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Riesz means of Fourier transforms and Fourier series on Hardy spaces

by

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Abstract. Elementary estimates for the Riesz kernel and for its derivative are given. Using these we show that the maximal operator of the Riesz means of a tempered distribution is bounded from $H_p(\mathbb{R})$ to $L_p(\mathbb{R})$ $(1/(\alpha+1) < p < \infty)$ and is of weak type $(1,1)$, where $H_p(\mathbb{R})$ is the classical Hardy space. As a consequence we deduce that the Riesz means of a function $f \in L_1(\mathbb{R})$ converge a.e. to f. Moreover, we prove that the Riesz means are uniformly bounded on $H_p(\mathbb{R})$ whenever $1/(\alpha+1) < p < \infty$. Thus, in case $f \in H_p(\mathbb{R})$, the Riesz means converge to f in $H_p(\mathbb{R})$ norm $(1/(\alpha+1) < p < \infty)$. The same results are proved for the conjugate Riesz means and for Fourier series of distributions.

1. Introduction. The Hardy–Lorentz spaces $H_{p,q}(\mathbb{R})$ of tempered distributions on the real line are endowed with the $L_{p,q}(\mathbb{R})$ Lorentz norms of the non-tangential maximal function. Of course, $H_p(\mathbb{R}) = H_{p,p}(\mathbb{R})$ are the usual Hardy spaces $(0 < p \le \infty)$.

In this paper the Riesz means $\sigma_T^{\alpha,\gamma} f$ of tempered distributions are considered. Usually the cases $\gamma = 1, 2$ are investigated. It can be found in Stein–Weiss [12] and Butzer–Nessel [4] that the Riesz means $\sigma_T^{\alpha,\gamma} f$ $(\gamma = 1,2)$ of a function $f \in L_1(\mathbb{R})$ converge a.e. to f as $T \to \infty$. In the special case $\alpha = \gamma = 1$ the Riesz means are called Fejér means. The author [15] proved that the maximal Fejér operator $\sigma_*^{1,1} := \sup_{T>0} |\sigma_T^{1,1}|$ is bounded from $H_p(\mathbb{R})$ to $L_p(\mathbb{R})$ provided that $1/2 < p < \infty$ (for $p = 1$ see also Móricz [9]) and is of weak type $(1,1)$, i.e.

$$\sup_{\varrho>0} \varrho\lambda(\sigma_*^{1,1} f > \varrho) \le C\|f\|_1 \quad (f \in L_1(\mathbb{R}))$$

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(this last result can also be found in Zygmund [17] and Móricz [9]). Similar theorems for the (C,α) summability of Fourier series are given in Weisz [13].

In this paper we sharpen and generalize these results. First we prove two estimates for the Riesz kernel and for its derivative with elementary methods. Next we show that the maximal operator $\sigma_*^{\alpha,\gamma}$ is bounded from $H_{p,q}(\mathbb{R})$ to $L_{p,q}(\mathbb{R})$ whenever $0<\alpha\le 1\le\gamma$, $1/(\alpha+1)<p<\infty$ and $0<q\le\infty$, and is of weak type $(1,1)$. We introduce the conjugate distribution $\widetilde{f}$, the conjugate Riesz means $\widetilde{\sigma}_T^{\alpha,\gamma}f$ and the conjugate maximal operator $\widetilde{\sigma}_*^{\alpha,\gamma}$. We deduce that the operator $\widetilde{\sigma}_*^{\alpha,\gamma}$ is also of type $(H_{p,q}(\mathbb{R}),L_{p,q}(\mathbb{R}))$ $(1/(\alpha+1)<p<\infty,\ 0<q\le\infty;\ 0<\alpha\le 1\le\gamma)$ and of weak type $(1,1)$. We extend these results also for $\alpha>1$.

A usual density argument then implies that, besides the convergence results mentioned above, the conjugate Riesz means $\widetilde{\sigma}_T^{\alpha,\gamma}f$ converge a.e. to $\widetilde{f}$ as $T\to\infty$, provided that $f\in L_1(\mathbb{R})$. Note that $\widetilde{f}$ is not necessarily integrable whenever f is.

We also prove that the operators $\sigma_T^{\alpha,\gamma}$ and $\widetilde{\sigma}_T^{\alpha,\gamma}$ $(T>0)$ are uniformly bounded in T from $H_{p,q}(\mathbb{R})$ to $H_{p,q}(\mathbb{R})$ if $1/(\alpha+1)<p<\infty$, $0<q\le\infty$. From this it follows that $\sigma_T^{\alpha,\gamma}f\to f$ and $\widetilde{\sigma}_T^{\alpha,\gamma}f\to\widetilde{f}$ in $H_{p,q}(\mathbb{R})$ norm as $T\to\infty$, whenever $f\in H_{p,q}(\mathbb{R})$ and $1/(\alpha+1)<p<\infty$, $0<q\le\infty$.

We also consider the Riesz means of Fourier series of distributions on the unit circle and prove all the results above in this context.

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2. Hardy spaces on the real line and Hilbert transforms. Let $\mathbb{R}$ denote the real line and λ be the Lebesgue measure. We also use the notation $|I|$ for the Lebesgue measure of the set I. We briefly write $L_p(\mathbb{R})$ for the real $L_p(\mathbb{R},\lambda)$ space with norm (or quasinorm) $\|f\|_p:=(\int_{\mathbb{R}}|f|^p\,d\lambda)^{1/p}$ $(0<p\le\infty)$.

The *distribution function* of a Lebesgue-measurable function f is defined by

$$\lambda(|f|>\varrho):=\lambda(\{x:|f(x)|>\varrho\})\quad(\varrho\ge 0).$$

The *weak* $L_p(\mathbb{R})$ space $L_p^*(\mathbb{R})$ $(0<p<\infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*(\mathbb{R})}:=\sup_{\varrho>0}\varrho\lambda(|f|>\varrho)^{1/p}<\infty$$

and we set $L_\infty^*(\mathbb{R})=L_\infty(\mathbb{R})$.

The spaces $L_p^*(\mathbb{R})$ are special cases of the more general Lorentz spaces $L_{p,q}(\mathbb{R})$. In their definition another concept is used. For a measurable function f the *non-increasing rearrangement* is defined by

$$\check{f}(t):=\inf\{\varrho:\lambda(|f|>\varrho)\le t\}.$$

The *Lorentz space* $L_{p,q}(\mathbb{R})$ is defined as follows: for $0<p<\infty$ and $0<q<\infty$,

$$\|f\|_{p,q}:=\Big(\int_0^\infty \check{f}(t)^q t^{q/p}\,\frac{dt}{t}\Big)^{1/q}$$

while for $0<p\le\infty$,

$$\|f\|_{p,\infty}:=\sup_{t>0}t^{1/p}\check{f}(t).$$

Let

$$L_{p,q}(\mathbb{R}):=L_{p,q}(\mathbb{R},\lambda):=\{f:\|f\|_{p,q}<\infty\}.$$

One can show the following equalities:

$$L_{p,p}(\mathbb{R})=L_p(\mathbb{R}),\qquad L_{p,\infty}(\mathbb{R})=L_p^*(\mathbb{R})\qquad(0<p\le\infty)$$

(see e.g. Bennett–Sharpley [1] or Bergh–Löfström [2]).

Let f be a tempered distribution on $\mathcal{S}(\mathbb{R})$ (briefly $f\in\mathcal{S}'(\mathbb{R})$). The *Fourier transform* of f is denoted by $\widehat{f}$. If f is an integrable function then

$$\widehat{f}(u)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}f(x)e^{-\imath ux}\,dx\qquad(u\in\mathbb{R})$$

where $\imath=\sqrt{-1}$.

For $f\in\mathcal{S}'(\mathbb{R})$ and $t>0$ let

$$u(x,t):=(f*P_t)(x)$$

where $*$ denotes convolution and

$$P_t(x):=\frac{ct}{t^2+x^2}\qquad(x\in\mathbb{R})$$

is the Poisson kernel.

The *non-tangential maximal function* is defined by

$$u^*(x):=\sup_{|x'-x|<t}|u(x',t)|.$$

For $0<p,q\le\infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbb{R})$ consists of all tempered distributions f for which $u^*\in L_{p,q}(\mathbb{R})$ and we set

$$\|f\|_{H_{p,q}(\mathbb{R})}:=\|u^*\|_{p,q}.$$

Note that in case $p=q$ the usual definition of the Hardy spaces $H_{p,p}(\mathbb{R})=H_p(\mathbb{R})$ is obtained. It is known that if $f\in H_p(\mathbb{R})$ $(0<p<\infty)$ then $f(x)=\lim_{t\to 0}u(x,t)$ in the sense of distributions (see Fefferman–Stein [6]). Recall that $L_1(\mathbb{R})\subset H_{1,\infty}(\mathbb{R})$, more exactly,

$$\|f\|_{H_{1,\infty}(\mathbb{R})}=\sup_{\varrho>0}\varrho\lambda(u^*>\varrho)\le C\|f\|_1\qquad(f\in L_1(\mathbb{R})).\tag{1}$$

Moreover,

$$H_{p,q}(\mathbb{R})\sim L_{p,q}(\mathbb{R})\qquad(1<p<\infty,\ 0<q\le\infty)\tag{2}$$

where $\sim$ denotes equivalence of norms and spaces (see Fefferman–Stein [6], Stein [11], Fefferman–Rivière–Sagher [5]).

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Fefferman–Rivière–Sagher [5] and also Weisz [14]).

THEOREM A. *If a sublinear (resp. linear) operator V is bounded from $H_{p_0}(\mathbb{R})$ to $L_{p_0}(\mathbb{R})$ (resp. to $H_{p_0}(\mathbb{R})$) and from $L_{p_1}(\mathbb{R})$ to $L_{p_1}(\mathbb{R})$ ($p_0 \le 1 < p_1 \le \infty$) then it is also bounded from $H_{p,q}(\mathbb{R})$ to $L_{p,q}(\mathbb{R})$ (resp. to $H_{p,q}(\mathbb{R})$) if $p_0 < p < p_1$ and $0 < q \le \infty$.*

For a tempered distribution $f \in H_p(\mathbb{R})$ $(0 < p < \infty)$ the *Hilbert transform* or the *conjugate distribution* $\widetilde{f}$ is defined by

$$\widetilde{f} := f * \Phi \quad \text{where} \quad \widehat{\Phi}(u) = -\imath \operatorname{sign} u, \quad \Phi(x) = \frac{1}{\pi x}.$$

One can prove (see e.g. Fefferman–Stein [6]) that $\widetilde{f}$ is a well defined distribution, $\widetilde{f} \in H_p(\mathbb{R})$ and $(\widetilde{f})^{\sim} = -f$. Furthermore, Fefferman and Stein [6] showed that

$$\|f\|_{H_p(\mathbb{R})} \sim \|f\|_p + \|\widetilde{f}\|_p \quad (0 < p < \infty). \tag{3}$$

As is well known, if f is an integrable function then

$$\widetilde{f}(x) = \text{p.v.}\ \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-t)}{t}\, dt := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{\varepsilon < |t|} \frac{f(x-t)}{t}\, dt.$$

Moreover, the conjugate function $\widetilde{f}$ does exist almost everywhere, but it is not integrable in general.

3. Riesz means. Suppose first that $f \in L_p(\mathbb{R})$ for some $1 \le p \le 2$. It is known that if $\widehat{f} \in L_1$ then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(u) e^{\imath x u}\, du \quad (x \in \mathbb{R}).$$

This motivates the definition of the *Dirichlet integral* $s_t f$:

$$s_t f(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} \widehat{f}(u) e^{\imath x u}\, du \quad (t > 0).$$

The *conjugate Dirichlet integral* is defined by

$$\widetilde{s}_t f(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} (-\imath \operatorname{sign} u) \widehat{f}(u) e^{\imath x u}\, du \quad (t > 0).$$

It is easy to see that

$$s_t f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) \frac{2}{\sqrt{2\pi}} \frac{\sin tu}{u}\, du,$$

$$\widetilde{s}_t f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widetilde{f}(x-u) \frac{2}{\sqrt{2\pi}} \frac{\sin tu}{u}\, du.$$

For $\alpha, \gamma > 0$ the *Riesz* and *conjugate Riesz means* are defined by

$$\sigma_T^{\alpha,\gamma} f(x) := \frac{\alpha\gamma}{T} \int_0^T \left(1 - \left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1} \left(\frac{t}{T}\right)^{\gamma-1} s_t f(x)\, dt \quad (T > 0),$$

$$\widetilde{\sigma}_T^{\alpha,\gamma} f(x) := \frac{\alpha\gamma}{T} \int_0^T \left(1 - \left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha-1} \left(\frac{t}{T}\right)^{\gamma-1} \widetilde{s}_t f(x)\, dt \quad (T > 0),$$

respectively. Integrating by parts we get

$$\sigma_T^{\alpha,\gamma} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) K_T^{\alpha,\gamma}(u)\, du$$

where

$$K_T^{\alpha,\gamma}(u) := \frac{2}{\sqrt{2\pi}} \int_0^T \left(1 - \left(\frac{t}{T}\right)^{\gamma}\right)^{\alpha} \cos tu\, dt = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \left|\frac{t}{T}\right|^{\gamma}\right)^{\alpha} \cos tu\, dt \tag{4}$$

is the *Riesz kernel*. Similarly,

$$\widetilde{\sigma}_T^{\alpha,\gamma} f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widetilde{f}(x-u) K_T^{\alpha,\gamma}(u)\, du.$$

The Riesz means are called *typical means* if $\gamma = 1$, *Bochner–Riesz means* if $\gamma = 2$ and *Fejér means* if $\alpha = \gamma = 1$. One can prove (cf. Butzer–Nessel [4]) that

$$\sigma_T^{\alpha,\gamma} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \left|\frac{t}{T}\right|^{\gamma}\right)^{\alpha} \widehat{f}(t) e^{\imath x t}\, dt,$$

$$\widetilde{\sigma}_T^{\alpha,\gamma} f(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \left|\frac{t}{T}\right|^{\gamma}\right)^{\alpha} (-\imath \operatorname{sign} t) \widehat{f}(t) e^{\imath x t}\, dt.$$

We extend the definition of the Riesz means to tempered distributions as follows:

$$\sigma_T^{\alpha,\gamma} f := f * K_T^{\alpha,\gamma} \quad (T > 0).$$

One can show that $\sigma_T^{\alpha,\gamma} f$ is well defined for all tempered distributions $f \in H_p(\mathbb{R})$ $(0 < p \le \infty)$ and for all functions $f \in L_p(\mathbb{R})$ $(1 \le p \le \infty)$ (cf. Fefferman–Stein [6]). The extension of the conjugate Riesz means is

$$\widetilde{\sigma}_T^{\alpha,\gamma} f := \widetilde{f} * K_T^{\alpha,\gamma} \quad (T > 0).$$

The *maximal* and *maximal conjugate Riesz operators* are defined by

$$\sigma_*^{\alpha,\gamma} f := \sup_{T>0} |\sigma_T^{\alpha,\gamma} f| \quad \text{and} \quad \widetilde{\sigma}_*^{\alpha,\gamma} f := \sup_{T>0} |\widetilde{\sigma}_T^{\alpha,\gamma} f|,$$

respectively.

4. Estimates of Riesz kernels. In this section we prove some estimates for the Riesz kernels $K_T^{\alpha,\gamma}$ and for their derivatives with elementary methods.

LEMMA 1. *If* $0 < \alpha \le 1 \le \gamma$ *then*

$$|K_T^{\alpha,\gamma}(u)| \le \frac{C}{T^\alpha |u|^{\alpha+1}} \quad (u \ne 0)$$

where C *depends only on* α *and* γ.

Proof. Since $K_T^{\alpha,\gamma}$ is even, we can suppose that $u > 0$. Changing variables we get

$$K_T^{\alpha,\gamma}(u) = \frac{2}{\sqrt{2\pi}} \frac{1}{u} \int_0^{Tu} \left(1 - \left(\frac{x}{Tu}\right)^\gamma\right)^\alpha \cos x \, dx.$$

The lemma will be proved if we show that

$$\Big| \int_0^{Tu} ((Tu)^\gamma - x^\gamma)^\alpha \cos x \, dx \Big| \le C (Tu)^{\alpha(\gamma-1)}.$$

In other words, denoting Tu by A, we have to show that

$$\Big| \int_0^{A} (A^\gamma - x^\gamma)^\alpha \cos x \, dx \Big| \le C A^{\alpha(\gamma-1)}.$$

Choose $n \in \mathbb{N}$ such that $2n\pi \le A < 2(n+1)\pi$. Then

$$\Big| \int_{2n\pi}^{A} (A^\gamma - x^\gamma)^\alpha \cos x \, dx \Big| \le C A^{\alpha(\gamma-1)}$$

because

$$A^\gamma - x^\gamma = (A - x)\gamma \xi^{\gamma-1} \le (A - x)\gamma A^{\gamma-1} \quad (x < \xi < A)$$

by the Lagrange theorem. So, it is enough to prove that

$$\Big| \int_0^{2n\pi} (A^\gamma - x^\gamma)^\alpha \cos x \, dx \Big| \le C A^{\alpha(\gamma-1)}. \tag{5}$$

Let us change variables: $x = y + 2k\pi$, $x = -y + (2k+1)\pi$, $x = y + (2k+1)\pi$ and $x = -y + (2k+2)\pi$ on the intervals $[2k\pi, (4k+1)\pi/2]$, $[(4k+1)\pi/2, (2k+1)\pi]$, $[(2k+1)\pi, (4k+3)\pi/2]$ and $[(4k+3)\pi/2, (2k+2)\pi]$, respectively. Then we obtain

$$\int_{2k\pi}^{(2k+2)\pi} (A^\gamma - x^\gamma)^\alpha \cos x \, dx = \int_0^{\pi/2} g_k(x) \cos x \, dx \tag{6}$$

where

$$g_k(x) := (A^\gamma - (x + 2k\pi)^\gamma)^\alpha - (A^\gamma - (-x + (2k+1)\pi)^\gamma)^\alpha - (A^\gamma - (x + (2k+1)\pi)^\gamma)^\alpha + (A^\gamma - (-x + (2k+2)\pi)^\gamma)^\alpha.$$

It is easy to check that $g_k'(x) > 0$, which means that g_k is increasing and

$$f(A) := \sum_{k=0}^{n-1} g_k(0) \le \sum_{k=0}^{n-1} g_k(x) \quad (x \in [0, \pi/2]) \tag{7}$$

where $2n\pi \le A < 2(n+1)\pi$. Since $g_k(\pi/2) = 0$, we conclude that $g_k(0) < 0$ and $f(A) < 0$. We have

$$f(A) = \sum_{k=0}^{n-1} [(A^\gamma - (2k\pi)^\gamma)^\alpha - 2(A^\gamma - ((2k+1)\pi)^\gamma)^\alpha + (A^\gamma - ((2k+2)\pi)^\gamma)^\alpha].$$

Moreover,

$$f'(A) = \sum_{k=0}^{n-1} \alpha\gamma [(A^\gamma - (2k\pi)^\gamma)^{\alpha-1} A^{\gamma-1} - 2(A^\gamma - ((2k+1)\pi)^\gamma)^{\alpha-1} A^{\gamma-1} + (A^\gamma - ((2k+2)\pi)^\gamma)^{\alpha-1} A^{\gamma-1}].$$

Since the function $g(x) := (A^\gamma - x^\gamma)^{\alpha-1}$ $(0 \le x \le A)$ is convex, the expressions in square brackets are all positive. Hence $f'(A) > 0$ and f is increasing. Therefore

$$\begin{aligned} f(A) \ge f(2n\pi) &= \sum_{k=0}^{n-1} [((2n\pi)^\gamma - (2k\pi)^\gamma)^\alpha \\ &\quad - 2((2n\pi)^\gamma - ((2k+1)\pi)^\gamma)^\alpha + ((2n\pi)^\gamma - ((2k+2)\pi)^\gamma)^\alpha] \\ &= \pi^{\alpha\gamma} \Big[2 \sum_{k=0}^{2n-1} (-1)^k ((2n)^\gamma - k^\gamma)^\alpha - (2n)^{\alpha\gamma} \Big]. \end{aligned} \tag{8}$$

If

$$h(x) := ((2n)^\gamma - x^\gamma)^\alpha \quad (0 \le x \le 2n)$$

then we see immediately that h' is negative and decreasing. By the Lagrange theorem there exists $2k < \xi < 2k+1$ such that

$$((2n)^\gamma - (2k)^\gamma)^\alpha - ((2n)^\gamma - (2k+1)^\gamma)^\alpha = -h'(\xi) \ge -h'(2k).$$

Consequently, by (8),

$$\frac{f(A)}{\pi^{\alpha\gamma}} \ge 2\sum_{k=0}^{n-1} -h'(2k) - (2n)^{\alpha\gamma} \ge \int_0^{2n-2} -h'\,d\lambda - 2h'(0) - (2n)^{\alpha\gamma}$$
$$\ge -h(2n-2) + h(0) - (2n)^{\alpha\gamma} = -((2n)^\gamma - (2n-2)^\gamma)^\alpha.$$

Since

$$(2n)^\gamma - (2n-2)^\gamma = 2\gamma\xi^{\gamma-1} \le 2\gamma(2n)^{\gamma-1} \quad (2n-2<\xi<2n),$$

we conclude that

$$\frac{f(A)}{\pi^{\alpha\gamma}} \ge -(2\gamma)^\alpha(2n)^{(\gamma-1)\alpha} \ge -CA^{(\gamma-1)\alpha}. \tag{9}$$

Taking into account (6), (7) and (9), we proved (5), which completes the proof of Lemma 1. ∎

LEMMA 2. *If* $0<\alpha\le 1\le\gamma$ *then*

$$|(K_T^{\alpha,\gamma})'(u)| \le \frac{C}{T^{\alpha-1}|u|^{\alpha+1}} \quad (u\ne 0)$$

where C *depends only on* α *and* γ.

Proof. It is easy to see that

$$(K_T^{\alpha,\gamma})'(u) = -\frac{2}{\sqrt{2\pi}}\int_0^T \left(1-\left(\frac{t}{T}\right)^\gamma\right)^\alpha t\sin tu\,dt$$
$$= -\frac{2}{\sqrt{2\pi}}\frac{1}{u^2}\int_0^{Tu}\left(1-\left(\frac{x}{Tu}\right)^\gamma\right)^\alpha x\sin x\,dx.$$

Of course we can suppose again that $u>0$. Similarly to the proof of Lemma 1, it is enough to verify that

$$\Big|\int_0^{2n\pi}(A^\gamma - x^\gamma)^\alpha x\sin x\,dx\Big| \le CA^{\alpha(\gamma-1)+1} \tag{10}$$

where $A=Tu$ and $2n\pi\le A<2(n+1)\pi$.

Let us change the variables $x=y+2k\pi$ and $x=y+(2k+1)\pi$ on $[2k\pi,(2k+1)\pi]$ and $[(2k+1)\pi,(2k+2)\pi]$, respectively. Then

$$\int_{2k\pi}^{(2k+2)\pi}(A^\gamma-x^\gamma)^\alpha x\sin x\,dx = \int_0^\pi g_k(x)\sin x\,dx \tag{11}$$

where

$$g_k(x) := (A^\gamma-(x+2k\pi)^\gamma)^\alpha(x+2k\pi)$$
$$-(A^\gamma-(x+(2k+1)\pi)^\gamma)^\alpha(x+(2k+1)\pi).$$

Again, $g_k'(x)>0$ and g_k is increasing. Then

$$f_1(A) := \sum_{k=0}^{n-1} g_k(0) \le \sum_{k=0}^{n-1} g_k(x) \le \sum_{k=0}^{n-1} g_k(\pi) =: f_2(A) \quad (x\in[0,\pi]). \tag{12}$$

We have

$$f_1(A) = \sum_{k=0}^{n-1}[(A^\gamma-(2k\pi)^\gamma)^\alpha 2k\pi - (A^\gamma-((2k+1)\pi)^\gamma)^\alpha(2k+1)\pi].$$

One can show that f_1 is decreasing and so

$$f_1(A) \ge f_1((2n+2)\pi)$$
$$= \pi^{\alpha\gamma+1}\sum_{k=0}^{n-1}[((2n+2)^\gamma-(2k)^\gamma)^\alpha 2k$$
$$-((2n+2)^\gamma-(2k+1)^\gamma)^\alpha(2k+1)].$$

If

$$h(x) := ((2n+2)^\gamma - x^\gamma)^\alpha x \quad (0\le x\le 2n+2)$$

then, by an easy computation,

$$h''(x) = ((2n+2)^\gamma-x^\gamma)^{\alpha-2}$$
$$\times(x^{2\gamma-1}\alpha\gamma(\alpha\gamma+1) - x^{\gamma-1}(2n+2)^\gamma(\alpha\gamma^2+\alpha\gamma)) < 0.$$

Thus h' is decreasing and

$$h(2k)-h(2k+1) = -h'(\xi) \ge -h'(2k) \quad (2k<\xi<2k+1).$$

Consequently,

$$\frac{f_1(A)}{\pi^{\alpha\gamma+1}} \ge \sum_{k=0}^{n-1} -h'(2k) \ge -h'(0) + \frac{1}{2}\int_0^{2n-2} -h'\,d\lambda \tag{13}$$
$$\ge -(2n+2)^{\alpha\gamma} - \tfrac{1}{2}((2n+2)^\gamma-(2n-2)^\gamma)^\alpha(2n-2)$$
$$\ge -(2n+2)^{\alpha\gamma} - \tfrac{1}{2}(4\gamma(2n+2)^{\gamma-1})^\alpha(2n-2)$$
$$\ge -CA^{\alpha\gamma} - CA^{\alpha(\gamma-1)+1} \ge -CA^{\alpha(\gamma-1)+1}.$$

On the other hand,

$$f_2(A) = \sum_{k=0}^{n-1}[(A^\gamma-((2k+1)\pi)^\gamma)^\alpha(2k+1)\pi - (A^\gamma-((2k+2)\pi)^\gamma)^\alpha(2k+2)\pi].$$

f_2 is also decreasing and so $f_2(A)\le f_2(2n\pi)$. We can show with the same method that

$$f_2(A) \le CA^{\alpha(\gamma-1)+1}. \tag{14}$$

We can establish that (11)–(14) imply (10). The proof of Lemma 2 is complete. ∎

LEMMA 3. *If $0<\alpha\le 1\le\gamma$ then*
$$\int_{\mathbb{R}} |K_T^{\alpha,\gamma}|\,d\lambda \le C \quad (T>0)$$
where C depends only on α and γ.

Proof. It is easy to see that $|K_T^{\alpha,\gamma}|\le \frac{2}{\sqrt{2\pi}}T$. Thus
$$\int_0^{1/T} |K_T^{\alpha,\gamma}(u)|\,du \le C.$$
By Lemma 1,
$$\int_{1/T}^{\infty} |K_T^{\alpha,\gamma}(u)|\,du \le \frac{C}{T^\alpha}\int_{1/T}^{\infty}\frac{1}{u^{\alpha+1}}\,du \le C.$$
Since $K_T^{\alpha,\gamma}$ is even, this proves the lemma. ∎

5. The boundedness of the maximal Riesz operator. A bounded measurable function a is a *p-atom* if there exists an interval $I\subset\mathbb{R}$ such that

(i) $\int_I a(x)x^j\,dx=0$ where $j\in\mathbb{N}$ and $j\le[1/p-1]$, the integer part of $1/p-1$,

(ii) $\|a\|_\infty\le|I|^{-1/p}$,

(iii) $\{a\ne 0\}\subset I$.

An operator V which maps the set of distributions into the collection of measurable functions will be called *p-quasi-local* if there exists a constant $C_p>0$ such that
$$\int_{\mathbb{R}\setminus 4I}|Va|^p\,d\lambda\le C_p$$
for every p-atom a where I is the support of the atom and $4I$ is the interval with the same center as I and with length $4|I|$. The following result can be found in Weisz [13]:

THEOREM B. *Suppose that the operator V is sublinear and p-quasi-local for some $0<p\le 1$. If V is bounded from $L_{p_1}(\mathbb{R})$ to $L_{p_1}(\mathbb{R})$ for a fixed $1<p_1\le\infty$ then*
$$\|Vf\|_p\le C_p\|f\|_{H_p(\mathbb{R})}\quad (f\in H_p(\mathbb{R})).$$

Now we can formulate our main result.

THEOREM 1. *Assume that $0<\alpha\le 1\le\gamma$. Then*
$$\|\sigma_*^{\alpha,\gamma}f\|_{p,q}\le C_{p,q}\|f\|_{H_{p,q}(\mathbb{R})}\quad (f\in H_{p,q}(\mathbb{R}))\tag{15}$$

for every $1/(\alpha+1)<p<\infty$ and $0<q\le\infty$. In particular, if $f\in L_1(\mathbb{R})$ then
$$\lambda(\sigma_*^{\alpha,\gamma}f>\varrho)\le\frac{C}{\varrho}\|f\|_1\quad(\varrho>0).\tag{16}$$

Proof. It is easy to see that Lemma 3 implies
$$\|\sigma_*^{\alpha,\gamma}f\|_\infty\le C\|f\|_\infty\quad(f\in L_\infty(\mathbb{R})).$$
First we verify (15) for $p=q$ and for $1/(\alpha+1)<p\le 1$. To this end, by Theorem B, we have to prove that the operator $\sigma_*^{\alpha,\gamma}$ is p-quasi-local.

Let a be an arbitrary p-atom with support I and $2^{K-1}<|I|\le 2^K$ ($K\in\mathbb{Z}$). We can suppose that the center of I is zero. In this case
$$[-2^{K-2},2^{K-2}]\subset I\subset[-2^{K-1},2^{K-1}].$$
Obviously,
$$\begin{aligned}\int_{\mathbb{R}\setminus 4I}|\sigma_*^{\alpha,\gamma}a(x)|^p\,dx&\le\sum_{|i|=1}^{\infty}\int_{i2^K}^{(i+1)2^K}|\sigma_*^{\alpha,\gamma}a(x)|^p\,dx\\&\le\sum_{|i|=1}^{\infty}\int_{i2^K}^{(i+1)2^K}\sup_{T\ge r_i}|\sigma_T^{\alpha,\gamma}a(x)|^p\,dx\\&\quad+\sum_{|i|=1}^{\infty}\int_{i2^K}^{(i+1)2^K}\sup_{T<r_i}|\sigma_T^{\alpha,\gamma}a(x)|^p\,dx\\&=(A)+(B)\end{aligned}$$
where $r_i:=[2^{-K}/|i|^\delta]$ ($i\in\mathbb{N}$) with $\delta>0$ to be chosen later. We can suppose that $i\ge 1$.

For
$$A(x):=\int_{-\infty}^{x}a(t)\,dt\quad(x\in\mathbb{R})$$
we have $\operatorname{supp}A\subset I$, A is zero at the endpoints of I and $\|A\|_\infty\le|I|^{-1/p+1}$.

Lemma 1 implies
$$|\sigma_T^{\alpha,\gamma}a(x)|=\Big|\int_I a(t)K_T^{\alpha,\gamma}(x-t)\,dt\Big|\le|I|^{-1/p}\int_I\frac{C}{T^\alpha|x-t|^{\alpha+1}}\,dt.$$
By a simple calculation we get
$$\int_{-2^{K-1}}^{2^{K-1}}\frac{1}{|x-t|^{\alpha+1}}\,dt\le\frac{C2^K}{(i2^K-2^{K-1})^{\alpha+1}}\le\frac{C2^{-K\alpha}}{i^{\alpha+1}}$$

if $x \in [i2^K, (i+1)2^K)$ $(i \ge 1)$. Hence

$$|\sigma_T^{\alpha,\gamma} a(x)| \le C_p 2^{-K/p-K\alpha} T^{-\alpha} \frac{1}{i^{\alpha+1}}.$$

Using the value of r_i we conclude that

$$(\mathrm{A}) \le C_p \sum_{i=1}^{\infty} 2^K 2^{-K-pK\alpha} \left(\frac{2^{-K}}{i^{\delta}}\right)^{-p\alpha} \frac{1}{i^{(\alpha+1)p}} \le C_p \sum_{i=1}^{\infty} \frac{1}{i^{-\delta p\alpha + p(\alpha+1)}}.$$

This series is convergent if

$$\delta < \frac{p(\alpha+1)-1}{p\alpha}. \tag{17}$$

Now we consider (B). Integrating by parts we can see that

$$|\sigma_T^{\alpha,\gamma} a(x)| = \Big|\int_I A(t)(K_T^{\alpha,\gamma})'(x-t)\,dt\Big|.$$

Using Lemma 2 we obtain

$$|\sigma_T^{\alpha,\gamma} a(x)| \le |I|^{-1/p+1} \int_I \frac{C}{T^{\alpha-1}|x-t|^{\alpha+1}}\,dt \le C_p 2^{-K/p+K-K\alpha} T^{1-\alpha} \frac{1}{i^{\alpha+1}}$$

if $x \in [i2^K, (i+1)2^K)$. Hence

$$(\mathrm{B}) \le C_p \sum_{i=1}^{\infty} 2^K 2^{-K+p(K-K\alpha)} \left(\frac{2^{-K}}{i^{\delta}}\right)^{p(1-\alpha)} \frac{1}{i^{(\alpha+1)p}} \le C_p \sum_{i=1}^{\infty} \frac{1}{i^{\delta p(1-\alpha)+p(\alpha+1)}},$$

which is a convergent series if

$$\delta > \frac{1-p(\alpha+1)}{p(1-\alpha)} \tag{18}$$

whenever $\alpha < 1$. If $\alpha = 1$ then we get $p > 1/2$. (17) and (18) imply that $p > 1/(\alpha+1)$.

Thus we have proved (15) for $p = q > 1/(\alpha+1)$. Applying Theorem A we obtain (15). Let us specify this result for $p = 1$ and $q = \infty$. If $f \in L_1(\mathbb{R})$ then (1) implies

$$\|\sigma_*^{\alpha,\gamma} f\|_{1,\infty} = \sup_{\varrho>0} \varrho\lambda(\sigma_*^{\alpha,\gamma} f > \varrho) \le C\|f\|_{H_{1,\infty}(\mathbb{R})} \le C\|f\|_1,$$

which shows (16). The proof of the theorem is complete. ■

We can state the same for the maximal conjugate Riesz operator.

THEOREM 2. *Assume that* $0 < \alpha \le 1 \le \gamma$. *Then*

$$\|\widetilde{\sigma}_*^{\alpha,\gamma} f\|_{p,q} \le C_{p,q}\|f\|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R}))$$

for every $1/(\alpha+1) < p < \infty$ *and* $0 < q \le \infty$. *In particular, if* $f \in L_1(\mathbb{R})$ *then*

$$\lambda(\widetilde{\sigma}_*^{\alpha,\gamma} f > \varrho) \le \frac{C}{\varrho}\|f\|_1 \quad (\varrho > 0).$$

Proof. By (3), $\|f\|_{H_p(\mathbb{R})} = \|\widetilde{f}\|_{H_p(\mathbb{R})}$ $(0 < p < \infty)$. Using Theorem 1 for $p = q$ and the fact that $\widetilde{\sigma}_*^{\alpha,\gamma} f = \sigma_*^{\alpha,\gamma} \widetilde{f}$ we obtain

$$\|\widetilde{\sigma}_*^{\alpha,\gamma} f\|_p = \|\sigma_*^{\alpha,\gamma} \widetilde{f}\|_p \le C_p\|\widetilde{f}\|_{H_p(\mathbb{R})} = C_p\|f\|_{H_p(\mathbb{R})} \quad (f \in H_p(\mathbb{R}))$$

for every $1/(\alpha+1) < p < \infty$. Now Theorem 2 follows from Theorem A. ■

Since the set of those functions $f \in L_1(\mathbb{R})$ whose Fourier transform has a compact support is dense in $L_1(\mathbb{R})$ (see Wiener [16]), the weak type inequalities of Theorems 1 and 2 and the usual density argument (see Marcinkiewicz–Zygmund [8]) imply

COROLLARY 1. *If* $0 < \alpha \le 1 \le \gamma$ *and* $f \in L_1(\mathbb{R})$ *then*

$$\sigma_T^{\alpha,\gamma} f \to f \quad \text{a.e. as } T \to \infty,$$
$$\widetilde{\sigma}_T^{\alpha,\gamma} f \to \widetilde{f} \quad \text{a.e. as } T \to \infty.$$

Note that $\widetilde{f}$ is not necessarily integrable whenever f is.

Now we consider the norm convergence of $\sigma_T^{\alpha,\gamma} f$. It follows from (15) that $\sigma_T^{\alpha,\gamma} f \to f$ in $L_p(\mathbb{R})$ norm as $T \to \infty$ if $f \in L_p(\mathbb{R})$ $(1 < p < \infty)$. We are going to generalize this result.

THEOREM 3. *If* $0 < \alpha \le 1 \le \gamma$ *and* $T > 0$ *then*

$$\|\sigma_T^{\alpha,\gamma} f\|_{H_{p,q}(\mathbb{R})} \le C_{p,q}\|f\|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R}))$$

and

$$\|\widetilde{\sigma}_T^{\alpha,\gamma} f\|_{H_{p,q}(\mathbb{R})} \le C_{p,q}\|f\|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R}))$$

for every $1/(\alpha+1) < p < \infty$ *and* $0 < q \le \infty$.

Proof. Since $(\sigma_T^{\alpha,\gamma} f)^\sim = \widetilde{\sigma}_T^{\alpha,\gamma} f$, we see by Theorems 1 and 2 that

$$\|\sigma_T^{\alpha,\gamma} f\|_p \le C_p\|f\|_{H_p(\mathbb{R})} \quad (f \in H_p(\mathbb{R})),$$
$$\|(\sigma_T^{\alpha,\gamma} f)^\sim\|_p \le C_p\|f\|_{H_p(\mathbb{R})} \quad (f \in H_p(\mathbb{R})),$$

for all $T > 0$. (3) implies that

$$\|\sigma_T^{\alpha,\gamma} f\|_{H_p(\mathbb{R})} \le C_p\|f\|_{H_p(\mathbb{R})} \quad (f \in H_p(\mathbb{R});\ T > 0).$$

Hence

$$\|\widetilde{\sigma}_T^{\alpha,\gamma} f\|_{H_p(\mathbb{R})} \le C_p\|f\|_{H_p(\mathbb{R})} \quad (f \in H_p(\mathbb{R});\ T > 0).$$

Now Theorem A proves Theorem 3. ■

We suspect that the conclusions of Theorems 1–3 are not true for $p \le 1/(\alpha+1)$ though we could not find any counterexample.

COROLLARY 2. *Suppose that* $1/(\alpha+1) < p < \infty$ *and* $0 < q \le \infty$. *If* $f \in H_{p,q}(\mathbb{R})$ *then*

$$\sigma_T^{\alpha,\gamma} f \to f \quad \text{in } H_{p,q}(\mathbb{R}) \text{ norm as } T \to \infty,$$
$$\widetilde{\sigma}_T^{\alpha,\gamma} f \to \widetilde{f} \quad \text{in } H_{p,q}(\mathbb{R}) \text{ norm as } T \to \infty.$$

We will extend the results to $\alpha > 1$. In the next lemma we express the $\sigma_T^{1+h,\gamma}$ means by the $\sigma_T^{1,\gamma}$ means ($h > 0$).

LEMMA 4. *For* $h > 0$ *we have*

$$\sigma_T^{1+h,\gamma} f(x) = \frac{h(h+1)\gamma}{T} \int_0^T \left(1 - \left(\frac{s}{T}\right)^\gamma\right)^{h-1} \left(\frac{s}{T}\right)^{2\gamma-1} \sigma_s^{1,\gamma} f(x)\, ds. \tag{19}$$

Proof. The right hand side of (19) is equal to

$$\frac{h(h+1)\gamma}{T} \int_0^T \left(1 - \left(\frac{s}{T}\right)^\gamma\right)^{h-1} \left(\frac{s}{T}\right)^{2\gamma-1}$$
$$\times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) \frac{2}{\sqrt{2\pi}} \int_0^s \left(1 - \left(\frac{t}{s}\right)^\gamma\right) \cos tu\, dt\, du\, ds$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) \frac{2}{\sqrt{2\pi}} \int_0^T \cos tu$$
$$\times \frac{h(h+1)\gamma}{T} \int_t^T \left(1 - \left(\frac{s}{T}\right)^\gamma\right)^{h-1} \left(\frac{s}{T}\right)^{2\gamma-1} \left(1 - \left(\frac{t}{s}\right)^\gamma\right) ds\, dt\, du.$$

Integrating by parts we obtain

$$\frac{h(h+1)\gamma}{T} \int_t^T \left(1 - \left(\frac{s}{T}\right)^\gamma\right)^{h-1} \left(\frac{s}{T}\right)^{2\gamma-1} \left(1 - \left(\frac{t}{s}\right)^\gamma\right) ds$$
$$= \frac{h(h+1)\gamma}{T^{\gamma h+\gamma}} \int_t^T (T^\gamma - s^\gamma)^{h-1} s^{\gamma-1} (s^\gamma - t^\gamma)\, ds$$
$$= \frac{h+1}{T^{\gamma h+\gamma}} \int_t^T (T^\gamma - s^\gamma)^h \gamma s^{\gamma-1}\, ds$$
$$= \frac{1}{T^{\gamma h+\gamma}} (T^\gamma - t^\gamma)^{h+1} = \left(1 - \left(\frac{t}{T}\right)^\gamma\right)^{h+1},$$

which proves the lemma. ■

Lemma 4 implies $\sigma_*^{\alpha,\gamma} f \le C \sigma_*^{1,\gamma} f$ whenever $\alpha > 1$. This shows that Theorems 1 and 2 hold also for $\alpha > 1$. The extension of Theorem 3 can be proved in the same way.

COROLLARY 3. *If* $\alpha > 1$ *then all inequalities of Theorems* 1–3 *and all convergence results of Corollaries* 1 *and* 2 *hold for every* $1/2 < p < \infty$ *and* $0 < q \le \infty$.

In the next sections we verify the results above in the periodic case, i.e. for the Riesz summability of Fourier series.

6. Hardy spaces on the unit circle and conjugate functions.

The Lorentz spaces on the measure space $(\mathbb{T} := [-\pi, \pi), \lambda)$ are denoted by $L_{p,q}(\mathbb{T})$. Let f be a distribution on $C^\infty(\mathbb{T})$ (briefly $f \in \mathcal{D}'(\mathbb{T})$). The nth Fourier coefficient of f is defined by $\widehat{f}(n) := f(e^{-\imath nx})$. If f is an integrable function then

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-\imath nx}\, dx \quad (n \in \mathbb{N}).$$

For simplicity, we assume that for a distribution $f \in \mathcal{D}'(\mathbb{T})$ we have $\widehat{f}(0) = 0$.

For $f \in \mathcal{D}'(\mathbb{T})$ and $z := re^{\imath x}$ $(0 < r < 1)$ let

$$u(z) = u(re^{\imath x}) := f * P_r(x)$$

where $*$ denotes again convolution and

$$P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{\imath kx} = \frac{1 - r^2}{1 + r^2 - 2r\cos x} \quad (x \in \mathbb{T})$$

is the periodic Poisson kernel.

The *non-tangential maximal function* is defined by

$$u^*(x) := \sup_{z \in \Omega(x)} |u(z)|$$

where $\Omega(x)$ is the usual Stolz domain (see e.g. Kashin–Saakyan [7], or Weisz [13]).

For $0 < p, q \le \infty$ the *Hardy–Lorentz space* $H_{p,q}(\mathbb{T})$ consists of all distributions f for which $u^* \in L_{p,q}(\mathbb{T})$ and we set

$$\|f\|_{H_{p,q}(\mathbb{T})} := \|u^*\|_{p,q}.$$

Again, it is known that if $f \in H_p(\mathbb{T})$ then $f(x) = \lim_{r\to 1} u(re^{\imath x})$ in the sense of distributions (see Fefferman–Stein [6]).

We remark that the analogues of (1)–(3) and of Theorems A and B are true in this case (cf. Weisz [13] and the references there).

For a distribution

$$f \sim \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{\imath kx}$$

the *conjugate distribution* is defined by

$$\widetilde{f} \sim \sum_{k=-\infty}^{\infty} (-\imath \operatorname{sign} k) \widehat{f}(k) e^{\imath kx}.$$

As is well known, if f is an integrable function then

$$\widetilde{f}(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x-t)}{2\tan(t/2)}\,dt := \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon<|t|<\pi} \frac{f(x-t)}{2\tan(t/2)}\,dt.$$

Moreover, $(\widetilde{f})^{\sim} = -f$.

7. Riesz summability of Fourier series. The *Riesz means* of a distribution f are defined by

$$\sigma_n^{\alpha,\gamma} f(x) := \sum_{k=-n}^{n} \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} \widehat{f}(k) e^{\imath kx} =: f * \kappa_n^{\alpha,\gamma}(x)$$

where

$$\kappa_n^{\alpha,\gamma}(x) := \sum_{k=-n}^{n} \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} e^{\imath kx} \qquad (x \in \mathbb{T})$$

is the periodic Riesz kernel. Similarly, we introduce the *conjugate Riesz means* of a distribution f by

$$\widetilde{\sigma}_n^{\alpha,\gamma} f(x) := \sum_{k=-n}^{n} \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} (-\imath \operatorname{sign} k) \widehat{f}(k) e^{\imath kx} = \widetilde{f} * \kappa_n^{\alpha,\gamma}(x).$$

The *maximal* and *maximal conjugate Riesz operators* are defined by

$$\sigma_*^{\alpha,\gamma} f := \sup_{n\in\mathbb{N}} |\sigma_n^{\alpha,\gamma} f| \quad \text{and} \quad \widetilde{\sigma}^{\alpha,\gamma} f := \sup_{n\in\mathbb{N}} |\widetilde{\sigma}_n^{\alpha,\gamma} f|.$$

The sum

$$\sqrt{2\pi} \sum_{k=-\infty}^{\infty} K_{n+1}^{\alpha,\gamma}(x + 2k\pi)$$

is a periodic function, where $K_{n+1}^{\alpha,\gamma}$ is the non-periodic Riesz kernel. It is easy to see that the kth Fourier coefficient of this sum is equal to

$$(K_{n+1}^{\alpha,\gamma})^{\wedge}(k) = \begin{cases} \left(1 - \left|\frac{k}{n+1}\right|^{\gamma}\right)^{\alpha} & \text{if } |k| < n+1, \\ 0 & \text{if } |k| \ge n+1. \end{cases}$$

This means that

$$\kappa_n^{\alpha,\gamma}(x) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} K_{n+1}^{\alpha,\gamma}(x + 2k\pi) \qquad (x \in \mathbb{T}).$$

Hence, by Lemma 1,

$$|\kappa_n^{\alpha,\gamma}(x)| \le \sum_{k=-\infty}^{\infty} \frac{C}{(n+1)^{\alpha}|x + 2k\pi|^{\alpha+1}} \le \frac{C}{n^{\alpha}|x|^{\alpha+1}}.$$

The corresponding estimate for the derivative of $\kappa_n^{\alpha,\gamma}$ can be proved in the same way.

LEMMA 5. *If* $0 < \alpha \le 1 \le \gamma$ *then*

$$|\kappa_n^{\alpha,\gamma}(x)| \le \frac{C}{n^{\alpha}|x|^{\alpha+1}} \qquad (x \in \mathbb{T},\ x \ne 0),$$

$$|(\kappa_n^{\alpha,\gamma})'(x)| \le \frac{C}{n^{\alpha-1}|x|^{\alpha+1}} \qquad (x \in \mathbb{T},\ x \ne 0).$$

Using Lemma 5 we can prove the following results as in Section 5.

THEOREM 4. *Assume that* $0 < \alpha \le 1 \le \gamma$. *Then*

$$\|\sigma_*^{\alpha,\gamma} f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \qquad (f \in H_{p,q}(\mathbb{T})),$$

$$\|\widetilde{\sigma}_*^{\alpha,\gamma} f\|_{p,q} \le C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \qquad (f \in H_{p,q}(\mathbb{T})),$$

for every $1/(\alpha+1) < p < \infty$ *and* $0 < q \le \infty$. *In particular, if* $f \in L_1(\mathbb{T})$ *then*

$$\lambda(\sigma_*^{\alpha,\gamma} f > \varrho) \le \frac{C}{\varrho} \|f\|_1 \qquad (\varrho > 0),$$

$$\lambda(\widetilde{\sigma}_*^{\alpha,\gamma} f > \varrho) \le \frac{C}{\varrho} \|f\|_1 \qquad (\varrho > 0).$$

COROLLARY 4. *If* $0 < \alpha \le 1 \le \gamma$ *and* $f \in L_1(\mathbb{T})$ *then*

$$\sigma_n^{\alpha,\gamma} f \to f \quad \textit{a.e. as } n \to \infty,$$

$$\widetilde{\sigma}_n^{\alpha,\gamma} f \to \widetilde{f} \quad \textit{a.e. as } n \to \infty.$$

THEOREM 5. *If* $0 < \alpha \le 1 \le \gamma$ *then*

$$\|\sigma_n^{\alpha,\gamma} f\|_{H_{p,q}(\mathbb{T})} \le C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \qquad (f \in H_{p,q}(\mathbb{T})),$$

$$\|\widetilde{\sigma}_n^{\alpha,\gamma} f\|_{H_{p,q}(\mathbb{T})} \le C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \qquad (f \in H_{p,q}(\mathbb{T})),$$

for every $1/(\alpha+1) < p < \infty$ *and* $0 < q \le \infty$.

COROLLARY 5. *Suppose that* $1/(\alpha+1) < p < \infty$ *and* $0 < q \le \infty$. *If* $f \in H_{p,q}(\mathbb{T})$ *then*

$$\sigma_n^{\alpha,\gamma} f \to f \quad \textit{in } H_{p,q}(\mathbb{T}) \textit{ norm as } n \to \infty,$$

$$\widetilde{\sigma}_n^{\alpha,\gamma} f \to \widetilde{f} \quad \textit{in } H_{p,q}(\mathbb{T}) \textit{ norm as } n \to \infty.$$

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On inessential and improjective operators

by

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Abstract. We give several characterizations of the improjective operators, introduced by Tarafdar, and we characterize the inessential operators among the improjective operators. It is an interesting problem whether both classes of operators coincide in general. A positive answer would provide, for example, an intrinsic characterization of the inessential operators. We give several equivalent formulations of this problem and we show that the inessential operators acting between certain pairs of Banach spaces coincide with the improjective operators.

1. Introduction. An important class which occurs in the perturbation theory of Fredholm operators is that of *inessential operators*, introduced by Kleinecke [7] as the inverse image in $\mathcal{L}(X)$ by the quotient map

$$\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)$$

of the radical of the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$, where X is a Banach space, $\mathcal{L}(X)$ is the set of all (continuous linear) operators on X and $\mathcal{K}(X)$ is the subset of all compact operators.

Other authors [9, 10] have defined and studied inessential operators acting between different Banach spaces X, Y. Let $\mathcal{L}(X,Y)$ be the set of all (continuous linear) operators acting from X into Y. An operator $T \in \mathcal{L}(X,Y)$ is *Fredholm*, in symbols $T \in \Phi(X,Y)$, if its kernel $\ker(T)$ is finite-dimensional and its range $R(T)$ is finite-codimensional. The *inessential operators* can be defined by

$$\mathcal{I}n(X,Y) := \{T \in \mathcal{L}(X,Y) : I_X - ST \in \Phi(X) \text{ for every } S \in \mathcal{L}(Y,X)\},$$

where I_X is the identity operator in X and $\Phi(X) = \Phi(X,X)$. Equivalently [2],

$$\mathcal{I}n(X,Y) := \{T \in \mathcal{L}(X,Y) : I_Y - TS \in \Phi(Y) \text{ for every } S \in \mathcal{L}(Y,X)\}.$$

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