Riesz means of Fourier transforms and
Fourier series on Hardy spaces

by

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Abstract. Elementary estimates for the Riesz kernel and for its derivative are given. Using these we show that the maximal operator of the Riesz means of a tempered distribution is bounded from $H_p^c(\mathbb{R})$ to $L^p(\mathbb{R})$ $(1/\alpha + 1 < p < \infty)$ and is of weak type $(1,1)$, where $H_p^c(\mathbb{R})$ is the classical Hardy space. As a consequence we deduce that the Riesz means of a function $f \in L^1(\mathbb{R})$ converge a.e. to $f$. Moreover, we prove that the Riesz means are uniformly bounded on $H_p^c(\mathbb{R})$ whenever $1/(\alpha + 1) < p < \infty$. Thus, in case $f \in H_p(\mathbb{R})$, the Riesz means converge to $f$ in $H_p(\mathbb{R})$ norm $(1/(\alpha + 1) < p < \infty)$. The same results are proved for the conjugate Riesz means and for Fourier series of distributions.

1. Introduction. The Hardy–Lorentz spaces $H_{p,q}(\mathbb{R})$ of tempered distributions on the real line are endowed with the $L_{p,q}(\mathbb{R})$ Lorentz norms of the non-tangential maximal function. Of course, $H_p(\mathbb{R}) = H_{p,p}(\mathbb{R})$ are the usual Hardy spaces $(0 < p \leq \infty)$.

In this paper the Riesz means $\sigma_T^{\alpha,\gamma}f$ of tempered distributions are considered. Usually the cases $\gamma = 1, 2$ are investigated. It can be found in Stein–Weiss [12] and Butzer–Nessel [4] that the Riesz means $\sigma_T^{\alpha,\gamma}f$ $(\gamma = 1, 2)$ of a function $f \in L^1(\mathbb{R})$ converge a.e. to $f$ as $T \to \infty$. In the special case $\alpha = \gamma = 1$ the Riesz means are called Fejér means. The author [15] proved that the maximal Fejér operator $\sigma_T^{1,1} := \sup_{T>0} |\sigma_T^{1,1}|$ is bounded from $H_p(\mathbb{R})$ to $L_p(\mathbb{R})$ provided that $1/2 < p < \infty$ (for $p = 1$ see also Móricz [9]) and is of weak type $(1,1)$, i.e.

$$\sup_{\varphi > 0} \varphi \lambda(\sigma_T^{1,1}f > \varphi) \leq C \|f\|_1 \quad (f \in L^1(\mathbb{R}))$$

1991 Mathematics Subject Classification: Primary 42B05, 42A38, 42A24; Secondary 42B30.

Key words and phrases: Hardy spaces, $p$-atom, atomic decomposition, interpolation, Fourier transforms, Riesz means.

This research was supported by Bolyai Scholarship, by the Hungarian Scientific Research Funds (OTKA) No F019633 and by the Foundation for Research and Improvement in Higher Education No FKFP-0142.

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(this last result can also be found in Zygmund [17] and Móricz [9]). Similar theorems for the \((C, \alpha)\) summability of Fourier series are given in Weisz [13].

In this paper we sharpen and generalize these results. First we prove two estimates for the Riesz kernel and for its derivative with elementary methods. Next we show that the maximal operator \(\sigma_\alpha^n\) bounded from \(H_{p,q}(\mathbb{R})\) to \(L_{p,q}(\mathbb{R})\) whenever \(0 < \alpha \leq 1 \leq \gamma, 1/(\alpha + 1) < p < \infty\) and \(0 < q \leq \infty\), and is of weak type \((1,1)\). We introduce the conjugate distribution \(\tilde{f}\), the conjugate Riesz means \(\tilde{\sigma}_\alpha^n\) and the conjugate maximal operator \(\tilde{\sigma}_\alpha^n\). We deduce that the operator \(\tilde{\sigma}_\alpha^n\) is also of type \((H_{p,q}(\mathbb{R}), L_{p,q}(\mathbb{R}))\) \((1/(\alpha + 1) < p < \infty, 0 < q \leq \infty; 0 < \alpha \leq 1 \leq \gamma)\) and of weak type \((1,1)\). We extend these results also for \(\alpha > 1\).

A usual density argument then implies that, besides the convergence results mentioned above, the conjugate Riesz means \(\tilde{\sigma}_\alpha^n\) converge a.e. to \(\tilde{f}\) as \(T \to \infty\), provided that \(f \in L_1(\mathbb{R})\). Note that \(\tilde{f}\) is not necessarily integrable whenever \(f\) is.

We also prove that the operators \(\sigma_\alpha^n\) and \(\tilde{\sigma}_\alpha^n\) (\(T > 0\)) are uniformly bounded in \(T\) from \(H_{p,q}(\mathbb{R})\) to \(H_{p,q}(\mathbb{R})\) if \(1/(\alpha + 1) < p < \infty\), \(0 < q \leq \infty\).

From this it follows that \(\sigma_\alpha^n f \to f\) and \(\tilde{\sigma}_\alpha^n f \to \tilde{f}\) in \(H_{p,q}(\mathbb{R})\) norm as \(T \to \infty\), whenever \(f \in H_{p,q}(\mathbb{R})\) and \(1/(\alpha + 1) < p < \infty\), \(0 < q \leq \infty\).

We also consider the Riesz means of Fourier series of distributions on the unit circle and prove all the results above in this context.

I would like to thank the referees for reading the paper carefully.

2. Hardy spaces on the real line and Hilbert transforms. Let \(\mathbb{R}\) denote the real line and \(\lambda\) be the Lebesgue measure. We also use the notation \(|I|\) for the Lebesgue measure of the set \(I\). We briefly write \(L_p(\mathbb{R})\) for the real \(L_p(\mathbb{R}, \lambda)\) space with norm (or quasinorm) \(\|f\|_p := \left(\int |f|^p \, d\lambda\right)^{1/p} (0 < p \leq \infty)\).

The distribution function of a Lebesgue-measurable function \(f\) is defined by

\[ \lambda([f > \delta]) := \lambda(\{x: |f(x)| > \delta\}) \quad (\delta \geq 0). \]

The weak \(L_p(\mathbb{R})\) space \(L^*_p(\mathbb{R}) (0 < p < \infty)\) consists of all measurable functions \(f\) for which

\[ \|f\|_{L^*_p(\mathbb{R})} := \sup_{\delta > 0} \delta \lambda([f > \delta])^{1/p} < \infty. \]

and we set \(L^*_\infty(\mathbb{R}) = L_{\infty}(\mathbb{R})\).

The spaces \(L^*_p(\mathbb{R})\) are special cases of the more general Lorentz spaces \(L_{p,q}(\mathbb{R})\). In their definition another concept is used. For a measurable function \(f\) the non-increasing rearrangement is defined by

\[ f(t) := \inf\{\delta: \lambda([|f| > \delta]) \leq t\}. \]

The Lorentz space \(L_{p,q}(\mathbb{R})\) is defined as follows: for \(0 < p < \infty\) and \(0 < q < \infty\),

\[ \|f\|_{L_{p,q}(\mathbb{R})} := \left(\int_0^{\infty} \left(\int_0^t |f|^{q'} \, \frac{dt}{t}\right)^{p/q} \frac{dt}{t}\right)^{1/q} \]

while for \(0 < p \leq \infty\),

\[ \|f\|_{L_{p,\infty}(\mathbb{R})} := \sup_{t > 0} t^{1/p} f(t). \]

Let

\[ L_{p,q}(\mathbb{R}) := L_{p,q}(\mathbb{R}, \lambda) := \{f: \|f\|_{L_{p,q}(\mathbb{R})} < \infty\}. \]

One can show the following equalities:

\[ L_{p,p}(\mathbb{R}) = L_p(\mathbb{R}), \quad L_{p,\infty}(\mathbb{R}) = L_p^*(\mathbb{R}) \quad (0 < p \leq \infty) \]

(see e.g. Bennett–Sharpley [1] or Bergh–Lofström [2]).

Let \(f\) be a tempered distribution on \(S(\mathbb{R})\) (briefly \(f \in S'(\mathbb{R})\)). The Fourier transform of \(f\) is denoted by \(\hat{f}\). If \(f\) is an integrable function then

\[ \hat{f}(u) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f(x) e^{-iu x} \, dx \quad (u \in \mathbb{R}) \]

where \(i = \sqrt{-1}\).

For \(f \in S'(\mathbb{R})\) and \(t > 0\) let

\[ u(x, t) := (f * P_t)(x) \]

where \(*\) denotes convolution and

\[ P_t(x) := \frac{ct}{t^2 + x^2} \quad (x \in \mathbb{R}) \]

is the Poisson kernel.

The non-tangential maximal function is defined by

\[ u^*(x) := \sup_{|z' - z| < t} |u(x', t)|. \]

For \(0 < p, q \leq \infty\) the Hardy–Lorentz space \(H_{p,q}(\mathbb{R})\) consists of all tempered distributions \(f\) for which \(u^* \in L_{p,q}(\mathbb{R})\) and we set

\[ \|f\|_{H_{p,q}(\mathbb{R})} := \|u^*\|_{L_{p,q}(\mathbb{R})}. \]

Note that in case \(p = q\) the usual definition of the Hardy spaces \(H_{p,p}(\mathbb{R}) = H_p(\mathbb{R})\) is obtained. It is known that if \(f \in H_p(\mathbb{R}) (0 < p < \infty)\) then

\[ f(x) = \lim_{t \to 0^+} u(x, t) \]

in the sense of distributions (see Fefferman–Stein [6]).

Recall that \(L_1(\mathbb{R}) \subset H_{1,\infty}(\mathbb{R})\), more exactly,

\[ \|f\|_{H_{1,\infty}(\mathbb{R})} = \sup_{\delta > 0} \delta \lambda([u^* > \delta]) \leq C\|f\|_1 \quad (f \in L_1(\mathbb{R})). \]

Moreover,

\[ H_{p,q}(\mathbb{R}) \sim L_{p,q}(\mathbb{R}) \quad (1 < p < \infty, 0 < q \leq \infty) \]
where \( \sim \) denotes equivalence of norms and spaces (see Fefferman–Stein [6], Stein [11], Fefferman–Rivièrè–Sagher [5]).

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Fefferman–Rivièrè–Sagher [5] and also Weisz [14]).

**Theorem A.** If a sublinear (resp. linear) operator \( V \) is bounded from \( H^p_0(\mathbb{R}) \) to \( L^p(\mathbb{R}) \) (resp. to \( H^p(\mathbb{R}) \)) and from \( L^{p_1}(\mathbb{R}) \) to \( L^{p_2}(\mathbb{R}) \) (\( p_0 \leq p_1 \leq \infty \)) then it is also bounded from \( H^{p,q}(\mathbb{R}) \) to \( L^{p,q}(\mathbb{R}) \) (resp. to \( H^{p,q}(\mathbb{R}) \)) if \( p_0 < p < p_1 \) and \( 0 < q \leq \infty \).

For a tempered distribution \( f \in H^p(\mathbb{R}) \) (\( 0 < p < \infty \)) the Hilbert transform or the conjugate distribution \( \tilde{f} \) is defined by

\[
\tilde{f} := f \ast \varphi \quad \text{where} \quad \varphi(u) = -i \sin u, \quad \varphi(x) = \frac{1}{\pi x}.
\]

One can prove (see e.g. Fefferman–Stein [6]) that \( \tilde{f} \) is a well defined distribution, \( \tilde{f} \in H^p(\mathbb{R}) \) and \( (\tilde{f})' = -f \). Furthermore, Fefferman and Stein [6] showed that

\[
\|f\|_{H^p(\mathbb{R})} \sim \|f\|_p + \|\tilde{f}\|_p \quad (0 < p < \infty).
\]

As is well known, if \( f \) is an integrable function then

\[
\tilde{f}(x) = \text{p.v.} \left\{ \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-t)}{t} \, dt := \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\varepsilon < |t| < \varepsilon} \frac{f(x-t)}{t} \, dt \right\}.
\]

Moreover, the conjugate function \( \tilde{f} \) does exist almost everywhere, but it is not integrable in general.

**3. Riesz means.** Suppose first that \( f \in L^p(\mathbb{R}) \) for some \( 1 \leq p \leq 2 \). It is known that if \( \tilde{f} \in L^1(\mathbb{R}) \) then

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(u) e^{iux} \, du \quad (x \in \mathbb{R}).
\]

This motivates the definition of the Dirichlet integral \( s_t f \):

\[
s_t f(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} \tilde{f}(u) e^{iux} \, du \quad (t > 0).
\]

The conjugate Dirichlet integral is defined by

\[
\tilde{s}_t f(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} (-i \sin u) \tilde{f}(u) e^{iux} \, du \quad (t > 0).
\]

It is easy to see that

\[
s_t f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) \left\{ \frac{2}{\sqrt{2\pi}} \sin tu \right\} \, du,
\]

\[
\tilde{s}_t f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(x-u) \left\{ \frac{2}{\sqrt{2\pi}} \sin tu \right\} \, du.
\]

For \( \alpha, \gamma > 0 \) the Riesz and conjugate Riesz means are defined by

\[
\sigma_{T,\gamma}^\alpha f(x) := \alpha \int_{0}^{T} \left( 1 - \left( \frac{t}{T} \right)^\gamma \right)^\alpha -1 \tilde{s}_t f(x) \, dt \quad (T > 0),
\]

\[
\tilde{\sigma}_{T,\gamma}^\alpha f(x) := \alpha \int_{0}^{T} \left( 1 - \left( \frac{t}{T} \right)^\gamma \right)^\alpha -1 s_t f(x) \, dt \quad (T > 0),
\]

respectively. Integrating by parts we get

\[
\sigma_{T,\gamma}^\alpha f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) K_T^{\gamma,\alpha}(u) \, du
\]

where

\[
K_T^{\gamma,\alpha}(u) := \frac{2}{\sqrt{2\pi}} \int_{0}^{T} \left( 1 - \left( \frac{t}{T} \right)^\gamma \right)^\alpha \cos tu \, dt
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left( 1 - \left| \frac{t}{T} \right| \right)^\alpha \cos tu \, dt
\]

is the Riesz kernel. Similarly,

\[
\tilde{\sigma}_{T,\gamma}^\alpha f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(x-u) K_T^{\gamma,\alpha}(u) \, du.
\]

The Riesz means are called typical means if \( \gamma = 1 \), Bochner–Riesz means if \( \gamma = 2 \) and Fejér means if \( \alpha = \gamma = 1 \). One can prove (cf. Butzer–Nessel [4]) that

\[
\sigma_{T,\gamma}^\alpha f(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left( 1 - \left| \frac{t}{T} \right| \right)^\alpha \tilde{f}(t) e^{i\alpha t} \, dt,
\]

\[
\tilde{\sigma}_{T,\gamma}^\alpha f(x) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left( 1 - \left| \frac{t}{T} \right| \right)^\alpha (-i \sin \alpha t) \tilde{f}(t) e^{i\alpha t} \, dt.
\]

We extend the definition of the Riesz means to tempered distributions as follows:

\[
\sigma_{T,\gamma}^\alpha f = f \ast K_T^{\gamma,\alpha} \quad (T > 0).
\]
One can show that $\sigma_T^{\alpha,\gamma} f$ is well defined for all tempered distributions $f \in H_p(\mathbb{R})$ ($0 < p \leq \infty$) and for all functions $f \in L_p(\mathbb{R})$ ($1 \leq p \leq \infty$) (cf. Fefferman–Stein [6]). The extension of the conjugate Riesz means is

$$\tilde{\sigma}_T^{\alpha,\gamma} f := \tilde{f} * K_T^{\alpha,\gamma} \quad (T > 0).$$

The maximal and maximal conjugate Riesz operators are defined by

$$\sigma_T^{\alpha,\gamma} f := \sup_{T > 0} |\sigma_T^{\alpha,\gamma} f| \quad \text{and} \quad \tilde{\sigma}_T^{\alpha,\gamma} f := \sup_{T > 0} |\tilde{\sigma}_T^{\alpha,\gamma} f|,$$

respectively.

4. Estimates of Riesz kernels. In this section we prove some estimates for the Riesz kernels $K_T^{\alpha,\gamma}$ and for their derivatives with elementary methods.

**Lemma 1.** If $0 < \alpha \leq 1 \leq \gamma$ then

$$|K_T^{\alpha,\gamma}(u)| \leq \frac{C}{T^{\alpha+1}} \quad (u \neq 0)$$

where $C$ depends only on $\alpha$ and $\gamma$.

*Proof.* Since $K_T^{\alpha,\gamma}$ is even, we can suppose that $u > 0$. Changing variables we get

$$K_T^{\alpha,\gamma}(u) = \frac{2}{\sqrt{2\pi} u} \int_0^T \left(1 - \left(\frac{u}{T t}\right)^{\gamma}\right)^\alpha \cos x \, dx.$$

The lemma will be proved if we show that

$$\left| \int_0^T \left( (Tu)^{\gamma} - x^{\gamma} \right)^\alpha \cos x \, dx \right| \leq C(Tu)^{\alpha(\gamma-1)}.$$

In other words, denoting $Tu$ by $A$, we have to show that

$$\left| \int_0^A (A^{\gamma} - x^{\gamma})^\alpha \cos x \, dx \right| \leq CA^{\alpha(\gamma-1)}.$$

Choose $n \in \mathbb{N}$ such that $2n\pi \leq A < 2(2n+1)\pi$. Then

$$\left| \int_{2n\pi}^A (A^{\gamma} - x^{\gamma})^\alpha \cos x \, dx \right| \leq CA^{\alpha(\gamma-1)}$$

because

$$A^{\gamma} - x^{\gamma} = (A - x)^{\gamma} - 1 \leq (A - x)^{\gamma} A^{\gamma-1} \quad (A - x) \leq A$$

by the Lagrange theorem. So, it is enough to prove that

$$\left| \int_0^{2n\pi} (A^{\gamma} - x^{\gamma})^\alpha \cos x \, dx \right| \leq CA^{\alpha(\gamma-1)}.$$

Let us change variables: $x = y + 2n\pi, y = -y + (2k+1)\pi, z = y + (2k+1)\pi$ and $z = -y + (2k+1)\pi$ on the intervals $[2k\pi, (4k+1)\pi/2), [(4k+1)\pi/2, (2k+1)\pi), [2k+1\pi, (4k+3)\pi/2)$ and $[(4k+3)\pi/2, (2k+2)\pi]$, respectively. Then we obtain

$$\int_0^{2k\pi} (A^{\gamma} - x^{\gamma})^\alpha \cos x \, dx = \int_0^{\pi/2} g_k(x) \cos x \, dx$$

where

$$g_k(x) := (A^{\gamma} - (x + 2k\pi)^{\gamma})^\alpha - (A^{\gamma} - (-x + (2k+1)\pi)^{\gamma})^\alpha - (A^{\gamma} - (x + (2k+1)\pi)^{\gamma})^\alpha + (A^{\gamma} - (-x + (2k+2)\pi)^{\gamma})^\alpha.$$

It is easy to check that $g_k(x) > 0$, which means that $g_k$ is increasing and

$$f(\alpha) := \sum_{k=0}^{n-1} g_k(x) \leq \sum_{k=0}^{n-1} g_k(x) \quad (x \in [0, \pi/2])$$

where $2n\pi \leq A < 2(n+1)\pi$. Since $g_k(\pi/2) = 0$, we conclude that $g_k(0) < 0$ and $f(\alpha) < 0$. We have

$$f(\alpha) = \sum_{k=0}^{n-1} (A^{\gamma} - (2k\pi)^{\gamma})^\alpha - 2(A^{\gamma} - (2k+1)\pi)^{\gamma} + (A^{\gamma} - (2k+2)\pi)^{\gamma}$$

Moreover,

$$f'(\alpha) = \sum_{k=0}^{n-1} \alpha \gamma [(A^{\gamma} - (2k\pi)^{\gamma})^\alpha - 1 - 2(A^{\gamma} - (2k+1)\pi)^{\gamma} + (A^{\gamma} - (2k+2)\pi)^{\gamma}]$$

Since the function $g(x) := (A^{\gamma} - x^{\gamma})^\alpha$ ($0 \leq x \leq A$) is convex, the expressions in square brackets are all positive. Hence $f'(\alpha) > 0$ and $f$ is increasing. Therefore

$$\int_0^{2n\pi} \frac{\alpha \gamma [(A^{\gamma} - (2k\pi)^{\gamma})^\alpha - 1 - 2(A^{\gamma} - (2k+1)\pi)^{\gamma} + (A^{\gamma} - (2k+2)\pi)^{\gamma}]}{2\pi}$$

If

$$h(x) := (2n\pi)^{\gamma} - (2k\pi)^{\gamma} \quad (0 \leq x \leq 2n)$$

then we see immediately that $h'$ is negative and decreasing. By the Lagrange theorem there exists $2k < \xi < 2k + 1$ such that

$$((2n\pi)^{\gamma} - (2k\pi)^{\gamma})^\alpha = h'(\xi) \geq -h'(2k).$$
Consequently, by (8),
\[
\frac{f(A)}{\pi^\alpha \gamma} \geq 2 \sum_{k=0}^{n-1} -h'(2k) - (2n)^{\alpha \gamma} \geq \int_0^{2n-2} -h' d\lambda - 2h(0) - (2n)^{\alpha \gamma} \\
\geq -h(2n-2) - h(0) - (2n)^{\alpha \gamma} = -((2n)^{\gamma} - (2n-2)^{\gamma})^\alpha.
\]

Since
\[
(2n)^{\gamma} - (2n-2)^{\gamma} = 2\gamma \xi^\gamma - 2\gamma(2n)^{\gamma-1} \leq 2\gamma(2n)^{\gamma-1} \quad (2n - 2 < \xi < 2n),
\]
we conclude that
\[
\frac{f(A)}{\pi^\alpha \gamma} \geq -(2\gamma)^\alpha (2n)^{\gamma-1} \geq -C(\gamma^{-1})^\alpha.
\]

Taking into account (8), (7) and (9), we proved (5), which completes the proof of Lemma 1. 

**Lemma 2.** If \(0 < \alpha \leq 1 \leq \gamma\) then
\[
\|(K_T^{\alpha,\gamma}(u))'\| \leq \frac{C}{T^{\alpha-1}|u|^\alpha+1} \quad (u \neq 0)
\]
where \(C\) depends only on \(\alpha\) and \(\gamma\).

**Proof.** It is easy to see that
\[
(K_T^{\alpha,\gamma}(u))' = -\frac{2}{\sqrt{2\pi}} \int_0^T \left(1 - \left(\frac{t}{T}\right)^\gamma\right)^\alpha t \sin tu dt
\]
\[
= -\frac{2}{\sqrt{2\pi}} \frac{1}{u^2} \int_0^u \left(1 - \left(\frac{t}{Tu}\right)^\gamma\right)^\alpha t \sin tu dt.
\]

Of course we can suppose again that \(u > 0\). Similarly to the proof of Lemma 1, it is enough to verify that
\[
\int_0^{2n\pi} (A^\gamma - x^\gamma)^\alpha x \sin x dx \leq CA^{\alpha(\gamma-1)+1}
\]
where \(A = Tu\) and \(2n\pi \leq A < 2(n+1)\pi\).

Let us change the variables \(x = y + 2k\pi\) and \(x = y + (2k+1)\pi\) on \([2k\pi, (2k+1)\pi]\) and \([(2k+1)\pi, (2k+2)\pi]\), respectively. Then
\[
\int_0^{(2k+2)\pi} (A^\gamma - x^\gamma)^\alpha x \sin x dx = \int_0^{2k\pi} g_k(x) \sin x dx
\]
where
\[
g_k(x) := (A^\gamma - (x + 2k\pi)^\gamma)^\alpha (x + 2k\pi) - (A^\gamma - (x + (2k+1)\pi)^\gamma)^\alpha (x + (2k+1)\pi).
\]

Again, \(g_k(x) > 0\) and \(g_k\) is increasing. Then
\[
(12) \quad f_1(A) := \sum_{k=0}^{n-1} g_k(0) \leq \sum_{k=0}^{n-1} g_k(x) \leq \sum_{k=0}^{n-1} g_k(\pi) =: f_2(A) \quad (x \in [0, \pi]).
\]

We have
\[
f_1(A) = \sum_{k=0}^{n-1} [(A^\gamma - (2k\pi)^\gamma)^\alpha 2k\pi - (A^\gamma - ((2k+1)\pi)^\gamma)^\alpha (2k+1)\pi].
\]

One can show that \(f_1\) is decreasing and so
\[
f_1(A) \geq f_1((2n+2)\pi) = \pi^{\alpha+1} \sum_{k=0}^{n-1} \left[\left(2n+2\pi\right)^\gamma - (2k\pi)^\gamma\right]^{\alpha} 2k
\]
\[
- ((2n+2\pi)^\gamma - (2k+1\pi)^\gamma)^\alpha (2k+1)\pi.
\]

If
\[
h(x) := \left((2n+2\pi)^\gamma - x^\gamma\right)^\alpha x \quad (0 \leq x \leq 2n+2)
\]
then, by an easy computation,
\[
h''(x) = \left((2n+2\pi)^\gamma - x^\gamma\right)^{\alpha-2}
\]
\[
\times (x^{\gamma-1} - x^\gamma) = (\gamma \alpha - 1) - x^{\gamma-1} (2n+2\pi)^{\gamma-1} (\gamma \alpha + \alpha \gamma) < 0.
\]

Thus \(h'\) is decreasing and
\[
h(2k) - h(2k+1) = -h'(\xi) \geq -h'(2k) \quad (2k < \xi < 2k+1).
\]

Consequently,
\[
(13) \quad \frac{f_1(A)}{\pi^{\alpha+\gamma-1}} \geq \sum_{k=0}^{n-1} -h'(2k) \geq -h'(0) + \frac{1}{2} \int_0^{2n-2} -h' d\lambda
\]
\[
\geq -\left(2n+2\pi\right)^{\gamma-1} - \frac{1}{2} \int_0^{2n-2} -h' d\lambda
\]
\[
\geq -\left(2n+2\pi\right)^{\gamma-1} - \frac{1}{2} \int_0^{2n-2} -h' d\lambda
\]
\[
\geq -C(\gamma-1)^{\alpha} - C(\gamma-1)^{\alpha+1} \geq -C(\gamma-1)^{\alpha+1}.
\]

On the other hand,
\[
f_2(A) = \sum_{k=0}^{n-1} [(A^\gamma - ((2k+1)\pi)^\gamma)^\alpha (2k+1)\pi - (A^\gamma - ((2k+2)\pi)^\gamma)^\alpha (2k+2)\pi].
\]

\(f_2\) is also decreasing and \(f_2(A) \leq f_2(2n\pi)\). We can show with the same method that
\[
(14) \quad f_2(A) \leq C(\gamma-1)^{\alpha+1}.
\]

We can establish that (11)–(14) imply (10). The proof of Lemma 2 is complete. 

Lemma 3. If $0 < \alpha \leq 1 \leq \gamma$ then
\[ \int_{\mathbb{R}} |K_T^{\alpha, \gamma}| \, d\lambda \leq C \quad (T > 0) \]
where $C$ depends only on $\alpha$ and $\gamma$.

Proof. It is easy to see that $|K_T^{\alpha, \gamma}| \leq \frac{2}{\sqrt{2n}} T.$ Thus
\[ \int_{0}^{1/T} |K_T^{\alpha, \gamma}(u)| \, du \leq C. \]
By Lemma 1,
\[ \int_{1/T}^{\infty} |K_T^{\alpha, \gamma}(u)| \, du \leq \frac{C}{T^\alpha} \int_{1/T}^{\infty} u^{-\alpha+1} \, du \leq C. \]
Since $K_T^{\alpha, \gamma}$ is even, this proves the lemma. \hfill \blacksquare

5. The boundedness of the maximal Riesz operator. A bounded measurable function $a$ is a $p$-atom if there exists an interval $I \subset \mathbb{R}$ such that
(i) $\int_{I} a(x) x^j \, dx = 0$ where $j \in \mathbb{N}$ and $j \leq \lfloor 1/p - 1 \rfloor$, the integer part of $1/p - 1$,
(ii) $|a|_{\infty} \leq |I|^{-1/p}$,
(iii) $\{a \neq 0\} \subset I$.

An operator $V$ which maps the set of distributions into the collection of measurable functions will be called $p$-quasi-local if there exists a constant $C_p > 0$ such that
\[ \int_{\mathbb{R} \setminus 4I} |V(a)|^p \, d\lambda \leq C_p \]
for every $p$-atom $a$ where $I$ is the support of the atom and $4I$ is the interval with the same center as $I$ and with length $4|I|$. The following result can be found in Weiss [13]:

Theorem B. Suppose that the operator $V$ is sublinear and $p$-quasi-local for some $0 < p \leq 1$. If $V$ is bounded from $L_{p_1}(\mathbb{R})$ to $L_{p_2}(\mathbb{R})$ for a fixed $1 < p_1 \leq \infty$ then
\[ \|Vf\|_{p} \leq C_p \|f\|_{H_p(\mathbb{R})} \quad (f \in H_p(\mathbb{R})). \]

Now we can formulate our main result.

Theorem 1. Assume that $0 < \alpha \leq 1 \leq \gamma$. Then
\[ \|\sigma_T^{\gamma} f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R})) \]
for every $1/(\alpha + 1) < p < \infty$ and $0 < q \leq \infty$. In particular, if $f \in L_1(\mathbb{R})$ then
\[ \lambda(\sigma_T^{\gamma} f > \varrho) \leq \frac{C}{\varrho} \|f\|_1 \quad (\varrho > 0). \]

Proof. It is easy to see that Lemma 3 implies
\[ \|\sigma_T^{\gamma} f\|_{\infty} \leq C \|f\|_{\infty} \quad (f \in L_{\infty}(\mathbb{R})). \]
First we verify (15) for $p = q$ and for $1/(\alpha + 1) < p \leq 1$. To this end, by Theorem B, we have to prove that the operator $\sigma_T^{\gamma}$ is $p$-quasi-local.

Let $u$ be an arbitrary $p$-atom with support $I$ and $2K^{-1} < |I| \leq 2K$ ($K \in \mathbb{Z}$). We can suppose that the center of $I$ is zero. In this case
\[ [-2K^{-2}, 2K^{-2}] \subset I \subset [-2K^{-1}, 2K^{-1}]. \]
Obviously,
\[ \int_{\mathbb{R} \setminus 4I} |\sigma_T^{\gamma} a(x)|^p \, dx \leq \sum_{|i| = 1} \int_{r_i} |\sigma_T^{\gamma} a(x)|^p \, dx \]
\[ \leq \sum_{|i| = 1} \int_{T^2 r_i} \sup |\sigma_T^{\gamma} a(x)|^p \, dx \]
\[ + \sum_{|i| = 1} \int_{T^2 r_i} \sup |\sigma_T^{\gamma} a(x)|^p \, dx \]
\[ = (A) + (B) \]
where $r_i := [2^{-K}/|i|^2] (i \in \mathbb{N})$ with $\delta > 0$ to be chosen later. We can suppose that $i \geq 1$.

For
\[ A(x) := \int_{-\infty}^{x} a(t) \, dt \quad (x \in \mathbb{R}) \]
we have supp $A \subset I$, $A$ is zero at the endpoints of $I$ and $\|A\|_{\infty} \leq |I|^{-1/p + 1}$.

Lemma 1 implies
\[ |\sigma_T^{\gamma} a(x)| = \left| \int_{I} a(t) K_T^{\alpha}(x - t) \, dt \right| \leq |I|^{-1/p} \int_{I} \frac{C}{T^\alpha (|x - t|^\alpha + 1)} \, dt. \]
By a simple calculation we get
\[ \int_{-2K^{-1}}^{2K^{-1}} \frac{1}{|x - t|^{\alpha + 1}} \, dt \leq \frac{C 2^K}{(i2K^{-2} - 2K^{-1})^{\alpha + 1}} \leq \frac{C2^{-K\alpha}}{i^{\alpha + 1}}. \]
if \( x \in [2^K, (i+1)2^K) \) \((i \geq 1)\). Hence
\[
|\sigma_T^{\alpha,\gamma} a(x)| \leq C_2 2^{-K/p-K\alpha T^{-\alpha}} 1/2^{p+1}.
\]
Using the value of \( r_1 \) we conclude that
\[
(A) \leq C_p 2^{-K/p-K\alpha} \left( \frac{2^{-K/p-K\alpha}}{2^{p+1}} \right) 1/2^{p+1} \leq C_p \sum_{i=1}^{\infty} r_i^{p+1} 1/2^{p+1}.
\]
This series is convergent if
\[
\delta < \frac{p\alpha}{2^{p+1}}.
\]
(17)
Now we consider (B). Integrating by parts we can see that
\[
|\sigma_T^{\alpha,\gamma} a(x)| = \left| \int f(t)(K^{\alpha,\gamma})' (x-t) dt \right|.
\]
Using Lemma 2 we obtain
\[
|\sigma_T^{\alpha,\gamma} a(x)| \leq \left| f \right|^{-1/p+1} \left| \int f \right|^{-1/p-1} \leq C \int f \left( \frac{1}{2^{p+1}} \right) 1/2^{p+1}.
\]
if \( x \in [2^K, (i+1)2^K) \). Hence
\[
(B) \leq C_p 2^{-K/p-K\alpha} \left( \frac{2^{-K/p-K\alpha}}{2^{p+1}} \right) 1/2^{p+1} \leq C_p \sum_{i=1}^{\infty} r_i^{-1/p+1} 1/2^{p+1},
\]
which is a convergent series if
\[
\delta > \frac{1-p\alpha}{p(1-\alpha)}.
\]
(18)
whenever \( \alpha < 1 \). If \( \alpha = 1 \) then we get \( p > 1/2 \). (17) and (18) imply that
\( p > 1/(\alpha+1) \).
Thus we have proved (15) for \( p = q > 1/(\alpha+1) \). Applying Theorem A we obtain (15). Let us specify this result for \( p = 1 \) and \( q = \infty \). If \( f \in L_1(\mathbb{R}) \) then (1) implies
\[
\|\sigma_T^{\alpha,\gamma} f\|_{L_1(\mathbb{R})} = \sup_{\alpha \geq 0} \rho \lambda(\sigma_T^{\alpha,\gamma} f > \rho) \leq C \| f \|_{H_1(\mathbb{R})} \| f \|_{L_1(\mathbb{R})},
\]
which shows (16). The proof of the theorem is complete. ■

We can state the same for the maximal conjugate Riesz operator.

**Theorem 2.** Assume that \( 0 < \alpha \leq 1 \leq \gamma \). Then
\[
|\sigma_T^{\alpha,\gamma} f|_{p,q} \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R}))
\]
for every \( 1/(\alpha+1) < p < \infty \) and \( 0 < q \leq \infty \). In particular, if \( f \in L_1(\mathbb{R}) \) then
\[
\lambda(\sigma_T^{\alpha,\gamma} f > \rho) \leq \frac{C}{\rho} \| f \|_{L_1(\mathbb{R})} \quad (\rho > 0).
\]

**Proof.** By (3), \( \| f \|_{H_p(\mathbb{R})} = \| \tilde{f} \|_{H_p(\mathbb{R})} \) \((0 < p < \infty)\). Using Theorem 1 for \( p = q \) and the fact that \( \sigma_T^{\alpha,\gamma} f = \sigma_T^{\alpha,\gamma} \tilde{f} \) we obtain
\[
|\sigma_T^{\alpha,\gamma} f|_{p,q} = |\sigma_T^{\alpha,\gamma} \tilde{f}|_{p,q} \leq C_{p,q} |\sigma_T^{\alpha,\gamma} \tilde{f}|_{H_{p,q}(\mathbb{R})} = C_p \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_p(\mathbb{R}))
\]
for every \( 1/(\alpha+1) < p < \infty \). Now Theorem 2 follows from Theorem A. ■

Since the set of those functions \( f \in L_1(\mathbb{R}) \) whose Fourier transform has a compact support is dense in \( L_1(\mathbb{R}) \) (see Wiener [16]), the weak type inequalities of Theorems 1 and 2 and the usual density argument (see Marcinkiewicz–Zygmund [8]) imply

**Corollary 1.** If \( 0 < \alpha \leq 1 \leq \gamma \) and \( f \in L_1(\mathbb{R}) \) then
\[
\sigma_T^{\alpha,\gamma} f \to f \quad a.e. \quad T \to \infty,
\]
\[
\sigma_T^{\alpha,\gamma} f \to \tilde{f} \quad a.e. \quad T \to \infty.
\]

Note that \( \tilde{f} \) is not necessarily integrable whenever \( f \) is.

Now we consider the norm convergence of \( \sigma_T^{\alpha,\gamma} f \). It follows from (15) that \( \sigma_T^{\alpha,\gamma} f \to f \) in \( L_p(\mathbb{R}) \) norm as \( T \to \infty \) if \( f \in L_p(\mathbb{R}) \) \((1 < p < \infty)\). We are going to generalize this result.

**Theorem 3.** If \( 0 < \alpha \leq 1 \leq \gamma \) and \( T > 0 \) then
\[
|\sigma_T^{\alpha,\gamma} f|_{H_{p,q}(\mathbb{R})} = \sup_{\alpha > 0} \rho \lambda(\sigma_T^{\alpha,\gamma} f > \rho) \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R}))
\]
and
\[
|\sigma_T^{\alpha,\gamma} f|_{H_{p,q}(\mathbb{R})} = \sup_{\alpha > 0} \rho \lambda(\sigma_T^{\alpha,\gamma} f > \rho) \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R}))
\]
for every \( 1/(\alpha+1) < p < \infty \) and \( 0 < q \leq \infty \).

**Proof.** Since \( (\sigma_T^{\alpha,\gamma} f)^{\infty} = \sigma_T^{\alpha,\gamma} \tilde{f} \), we see by Theorems 1 and 2 that
\[
|\sigma_T^{\alpha,\gamma} f|_{p,q} \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_p(\mathbb{R}))
\]
\[
|\sigma_T^{\alpha,\gamma} f|_{p,q} \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_p(\mathbb{R}))
\]
for all \( T > 0 \). (3) implies that
\[
|\sigma_T^{\alpha,\gamma} f|_{H_{p,q}(\mathbb{R})} = \sup_{\alpha > 0} \rho \lambda(\sigma_T^{\alpha,\gamma} f > \rho) \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_p(\mathbb{R}); \quad T > 0).
\]
Hence
\[
|\sigma_T^{\alpha,\gamma} f|_{H_{p,q}(\mathbb{R})} \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_p(\mathbb{R}); \quad T > 0).
\]
Now Theorem A proves Theorem 3. ■

**Theorem A.** If \( 0 < \alpha \leq 1 \leq \gamma \) and \( T > 0 \) then
\[
|\sigma_T^{\alpha,\gamma} f|_{p,q} \leq C_{p,q} \| f \|_{H_{p,q}(\mathbb{R})} \quad (f \in H_{p,q}(\mathbb{R}))
\]
We suspect that the conclusions of Theorems 1–3 are not true for \( p \leq 1/(\alpha + 1) \) though we could not find any counterexample.

**Corollary 2.** Suppose that \( 1/(\alpha + 1) < p < \infty \) and \( 0 < q \leq \infty \). If \( f \in H_{p,q}(\mathbb{R}) \) then

\[
\sigma_{T}^{\gamma/2} f \to f \quad \text{in } H_{p,q}(\mathbb{R}) \text{ norm as } T \to \infty,
\]

\[
\sigma_{T}^{\gamma/2} f \rightharpoonup f \quad \text{in } H_{p,q}(\mathbb{R}) \text{ norm as } T \to \infty.
\]

We will extend the results to \( \alpha > 1 \). In the next lemma we express the \( \sigma_{T}^{\alpha + h,\gamma} \) means by the \( \sigma_{T}^{\alpha,\gamma} \) means (\( h > 0 \)).

**Lemma 4.** For \( h > 0 \) we have

\[
(19) \quad \sigma_{T}^{\alpha + h,\gamma} f(x) = \frac{h(h+1)\gamma}{T} \int_{0}^{T} \left( 1 - \left( \frac{s}{T} \right)^{\gamma} \right)^{h-1} \left( \frac{s}{T} \right)^{2\gamma-1} \sigma_{s}^{\alpha,\gamma} f(x) \, ds.
\]

**Proof.** The right hand side of (19) is equal to

\[
\frac{h(h+1)\gamma}{T} \int_{0}^{T} \left( 1 - \left( \frac{s}{T} \right)^{\gamma} \right)^{h-1} \left( \frac{s}{T} \right)^{2\gamma-1} \times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) \frac{2}{\sqrt{2\pi}} \int_{0}^{T} \left( 1 - \left( \frac{t}{s} \right)^{\gamma} \right) \cos tu \, dt \, du \, ds
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-u) \frac{2}{\sqrt{2\pi}} \int_{0}^{T} \cos tu \, du \, ds
\]

\[
\times \frac{h(h+1)\gamma}{T} \int_{t}^{T} \left( 1 - \left( \frac{s}{T} \right)^{\gamma} \right)^{h-1} \left( \frac{s}{T} \right)^{2\gamma-1} \left( 1 - \left( \frac{t}{s} \right)^{\gamma} \right) \, ds \, dt \, du.
\]

Integrating by parts we obtain

\[
\frac{h(h+1)\gamma}{T} \int_{t}^{T} \left( 1 - \left( \frac{s}{T} \right)^{\gamma} \right)^{h-1} \left( \frac{s}{T} \right)^{2\gamma-1} \left( 1 - \left( \frac{t}{s} \right)^{\gamma} \right) \, ds
\]

\[
= \frac{h(h+1)\gamma}{T^{\alpha+h+\gamma}} \int_{t}^{T} (T^\gamma - s^\gamma)^{h-1} s^\gamma (s^\gamma - t^\gamma) \, ds
\]

\[
= \frac{h(h+1)\gamma}{T^{\alpha+h+\gamma}} \int_{t}^{T} (T^\gamma - s^\gamma)^{h} s^\gamma \, ds
\]

\[
= \frac{1}{T^{\alpha+h+\gamma}} (T^\gamma - t^\gamma)^{h+1} = \left( 1 - \left( \frac{t}{T} \right)^{\gamma} \right)^{h+1},
\]

which proves the lemma. ■

Lemma 4 implies \( \sigma_{T}^{\alpha,\gamma} f \leq C \sigma_{T}^{\alpha,\gamma} f \) whenever \( \alpha > 1 \). This shows that Theorems 1 and 2 hold also for \( \alpha > 1 \). The extension of Theorem 3 can be proved in the same way.

**Corollary 3.** If \( \alpha > 1 \) then all inequalities of Theorems 1–3 and all convergence results of Corollaries 1 and 2 hold for every \( 1/2 < p < \infty \) and \( 0 < q \leq \infty \).

In the next sections we verify the results above in the periodic case, i.e. for the Riesz summability of Fourier series.

6. Hardy spaces on the unit circle and conjugate functions. The Lorentz spaces on the measure space \( (T := [-\pi, \pi), \lambda) \) are denoted by \( L_{p,q}(T) \). Let \( f \) be a distribution on \( C_{c}^\infty(T) \) (briefly \( f \in D'(T) \)). The rth Fourier coefficient of \( f \) is defined by \( \hat{f}(n) := f(e^{-in\pi}) \). If \( f \) is an integrable function then

\[
\hat{f}(n) = \frac{1}{2\pi} \int_{T} f(x) e^{-inx} \, dx \quad (n \in \mathbb{N}).
\]

For simplicity, we assume that for a distribution \( f \in D'(T) \) we have \( \hat{f}(0) = 0 \).

For \( f \in D'(T) \) and \( z := re^{i\theta} \) (\( 0 < r < 1 \)) let \( u(x) := u(re^{i\theta}) \) \( := f * P_r(z) \) where * denotes again convolution and

\[
P_r(z) := \sum_{k=\infty}^{\infty} r^{|k|} e^{ikz} = \frac{1 - r^2}{1 + r^2 - 2r \cos x} \quad (x \in T)
\]

is the periodic Poisson kernel.

The non-tangential maximal function is defined by

\[
u^*(x) := \sup_{r \in O(x)} |u(z)|
\]

where \( O(x) \) is the usual Stolz domain (see e.g. Kashin-Saakyan [7], or Weisz [13]).

For \( 0 < p, q \leq \infty \) the Hardy–Lorentz space \( H_{p,q}(T) \) consists of all distributions \( f \) for which \( u^* \in L_{p,q}(T) \) and we set

\[
\|f\|_{H_{p,q}(T)} := \|u^*\|_{p,q}.
\]

Again, it is known that if \( f \in H_{p,q}(T) \) then \( f(x) = \lim_{r \to 1} u(re^{i\theta}) \) in the sense of distributions (see Fefferman–Stein [6]).

We remark that the analogues of (1)–(3) and of Theorems A and B are true in this case (cf. Weisz [13] and the references there).
For a distribution
\[ f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}, \]
the conjugate distribution is defined by
\[ \tilde{f} \sim \sum_{k=-\infty}^{\infty} (-1)^{\text{sign } k} \hat{f}(k)e^{ikx}. \]
As is well known, if \( f \) is an integrable function then
\[ \tilde{f}(x) = \text{p.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} f(x-t) \frac{dt}{2\tan(t/2)} = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|<\varepsilon} f(x-t) \frac{dt}{2\tan(t/2)}. \]
Moreover, \( \tilde{f} = -f \).

7. Riesz summability of Fourier series. The Riesz means of a distribution \( f \) are defined by
\[ \sigma_{n}^{\alpha,\gamma} f(x) := \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right)^{\alpha} \hat{f}(k)e^{ikx} =: f \ast \kappa_{n}^{\alpha,\gamma}(x) \]
where
\[ \kappa_{n}^{\alpha,\gamma}(x) := \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right)^{\alpha} e^{ikx} \quad (x \in \mathbb{T}) \]
is the periodic Riesz kernel. Similarly, we introduce the conjugate Riesz means of a distribution \( f \) by
\[ \overline{\sigma}_{n}^{\alpha,\gamma} f(x) := \sum_{k=-n}^{n} \left( 1 - \frac{|k|}{n+1} \right)^{\alpha} (-1)^{\text{sign } k} \hat{f}(k)e^{ikx} = \tilde{f} \ast \kappa_{n}^{\alpha,\gamma}(x). \]
The maximal and maximal conjugate Riesz operators are defined by
\[ \sigma_{n}^{\alpha,\gamma} f := \sup_{n \in \mathbb{N}} |\sigma_{n}^{\alpha,\gamma} f| \quad \text{and} \quad \overline{\sigma}_{n}^{\alpha,\gamma} f := \sup_{n \in \mathbb{N}} |\overline{\sigma}_{n}^{\alpha,\gamma} f|. \]
The sum
\[ \sqrt{2\pi} \sum_{k=-\infty}^{\infty} K_{n+1}^{\alpha,\gamma}(x+2k\pi) \]
is a periodic function, where \( K_{n+1}^{\alpha,\gamma} \) is the non-periodic Riesz kernel. It is easy to see that the \( k \)-th Fourier coefficient of this sum is equal to
\[ (K_{n+1}^{\alpha,\gamma})^{\wedge}(k) = \begin{cases} \left( 1 - \frac{k}{n+1} \right)^{\alpha} & \text{if } |k| < n+1, \\ 0 & \text{if } |k| \geq n+1. \end{cases} \]
This means that
\[ \kappa_{n}^{\alpha,\gamma}(x) = \sqrt{2\pi} \sum_{k=-\infty}^{\infty} K_{n+1}^{\alpha,\gamma}(x+2k\pi) \quad (x \in \mathbb{T}). \]
Hence, by Lemma 1,
\[ |\kappa_{n}^{\alpha,\gamma}(x)| \leq \sum_{k=-\infty}^{\infty} \frac{C}{n^{\alpha}|x+2k\pi|^{\alpha+1}} \leq \frac{C}{n^{\alpha}|x|^{\alpha+1}}. \]
The corresponding estimate for the derivative of \( \kappa_{n}^{\alpha,\gamma} \) can be proved in the same way.

**Lemma 5.** If \( 0 < \alpha \leq 1 \leq \gamma \) then
\[ |\kappa_{n}^{\alpha,\gamma}(x)| \leq \frac{C}{n^{\alpha}|x|^{\alpha+1}} \quad (x \in \mathbb{T}, x \neq 0), \]
\[ |(\kappa_{n}^{\alpha,\gamma})'(x)| \leq \frac{C}{n^{\alpha-1}|x|^{\alpha+1}} \quad (x \in \mathbb{T}, x \neq 0). \]
Using Lemma 5 we can prove the following results as in Section 5.

**Theorem 4.** Assume that \( 0 < \alpha < 1 \leq \gamma \). Then
\[ \|\sigma_{n}^{\alpha,\gamma} f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \quad (f \in H_{p,q}(\mathbb{T})), \]
\[ \|\overline{\sigma}_{n}^{\alpha,\gamma} f\|_{p,q} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \quad (f \in H_{p,q}(\mathbb{T})), \]
for every \( 1/(\alpha+1) < p < \infty \) and \( 0 < q \leq \infty \). In particular, if \( f \in L_{1}(\mathbb{T}) \) then
\[ \lambda(\sigma_{n}^{\alpha,\gamma} f > \theta) \leq \frac{C}{\theta} \|f\|_{1} \quad (\theta > 0), \]
\[ \lambda(\overline{\sigma}_{n}^{\alpha,\gamma} f > \theta) \leq \frac{C}{\theta} \|f\|_{1} \quad (\theta > 0). \]

**Corollary 4.** If \( 0 < \alpha \leq 1 \leq \gamma \) and \( f \in L_{1}(\mathbb{T}) \) then
\[ \sigma_{n}^{\alpha,\gamma} f \rightarrow f \quad \text{a.e. as } n \rightarrow \infty, \]
\[ \overline{\sigma}_{n}^{\alpha,\gamma} f \rightarrow \tilde{f} \quad \text{a.e. as } n \rightarrow \infty. \]

**Theorem 5.** If \( 0 < \alpha < 1 \leq \gamma \) then
\[ \|\sigma_{n}^{\alpha,\gamma} f\|_{H_{p,q}(\mathbb{T})} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \quad (f \in H_{p,q}(\mathbb{T})), \]
\[ \|\overline{\sigma}_{n}^{\alpha,\gamma} f\|_{H_{p,q}(\mathbb{T})} \leq C_{p,q} \|f\|_{H_{p,q}(\mathbb{T})} \quad (f \in H_{p,q}(\mathbb{T})), \]
for every \( 1/(\alpha+1) < p < \infty \) and \( 0 < q \leq \infty \).

**Corollary 5.** Suppose that \( 1/(\alpha+1) < p < \infty \) and \( 0 < q \leq \infty \). If \( f \in H_{p,q}(\mathbb{T}) \) then
\[ \sigma_{n}^{\alpha,\gamma} f \rightarrow f \quad \text{in } H_{p,q}(\mathbb{T}) \text{ norm as } n \rightarrow \infty, \]
\[ \overline{\sigma}_{n}^{\alpha,\gamma} f \rightarrow \tilde{f} \quad \text{in } H_{p,q}(\mathbb{T}) \text{ norm as } n \rightarrow \infty. \]
On inessential and imprjective operators

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Abstract. We give several characterizations of the imprjective operators, introduced by Tarafdar, and we characterize the inessential operators among the imprjective operators. It is an interesting problem whether both classes of operators coincide in general. A positive answer would provide, for example, an intrinsic characterization of the inessential operators. We give several equivalent formulations of this problem and we show that the inessential operators acting between certain pairs of Banach spaces coincide with the imprjective operators.

1. Introduction. An important class which occurs in the perturbation theory of Fredholm operators is that of inessential operators, introduced by Klinecke [7] as the inverse image in \( \mathcal{L}(X) \) by the quotient map

\[
\pi : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{K}(X)
\]

of the radical of the Calkin algebra \( \mathcal{L}(X)/\mathcal{K}(X) \), where \( X \) is a Banach space, \( \mathcal{L}(X) \) is the set of all (continuous linear) operators on \( X \) and \( \mathcal{K}(X) \) is the subset of all compact operators.

Other authors [9, 10] have defined and studied inessential operators acting between different Banach spaces \( X, Y \). Let \( \mathcal{L}(X, Y) \) be the set of all (continuous linear) operators acting from \( X \) into \( Y \). An operator \( T \in \mathcal{L}(X, Y) \) is Fredholm, in symbols \( T \in \mathcal{F}(X, Y) \), if its kernel \( \ker(T) \) is finite-dimensional and its range \( \text{R}(T) \) is finite-codimensional. The inessential operators can be defined by

\[
\mathcal{I}_n(X, Y) := \{ T \in \mathcal{L}(X, Y) : I_X - ST \in \mathcal{F}(X) \text{ for every } S \in \mathcal{L}(Y, X) \},
\]

where \( I_X \) is the identity operator in \( X \) and \( \mathcal{F}(X) = \mathcal{F}(X, X) \). Equivalently [2],

\[
\mathcal{I}_n(X, Y) := \{ T \in \mathcal{L}(X, Y) : I_Y - TS \in \mathcal{F}(Y) \text{ for every } S \in \mathcal{L}(Y, X) \}.
\]