{ is surjective for no infinite-codimensional, invariant subspace } M \text{ of } T\}$$

THEOREM 3.5. *Let $\mathcal{A}$ be a quasi-operator ideal and let Y be a Banach space.*

(a) *If $\mathcal{A}(Y) \subset \Omega_+(Y)$, then $\mathcal{A}(Y,Z) \subset \mathcal{I}n(Y,Z)$ for every Banach space Z.*

(b) *If $\mathcal{A}(Y) \subset \Omega_-(Y)$, then $\mathcal{A}(X,Y) \subset \mathcal{I}n(X,Y)$ for every Banach space X.*

Proof. (a) Suppose that there exists $T \in \mathcal{A}(Y,Z) \setminus \mathcal{I}n(Y,Z)$. By Theorem 1.1, we can find $S \in \mathcal{L}(Z,Y)$ so that $M := \ker(I_Y - ST)$ is infinite-dimensional. However, since $ST \in \mathcal{A}(Y) \subset \Omega_+(Y)$ and ST coincides with the identity on M, we find that M is finite-dimensional; a contradiction.

(b) Suppose that there exists $T \in \mathcal{A}(X,Y) \setminus \mathcal{I}n(X,Y)$. By Theorem 1.1, we can find $S \in \mathcal{L}(Y,X)$ so that $N := \overline{R(I_Y - TS)}$ is infinite-codimensional. However, since $TS \in \mathcal{A}(Y) \subset \Omega_-(Y)$, N is an invariant subspace of TS and $(TS)^N$ is surjective, we deduce that N is finite-codimensional; a contradiction. ∎

In the case $\mathcal{A} = \mathcal{I}mp$, we obtain further characterizations of pairs X, Y satisfying $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$.

COROLLARY 3.6. *For a Banach space X the following statements are equivalent:*

(a) $\mathcal{I}n(X) = \mathcal{I}mp(X)$;
(b) $\mathcal{I}mp(X) \subset \Omega_+(X)$;
(c) $\mathcal{I}mp(X) \subset \Omega_-(X)$;
(d) $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ *for every Banach space Y.*

Proof. For the implications (a)⇒(b) and (a)⇒(c), it is enough to observe that $\mathcal{I}n(X) \subset \Omega_+(X) \cap \Omega_-(X)$. Indeed, if $T \in \mathcal{L}(X) \setminus \mathcal{I}n(X)$, then using Theorem 1.1 we can find an operator $S \in \mathcal{L}(X)$ so that $M := \ker(I_X - ST)$ is infinite-dimensional. Since the restriction of T to M is an isomorphism and $T(M) \subset M$, we conclude $T \notin \Omega_+$. Analogously, we can conclude $T \notin \Omega_-$.

(b)⇒(d) follows from Theorem 3.5 and $\mathcal{I}n(X,Y) \subset \mathcal{I}mp(X,Y)$.

(c)⇒(d) follows from Theorem 3.5, the inclusion $\mathcal{I}n(Y,X) \subset \mathcal{I}mp(Y,X)$ and the equivalence $\mathcal{I}n(Y,X) = \mathcal{I}mp(Y,X) \Leftrightarrow \mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$, proved in Proposition 3.4.

(d)⇒(a) is trivial. ∎

Corollary 3.6 shows that Question 1 is equivalent to the following one.

QUESTION 4. Is it true that $\mathcal{I}n(X) = \mathcal{I}mp(X)$ for every Banach space X?

Let X be a complex Banach space X and $T \in \mathcal{L}(X)$. Recall [6, §48, page 203] that a subset σ of the spectrum $\sigma(T)$ of T is a *spectral set* if both σ and $\sigma(T) \setminus \sigma$ are closed.

PROPOSITION 3.7. *Let X be an infinite-dimensional complex Banach space and let $T \in \mathcal{I}mp(X)$. Then we have*

(a) $0 \in \sigma(T)$;

(b) *if $\sigma \subset \sigma(T)$ is a spectral set and $0 \notin \sigma$, then the spectral projection associated with σ has finite-dimensional range.*

Proof. (a) Clearly, an invertible operator in an infinite-dimensional space cannot be improjective.

(b) Let P denote the spectral projection associated with σ (see [6, §49]). Then taking $M := R(P)$, we have $T(M) = M$ and the restriction TJ_M is an isomorphism. Since $T \in \mathcal{I}mp$ we deduce that M is finite-dimensional. ∎

Next we show that, for complex Banach spaces, Question 4 can be formulated in terms of the spectral properties of improjective operators.

PROPOSITION 3.8. *Let X be a complex Banach space. Then $\mathcal{I}mp(X) = \mathcal{I}n(X)$ if and only if for every $T \in \mathcal{I}mp(X)$, the spectrum $\sigma(T)$ is either a finite set or a sequence which clusters at 0.*

Proof. The direct implication follows from a well-known property of the spectrum of inessential operators [1].

For the converse, fix an operator $T \in \mathcal{I}mp(X)$. For every $S \in \mathcal{L}(X)$, we have $ST \in \mathcal{I}mp(X)$. Now, by the hypothesis and Proposition 3.7, either $I_X - ST$ is bijective, or 1 is an isolated point in $\sigma(ST)$ and the spectral projection associated with the spectral set $\{1\}$ has finite-dimensional range. In any case, $\ker(I_X - ST)$ is finite-dimensional, and applying Theorem 1.1, we conclude that T is inessential. ∎

Proposition 3.8 shows that, for complex Banach spaces, Question 4 is equivalent to the first part of the following one.

QUESTION 5. Let X be a complex Banach space X.

(a) Is it true that $\sigma(T)$ is either a finite set or a sequence which clusters at 0, for every $T \in \mathcal{I}mp(X)$?

(b) We remark that we do not know if the assertion of Proposition 3.8 is valid for a single operator, instead of the whole set $\mathcal{I}mp(X)$. Therefore, we ask: Assume that the spectrum of $T \in \mathcal{I}mp(X)$ is either a finite set or a sequence which clusters at 0. Is T inessential?

4. Some examples. Here we present several examples of pairs X, Y of Banach spaces for which we have the equality $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$. Note that, by Proposition 3.4, this is equivalent to $\mathcal{I}n(Y,X) = \mathcal{I}mp(Y,X)$. Some of these examples correspond to the case in which one of the spaces admits many projections, like the subprojective spaces and the superprojective spaces, and some others to the opposite case in which one of the spaces admits only trivial projections.

First we recall some results of [2, 4] which give examples of pairs X, Y such that $\mathcal{L}(X,Y) = \mathcal{I}n(X,Y)$, or equivalently, $\mathcal{L}(Y,X) = \mathcal{I}n(Y,X)$ [4, Proposition 1]. We refer to [4] for the definitions of the concepts involved in the results.

THEOREM 4.1 ([2, Theorem 2.3], [4, Theorem 1]). *We have the equalities*
$$\mathcal{L}(X,Y) = \mathcal{I}mp(X,Y) = \mathcal{I}n(X,Y)$$
in the following cases:

(a) *X is reflexive and Y has the Dunford–Pettis property*;

(b) *X has the reciprocal Dunford–Pettis property and Y has the Schur property*;

(c) *X contains no copies of ℓ_∞ and $Y = \ell_\infty$, H^∞ or $C(K)$ with K σ-stonian*;

(d) *X contains no copies of c_0 and $Y = C(K)$*;

(e) *X contains no complemented copies of c_0 and $Y = C[0,1]$*;

(f) *X contains no complemented copies of ℓ_1 and $Y = L_1(\mu)$*;

(g) *X contains no complemented copies of ℓ_p and $Y = \ell_p$, $1 \le p \le \infty$*;

(h) *X contains no complemented copies of ℓ_p or ℓ_2 and $Y = L_p[0,1]$, $1 < p < \infty$.*

We observe that Theorem 4.1 includes and extends the list of examples of Tarafdar [11, 12] of pairs X, Y for which $\mathcal{L}(X,Y) = \mathcal{I}mp(X,Y)$.

It was proved by Tarafdar [12, Theorem 3.2] that $\mathcal{L}(X,Y) = \mathcal{I}mp(X,Y)$ if and only if no infinite-dimensional complemented subspace of X is isomorphic to a complemented subspace of Y. So, the following question is natural.

QUESTION 6. Let X and Y be Banach spaces such that $\mathcal{L}(X,Y) = \mathcal{I}mp(X,Y)$. Is it true that every complemented subspace of $X \times Y$ is isomorphic to the product of a complemented subspace of X and a complemented subspace of Y?

The answer is affirmative when $\mathcal{L}(X,Y) = \mathcal{I}n(X,Y)$ [4, Theorem 3].

DEFINITION 4.2. A Banach space X is said to be *subprojective* if every infinite-dimensional closed subspace of X contains an infinite-dimensional subspace which is complemented in X. The space X is said to be *superprojective* if every infinite-codimensional closed subspace of X is contained in an infinite-codimensional subspace which is complemented in X.

The spaces ℓ_p $(1 < p < \infty)$ are subprojective and superprojective, and the spaces $L_p[0,1]$ are subprojective for $2 \le p < \infty$ and superprojective for $1 < p \le 2$. Moreover, ℓ_1 and c_0 are subprojective, but $L_1[0,1]$ and $C[0,1]$ are neither subprojective nor superprojective [15]. For further information on subprojective and superprojective spaces, we refer to [2].

If one of the spaces is subprojective or superprojective, the improjective operators are inessential.

THEOREM 4.3. *Assume that one of the spaces X, Y is subprojective or superprojective. Then $\mathcal{I}mp(X,Y) = \mathcal{I}n(X,Y)$.*

Proof. Assume first that Y is superprojective. If $T \in \mathcal{L}(X,Y)$ is not inessential, then we can find an operator $S \in \mathcal{L}(Y,X)$ such that $M := R(I_Y - TS)$ is infinite-codimensional in Y. We take an infinite-codimensional, complemented subspace N of Y containing M, and we select a projection P with $\ker(P) = N$.

We see that $R(P)$ is infinite-dimensional. Moreover, since $P(I_Y - TS) = 0$, we find that PTS restricted to $R(P)$ coincides with the identity operator. Then PTS is not improjective, and by Proposition 3.1, T is not improjective.

Now we consider the case in which X is subprojective. If $T \in \mathcal{L}(X,Y)$ is not inessential, then we can find an operator $S \in \mathcal{L}(Y,X)$ such that $M :=$

$\ker(I_X - ST)$ is infinite-dimensional in X. We take an infinite-dimensional, complemented subspace N of X contained in M.

Since ST restricted to the subspace N coincides with the identity, we see that $T(N)$ is closed, the restriction $SJ_{T(N)}$ is an isomorphism and $S(T(N))$ is complemented. Hence, by Lemma 2.2, $T(N)$ is complemented. Since TJ_N is an isomorphism, we conclude that T is not improjective.

For the remaining cases it is enough to apply Proposition 3.4 and the previously proved cases. ∎

Recall that an operator $T \in \mathcal{L}(X,Y)$ is *strictly singular* if no restriction TJ_M of T to an infinite-dimensional closed subspace M of X is an isomorphism. The operator T is *strictly cosingular* if there is no infinite-codimensional closed subspace N of Y such that $R(T) + N = Y$. We denote by $\mathcal{SS}$ and $\mathcal{SC}$ the classes of all strictly singular and strictly cosingular operators, respectively. These classes are closed operator ideals [10, 1.9, 1.10, 4.2.7]. Moreover, both $\mathcal{SS}$ and $\mathcal{SC}$ are contained in $\mathcal{I}n$ [10, 26.7.3]. In particular, the inclusion

$$\mathcal{SS}(X,Y) \cup \mathcal{SC}(X,Y) \subset \mathcal{I}mp(X,Y)$$

holds for each pair X, Y of Banach spaces.

In the following result we show that, in some cases in which one of the spaces is subprojective or superprojective, the improjective operators coincide with the strictly singular or the strictly cosingular operators. We observe that part (a) was proved before in [11, Theorem 1.3].

THEOREM 4.4. (a) *If Y is subprojective, then $\mathcal{I}mp(X,Y) = \mathcal{SS}(X,Y)$.*
(b) *If X is superprojective, then $\mathcal{I}mp(X,Y) = \mathcal{SC}(X,Y)$.*

Proof. (a) Assume that Y is subprojective and $T \in \mathcal{L}(X,Y)$ is not strictly singular. Take an infinite-dimensional closed subspace M of X such that TJ_M is an isomorphism. Since Y is subprojective, we can take an infinite-dimensional subspace N of $T(M)$ which is complemented in Y. Then $A := (T|_M)^{-1}(N)$ is an infinite-dimensional closed subspace of X contained in M. In particular, TJ_A is an isomorphism. Moreover, $T(A) = N$ is complemented in Y. Hence T is not improjective.

(b) Assume that X is superprojective and $T \in \mathcal{L}(X,Y)$ is not strictly cosingular. Take an infinite-codimensional closed subspace N of Y such that $Q_N T$ is surjective. Then $\ker(Q_N T) = T^{-1}(N)$ is closed and infinite-codimensional; indeed, if A is an algebraic complement of $T^{-1}(N)$ in X, then $Y = N \oplus T(A)$.

Since X is superprojective, we can take an infinite-codimensional complemented subspace L of X containing $T^{-1}(N)$. Now, since $Q_N T$ is surjective and $\ker(Q_N T) \subset L$, we see that $Q_N T(L)$ is closed; hence $B := Q_N^{-1}(Q_N T(L)) = T(L) + N$ is closed as well.

Note that $Q_B T$ is surjective and $T^{-1}(B) = L$; in particular, B is infinite-codimensional. Therefore, by Theorem 2.3 we conclude that T is not improjective. ∎

The following examples show that we cannot change the order of the spaces X, Y in Theorem 4.4. In (a), if X is subprojective, then $\mathcal{I}mp(X,Y) = \mathcal{SS}(X,Y)$ is not true in general, and analogously in (b).

EXAMPLE 4.5. (a) The space ℓ_2 is subprojective and, by Theorem 4.1(c), we have

$$\mathcal{L}(\ell_2, \ell_\infty) = \mathcal{I}mp(\ell_2, \ell_\infty) = \mathcal{I}n(\ell_2, \ell_\infty);$$

however, $\mathcal{L}(\ell_2, \ell_\infty) \neq \mathcal{SS}(\ell_2, \ell_\infty)$, because ℓ_∞ contains a closed subspace isomorphic to ℓ_2 [3, Theorem IV.II.2].

Another example may be derived from the fact that the natural inclusion of $L_2[0,1]$ in $L_1[0,1]$ is not strictly singular, because it is an isomorphism on the closed subspace generated by the Rademacher functions [3, Proposition VI.1.1]. However, by Theorem 4.1(f), we have $\mathcal{L}(L_2[0,1], L_1[0,1]) = \mathcal{I}mp(L_2[0,1], L_1[0,1])$.

(b) The space ℓ_2 is superprojective and, by [3, Proposition IV.I.2], every operator $T \in \mathcal{L}(\ell_1, \ell_2)$ is strictly singular. Therefore, we have

$$\mathcal{L}(\ell_1, \ell_2) = \mathcal{I}mp(\ell_1, \ell_2) = \mathcal{I}n(\ell_1, \ell_2).$$

However, $\mathcal{L}(\ell_1, \ell_2) \neq \mathcal{SC}(\ell_1, \ell_2)$, because ℓ_1 has a quotient isomorphic to ℓ_2 [3, Theorem IV.II.1].

Next we consider some examples X, Y for which $\mathcal{I}n(X,Y) = \mathcal{I}mp(X,Y)$ because one of the spaces has very few projections.

DEFINITION 4.6. A Banach space X is said to be *indecomposable* if it does not contain pairs of infinite-dimensional closed subspaces M, N so that $X = M \oplus N$.

Recall that an operator $T \in \mathcal{L}(X,Y)$ is *upper semi-Fredholm*, in symbols $T \in \Phi_+(X,Y)$, if its range $R(T)$ is closed and its kernel $\ker(T)$ is finite-dimensional; T is *lower semi-Fredholm*, in symbols $T \in \Phi_-(X,Y)$, if $R(T)$ is finite-codimensional (hence closed [13, Theorem IV.5.10]).

The following result, essentially proved by Weis ([14, Corollary 2.3] contains the hard implications), characterizes the class of Banach spaces such that any operator either is semi-Fredholm or belongs to one of the operator ideals $\mathcal{SS}$ or $\mathcal{SC}$, in terms of the decomposability of their closed subspaces or quotients. We include a proof for the convenience of the reader and for future reference to some of the steps.

THEOREM 4.7 [14]. *Let Y be a Banach space.*

(a) *We have $\mathcal{L}(Y,Z) = \mathcal{SS}(Y,Z) \cup \Phi_+(Y,Z)$ for every Banach space Z if and only if all the closed subspaces of Y are indecomposable.*

(b) *We have $\mathcal{L}(X,Y) = \mathcal{SC}(X,Y) \cup \Phi_-(X,Y)$ for every Banach space X if and only if all the quotients of Y are indecomposable.*

Proof. (a) Assume that $T \in \mathcal{L}(Y,Z)$, but $T \notin \mathcal{SS} \cup \Phi_+$. Since T is not strictly singular, we can find a number $\delta > 0$ and an infinite-dimensional closed subspace M of Y such that $\|Tx\| \ge \delta\|x\|$ for every $x \in M$. Now, since T is not upper semi-Fredholm, there exists an infinite-dimensional closed subspace N of Y so that $\|Tx\| < (\delta/2)\|x\|$ for every $x \in N$. It is immediate to check that $M \cap N = \{0\}$ and $M + N$ is closed; hence the closed subspace $M \oplus N$ of Y is not decomposable.

Conversely, assume that not all the closed subspaces of Y are indecomposable. Then we can find two closed infinite-dimensional subspaces M, N of Y such that $M \cap N = \{0\}$ and $M + N$ is closed. Thus, the quotient map $Q_N \in \mathcal{L}(Y, Y/N)$ does not belong to $\mathcal{SS} \cup \Phi_+$, because Q_N is an isomorphism on M and N is infinite-dimensional.

(b) Assume that $T \in \mathcal{L}(X,Y)$, but $T \notin \mathcal{SC} \cup \Phi_-$. Since T is not strictly cosingular, we can find an infinite-codimensional closed subspace M of Y such that $Q_M T$ is surjective. By the open mapping theorem, there exists a number $\delta > 0$ so that $\delta B_{Y/M} \subset Q_M T(B_X)$; hence $\|T^* f\| \ge \delta\|f\|$ for every $f \in M^\perp$.

On the other hand, since T is not lower semi-Fredholm, there exists an infinite-codimensional closed subspace N of Y so that $\|Q_N T x\| < (\delta/2)\|x\|$ for every $x \in X$; hence $\|T^* f\| < (\delta/2)\|f\|$ for every $f \in N^\perp$. We have $M^\perp \cap N^\perp = \{0\}$ and $M^\perp + N^\perp$ is closed. Then $M + N = Y$ and $M \cap N$ is infinite-codimensional in M and in N, hence the quotient $Y/(M \cap N) = M/(M \cap N) \oplus N/(M \cap N)$ is decomposable. Therefore Y is not quotient hereditarily indecomposable.

Conversely, if not all the quotients of Y are indecomposable, then we can find closed subspaces U, M and N of Y such that $U = M \cap N$, $M + N = Y$ and both M/U and N/U are infinite-dimensional, i.e., $Y/U = M/U \oplus N/U$ is not indecomposable. Then the natural injection $J_M \in \mathcal{L}(M, Y)$ does not belong to $\mathcal{SC} \cup \Phi_-$, because $R(J_M) + N = Y$ and both M and N are infinite-codimensional. ■

We observe that, at the time Weis proved this result, the existence of Banach spaces satisfying the hypothesis of Theorem 4.7 was an open problem. However, Gowers and Maurey [5] have recently constructed examples of spaces satisfying these conditions.

EXAMPLE 4.8 [5]. There exists a reflexive Banach space X_{GM} such that all its closed subspaces are indecomposable. Hence, all the quotients of X_{GM}^* are indecomposable.

Now we can show further examples of Banach spaces X, Y for which $\mathcal{I}mp(X,Y) = \mathcal{I}n(X,Y)$. Given a complex Banach space and an operator $T \in \mathcal{L}(X)$, we define

$$\sigma_+(T) := \{z \in \mathbb{C} : zI_X - T \notin \Phi_+\}, \quad \sigma_-(T) := \{z \in \mathbb{C} : zI_X - T \notin \Phi_-\}.$$

We need the following fact.

LEMMA 4.9. *Let X be an infinite-dimensional complex Banach space. Then for every operator $T \in \mathcal{L}(X)$, the sets $\sigma_+(T)$ and $\sigma_-(T)$ are compact and non-empty.*

Proof. It is well known that $\sigma_e(T) := \{z \in \mathbb{C} : zI_X - T \notin \Phi\}$ is a non-empty compact set, because it coincides with the spectrum of the image of T in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ [6, §53]. Moreover, by the stability of the index of a semi-Fredholm operator under small perturbations [8, Proposition 2.c.9], the boundary of $\sigma_e(T)$ is contained both in $\sigma_+(T)$ and in $\sigma_-(T)$. ■

Part (a) of the following result was proved by Gowers and Maurey [5]. However, their proof is quite long. Here we present a much shorter proof, based on Lemma 4.9 and some ideas in [5].

PROPOSITION 4.10. *Let X be an infinite-dimensional Banach space.*

(a) *If every closed subspace of X is indecomposable, then*

$$\mathcal{L}(X) = \{\mathbb{K} I_X\} \oplus \mathcal{SS}(X).$$

(b) *If every quotient of X is indecomposable, then*

$$\mathcal{L}(X) = \{\mathbb{K} I_X\} \oplus \mathcal{SC}(X).$$

Proof. (a) First we assume that $\mathbb{K} = \mathbb{C}$, the complex field. Let $T \in \mathcal{L}(X)$. By Lemma 4.9, there exists $z_0 \in \mathbb{C}$ such that $z_0 I_X - T \notin \Phi_+$. Since all the closed subspaces of X are indecomposable, by Theorem 4.7, $z_0 I_X - T$ is strictly singular, and the result is clear.

In the case $\mathbb{K} = \mathbb{R}$, denote by S the natural extension of T to the complexification $X_{\mathbb{C}}$ of X. Observe that $X_{\mathbb{C}}$ can be represented by a product $X \times X$. We refer to [8, Proof of Theorem 2.c.13] for a brief description. Using this representation, it is easy to check that

$$(zI_{X_{\mathbb{C}}} - S)(x,y) = (\overline{z} I_{X_{\mathbb{C}}} - S)(y,x), \quad (x,y) \in X \times X,$$

where $\overline{z}$ is the complex conjugate of z. In particular, $zI_{X_{\mathbb{C}}} - S \in \Phi_+$ if and only if $\overline{z} I_{X_{\mathbb{C}}} - S \in \Phi_+$. Moreover, $S \in \mathcal{SS}$ if and only if $T \in \mathcal{SS}$. Hence, by Theorem 4.7, the set $\sigma_+(S)$ contains at most one real number.

Assume that there exist numbers $z, w \in \sigma_+(S)$ such that $z \neq w \neq \overline{z}$, and define $T_w := T^2 - 2\,\mathrm{Re}\, wT + |w|^2 I_X$. Then we have

$$(S - \overline{w} I_{X_C})(S - w I_{X_C})(x, y) = (T_w x, T_w y).$$

Therefore, T_z and T_w are not upper semi-Fredholm, hence they are strictly singular, and we get $T_z - T_w = aT + bI_X \in \mathcal{SS}$, for some $a, b \in \mathbb{R}$, not both 0.

Note that $a \neq 0$. Otherwise, we would have $I_X \in \mathcal{SS}$, hence X finite-dimensional, in contradiction with the hypothesis. Therefore, there is a real number t so that $tI_X - T \in \mathcal{SS}$, and the result is proved.

(b) The proof is very similar. ∎

PROPOSITION 4.11. *Let X, Y be Banach spaces. Assume that each closed subspace of X is indecomposable, or each quotient of X is indecomposable. Then $\mathcal{I}mp(X, Y) = \mathcal{I}n(X, Y)$.*

Proof. By Corollary 3.6 it is enough to show that $\mathcal{I}mp(X) = \mathcal{I}n(X)$. So we fix $T \in \mathcal{L}(X) \setminus \mathcal{I}n$. By Proposition 4.10, in both cases we can write $T = \lambda I_X + S$, where $S \in \mathcal{L}(X)$ is inessential. Hence $\lambda \neq 0$ and T is a Fredholm operator. Thus $T \notin \mathcal{I}mp$. ∎

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Dipartimento di Matematica ed Applicazioni
Facoltà di Ingegneria
Università di Palermo
Viale delle Scienze
I-90128 Palermo, Italy
E-mail: paiena@unipa.it

Departamento de Matemáticas
Facultad de Ciencias
Universidad de Cantabria
E-39071 Santander, Spain
E-mail: gonzalem@ccaix3.unican.es

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