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School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455
U.S.A.
E-mail: forzani@intec.unl.edu.ar

Departamento de Matemáticas
Universidad Autónoma de Madrid
Ciudad Universitaria de Canto Blanco
28049 Madrid, Spain

Current adress:
Departamento de Matemáticas
Facultad de Ciencias Exactas
Universidad Nacional Salta
Bs. As. 177
4400 Salta, Argentina
E-mail: scotto@ciunsa.unsa.edu.ar

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Injective semigroup-algebras

by

J. J. GREEN (Sheffield)

Abstract. Semigroups S for which the Banach algebra $\ell^1(S)$ is injective are investigated and an application to the work of O. Yu. Aristov is described.

1. Introduction. Injective Banach algebras were introduced by Varopoulos in [12] and have continued to attract investigation some 25 years later. In this note we make some progress towards a structural description of the semigroups S for which the Banach algebra $\ell^1(S)$ is injective.

To introduce our notation suppose that A and B are Banach algebras. We write $A \otimes B$ for the algebraic tensor product over $\mathbb{C}$, and $A \otimes_\varepsilon B$ for $A \otimes B$ equipped with (but not completed in) the injective tensor norm

$$\Big\| \sum_{i=1}^n a_i \otimes b_i \Big\|_\varepsilon := \sup\Big\{ \Big| \sum_{i=1}^n f(a_i) g(b_i) \Big| : f \in A_1^*,\ g \in B_1^* \Big\}.$$

Tensor products and tensor norms are given a detailed treatment in [5], while [3, §42] provides an introduction. Following Varopoulos we will say that a Banach algebra A is *injective* if the mapping

$$R_A : A \otimes_\varepsilon B \to A, \qquad \sum_{i=1}^n a_i \otimes b_i \mapsto \sum_{i=1}^n a_i b_i,$$

often called the *product morphism*, is bounded.

If S is a semigroup we write $\mathbb{C}[S]$ for the algebra of formal sums

$$x = \sum_{s\in S} \xi_s s \tag{1}$$

for which only finitely many of the $\xi_s \in \mathbb{C}$ are non-zero. When equipped with the ℓ^1 norm

$$\|x\|_1 := \sum_{s\in S} |\xi_s|,$$

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$\mathbb{C}[S]$ is a normed algebra whose completion is the ℓ^1 *semigroup-algebra* universally denoted by $\ell^1(S)$. We will always assume that our semigroups are countable and we use the notational convention that a semigroup S has an unspecified but fixed enumeration of its elements, i.e. $S = \{s_i : i \in \mathbb{N}\}$.

2. Necessary conditions. It is well known that an injective Banach algebra is an operator algebra [10, Th. 4.2.26] and so Arens regular [4]. Thus necessary conditions for a semigroup S to have $\ell^1(S)$ injective follow immediately from the characterization of the semigroups S for which $\ell^1(S)$ is Arens regular [11], [13], [2]. Indeed, the title of [11] indicates that the Arens regularity of $\ell^1(S)$ places strong restrictions on the structure of S, hence the injectivity of $\ell^1(S)$ more so.

The following lemma enables us to utilise the results above but at the same time to exploit the stronger hypothesis of injectivity.

LEMMA 2.1. *Suppose that S and T are semigroups and that $\ell^1(T)$ is not injective. Suppose further that there are finite subsets $T_1, T_2, \ldots$ with*

$$T_1 \subseteq T_2 \subseteq \ldots \subseteq T$$

whose union is T and, if $m = m(n)$ denotes the smallest integer such that $T_n^2 \subseteq T_m$, that there are maps

$$\psi_n : T_{m(n)} \to S \quad (n \in \mathbb{N})$$

with

$$\psi_n(a)\psi_n(b) = \psi_n(ab) \quad (a, b \in T_n,\ n \in \mathbb{N}).$$

Then $\ell^1(S)$ is not injective.

Proof. If $K > 0$ is given then, since $\ell^1(T)$ is not injective, there is some $u \in \ell^1(T) \otimes_\varepsilon \ell^1(T)$ with $\|u\|_\varepsilon \le 1$ and $\|R_{\ell^1(T)}(u)\|_1 \ge K$. Indeed, by a density argument, we assume that u has a representation as a finite sum

$$u = \sum_{i,j} \xi_{i,j} a_i \otimes b_j \quad (a_i, b_j \in T)$$

and take n to be a number such that a_i and b_j are in T_n whenever $\xi_{i,j} \in \mathbb{C}$ is non-zero. The map ψ_n has an obvious linearisation, which we also denote ψ_n when we define

$$v = \sum_{i,j} \xi_{i,j}\psi_n(a_i) \otimes \psi_n(b_j) \in \ell^1(S) \otimes_\varepsilon \ell^1(S).$$

Then we have

$$(2) \qquad \|v\|_\varepsilon = \sup\Big\{\Big|\sum_{i,j} \xi_{i,j} f(\psi_n(a_i))g(\psi_n(b_j))\Big| : f, g \in (\ell^1(S))^*_1\Big\}$$
$$\le \sup\Big\{\Big|\sum_{i,j} \xi_{i,j} F(a_i)G(b_j)\Big| : F, G \in (\ell^1(T))^*_1\Big\} = \|u\|_\varepsilon$$

since $f \circ \psi_n$ and $g \circ \psi_n$ are linear functionals on $\ell^1(T_m)$ of norm no greater than one, and so may be extended to such on $\ell^1(T)$ by the Hahn–Banach theorem. Then

$$R_{\ell^1(S)}(v) = \sum_{i,j} \xi_{i,j}\psi_n(a_i)\psi_n(b_j) = \psi_n(R_{\ell^1(T)}(u))$$

so that, by (2),

$$\|R_{\ell^1(S)}(v)\|_1 = \|\psi_n(R_{\ell^1(T)}(u))\|_1 \ge K\|u\|_\varepsilon \ge \|v\|_\varepsilon. \ \blacksquare$$

Our first application of the lemma is to show that semigroups S with $\ell^1(S)$ injective are "uniformly periodic".

PROPOSITION 2.2. *If S is a semigroup with $\ell^1(S)$ is injective then there is a number $N \in \mathbb{N}$ such that*

$$\operatorname{card}\{s^n : n \in \mathbb{N}\} \le N \quad (s \in S).$$

In particular, such a semigroup is periodic.

Proof. If there is no such N then for each $n \in \mathbb{N}$ we can find some $s \in S$ such that $s, s^2, \ldots, s^{2n}$ are distinct. Writing $T_n = \{1, \ldots, n\}$ (considered as a subset of the semigroup of $\mathbb{N}$ with addition as product) and defining

$$\psi_n : T_{2n} \to S, \quad i \mapsto s^i,$$

we see that the conditions of the lemma are met once we have shown that the semigroup $T = (\mathbb{N}, +)$ has a semigroup algebra which is not injective. But it is not even Arens regular, as is shown by a straightforward application of [2, Th. 2.7]. $\blacksquare$

The hypothesis of injectivity in Proposition 2.2 cannot be weakened to that of Arens regularity. To see this we observe the following fact, whose proof, again, is a consequence of [2, Th. 2.7].

PROPOSITION 2.3. *Let S be a semigroup with zero θ such that for each $s \in S$ there are only finitely many $t \in S$ such that $st \ne \theta$ and only finitely many $r \in S$ such that $rs \ne \theta$. Then $\ell^1(S)$ is Arens regular.*

The conditions of Proposition 2.3 are met by the semigroup S which is the zero direct product [7, Ch. 3, Sect. 3] of a sequence of cyclic groups of increasing order. So we find a semigroup that clearly does not satisfy the conditions of Proposition 2.2, but whose semigroup algebra is not Arens regular.

The second application of Lemma 2.1 concerns the set $E(S)$ of idempotents in a semigroup S. Let $\le$ denote the partial order on $E(S)$ defined by

$$e \le f \quad \text{if and only if} \quad ef = fe = e.$$

PROPOSITION 2.4. *Let S be a semigroup such that $\ell^1(S)$ is injective. Then there is a number $N \in \mathbb{N}$ such that no chain of idempotents in $E(S)$ exceeds N in length.*

Proof. If there is no such N then, for each $n \in \mathbb{N}$, we can find some chain of n idempotents, say $e_n \le e_{n-1} \le \ldots \le e_1$. Writing $T_n = \{1, \ldots, n\}$ (considered as a subset of the semigroup of $\mathbb{N}$ with the max product) and defining

$$\psi_n : T_n \to S, \quad i \mapsto e_{n-i+1},$$

we see that the conditions of Lemma 2.1 are met once we show that the semigroup $(\mathbb{N}, \max)$ has a semigroup algebra which is not injective. Again [2, Th. 2.7] shows that it is not even Arens regular. ■

One may show that Arens regularity cannot replace injectivity as the hypothesis of Proposition 2.4. The method is similar to the above — consider the semigroup which is a zero direct sum of a sequence of chains of increasing order. Notice, however, that the Arens regularity of $\ell^1(S)$ implies that the chains of idempotents in S must at least be finite, else S has a sub-semigroup isomorphic to $(\mathbb{N}, \max)$.

The contrast between the associations of Arens regularity with finiteness and injectivity with uniform boundedness seems a theme of subject and is maintained in the next section.

3. Sufficient conditions for semigroups with zero. For semigroups S with zero there are some conditions that force the injectivity of $\ell^1(S)$: conditions which prescribe the *sparsity* of non-zero products in S. Our approach to these is via a well-known algebraic construction.

If S is a semigroup with zero θ then we will write $\mathbb{C}_{\mathrm{r}}[S]$ for the *reduced semigroup-algebra* of S, the linear algebra $\mathbb{C}[S]/\mathbb{C}[\{\theta\}]$, and denote by $\ell^1_{\mathrm{r}}(S)$ the completion of $\mathbb{C}_{\mathrm{r}}[S]$ in the ℓ^1 norm

$$\Big\| \sum_{s \in S \setminus \{\theta\}} \xi_s s \Big\|_1 := \sum_{s \in S \setminus \{\theta\}} |\xi_s|.$$

Our interest in such algebras lies in the following fact, whose proof is, but for a change in notation, essentially the argument used by Varopoulos in [12] (and attributed there to S. Kaijser) to show that ℓ^1 is injective.

LEMMA 3.1 (Varopoulos, 1972). *Let $S = \{\theta, e_1, e_2, \ldots\}$ be a countable semigroup with zero θ and suppose that*

$$u = \sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j \in \ell^1_{\mathrm{r}}(S) \otimes_\varepsilon \ell^1_{\mathrm{r}}(S).$$

Then for any permutation σ on $\{1, \ldots, m\}$,

$$\sum_{i=1}^{m} |\xi_{i,\sigma(i)}| \le \|u\|_\varepsilon.$$

PROPOSITION 3.2. *Let S be a countable semigroup with zero θ. Suppose that there is some $K \in \mathbb{N}$ such that for each non-zero $s \in S$ there are at most K elements $t \in S$ with $st \ne \theta$ and at most K elements $r \in S$ with $rs \ne \theta$. Then $\ell^1_{\mathrm{r}}(S)$ is injective and $\|R_{\ell^1_{\mathrm{r}}(S)}\| \le K$.*

Proof. We write $S = \{\theta, e_1, e_2, \ldots\}$ and suppose that $u \in \ell^1_{\mathrm{r}}(S) \otimes_\varepsilon \ell^1_{\mathrm{r}}(S)$ is of the form

$$u = \sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j. \tag{3}$$

We set $M = \max\{n : e_i e_j = e_n \text{ for some } i, j = 1, \ldots, m\}$ so that

$$R_{\ell^1_{\mathrm{r}}(S)}(u) = \sum_{i,j=1}^{m} \xi_{i,j} e_i e_j = \sum_{k=1}^{M} \Big(\sum_{e_i e_j = e_k} \xi_{i,j} \Big) e_k,$$

from which we obtain the inequality

$$\|R_{\ell^1_{\mathrm{r}}(S)}(u)\| \le \sum_{\substack{1 \le i,j \le m \\ e_i e_j \ne \theta}} |\xi_{i,j}|. \tag{4}$$

Setting

$$\lambda_{i,j} = \begin{cases} 0 & \text{if } e_i e_j = \theta, \\ |\xi_{i,j}| & \text{otherwise,} \end{cases}$$

we see that the right-hand side of (4) is the summation over the elements of the $m \times m$ matrix $\Lambda = [\lambda_{i,j}]$, a matrix which has at most K non-zero elements in each row and in each column. Such a matrix can be written as the sum of exactly K matrices with at most one non-zero element of Λ in each row and in each column (this is shown in Mirsky's book [9, Th. 11.1.6]) and so the right-hand side of (4) is the sum of exactly K sums of the form $\sum_{i=1}^{m} |\xi_{i,\sigma(i)}|$. Hence, applying Lemma 3.1, we find that

$$\|R_{\ell^1_{\mathrm{r}}(S)}(u)\| \le K \|u\|_\varepsilon$$

for all u of the form (3). The result now follows from the fact that such elements are dense in $\ell^1_{\mathrm{r}}(S) \otimes_\varepsilon \ell^1_{\mathrm{r}}(S)$. ■

Notice that Proposition 3.2 applied to the semigroup $S = \{\theta, e_1, e_2, \ldots\}$ with product

$$e_i e_j = \begin{cases} e_i & \text{if } i = j, \\ \theta & \text{otherwise,} \end{cases}$$

shows that $\ell^1_{\mathrm{r}}(S)$, which is clearly isomorphic with ℓ^1, is injective. Thus we recover the result and implicit bound described in Varopoulos [12].

We can apply Proposition 3.2 to the subject of this article by use of the following theorem.

THEOREM 3.3. *Let S be a countable semigroup with zero θ and such that $\ell^1_{\mathrm{r}}(S)$ is injective. Then $\ell^1(S)$ is injective and*

$$\|R_{\ell^1(S)}\| \le 6\|R_{\ell^1_{\mathrm{r}}(S)}\| + 1.$$

Proof. We will write $S = \{e_0, e_1, \ldots\}$, where $e_0 = \theta$, for simplicity of notation. If

$$u = \sum_{i,j=0}^{m} \xi_{i,j} e_i \otimes e_j \in \ell^1(S) \otimes_\varepsilon \ell^1(S) \tag{5}$$

then $|\sum_{i,j=0}^{m} \xi_{i,j}| \le \|u\|_\varepsilon$ and since

$$\sum_{e_ie_j=e_0} \xi_{i,j} = \sum_{k=0}^{\infty}\Big(\sum_{e_ie_j=e_k} \xi_{i,j}\Big) - \sum_{k=1}^{\infty}\Big(\sum_{e_ie_j=e_k} \xi_{i,j}\Big)$$

we find that

$$\Big|\sum_{e_ie_j=e_0} \xi_{i,j}\Big| \le \Big|\sum_{i,j=0}^{m} \xi_{i,j}\Big| + \sum_{k=1}^{\infty}\Big|\sum_{e_ie_j=e_k} \xi_{i,j}\Big| \le \|u\|_\varepsilon + \sum_{k=1}^{\infty}\Big|\sum_{e_ie_j=e_k} \xi_{i,j}\Big|,$$

which gives

$$\|R_{\ell^1(S)}(u)\|_1 = \sum_{k=0}^{\infty}\Big|\sum_{e_ie_j=e_k} \xi_{i,j}\Big| \le \|u\|_\varepsilon + 2\sum_{k=1}^{\infty}\Big|\sum_{e_ie_j=e_k} \xi_{i,j}\Big|. \tag{6}$$

Now

$$\begin{aligned}\sum_{k=1}^{\infty}\Big|\sum_{e_ie_j=e_k} \xi_{i,j}\Big| &= \Big\|R_{\ell^1_{\mathrm{r}}(S)}\Big(\sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j\Big)\Big\|_1 \\ &\le \|R_{\ell^1_{\mathrm{r}}(S)}\| \Big\|\sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j\Big\|_{\ell^1_{\mathrm{r}}(S)\otimes_\varepsilon \ell^1_{\mathrm{r}}(S)} \\ &= \|R_{\ell^1_{\mathrm{r}}(S)}\| \Big\|\sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j\Big\|_{\ell^1(S)\otimes_\varepsilon \ell^1(S)},\end{aligned} \tag{7}$$

since injective tensor products preserve subspaces [5, §4.3], and since

$$\sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j = u - \Big(\sum_{i=0}^{m} \xi_{i,0} e_i\Big) \otimes e_0 - e_0 \otimes \Big(\sum_{j=1}^{m} \xi_{0,j} e_j\Big)$$

we have

$$\begin{aligned}\Big\|\sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j\Big\| &\le \|u\|_\varepsilon + \|e_0\|_1\Big(\Big\|\sum_{i=0}^{m} \xi_{i,0} e_i\Big\|_1 + \Big\|\sum_{j=1}^{m} \xi_{0,j} e_j\Big\|_1\Big) \\ &= \|u\|_\varepsilon + \sum_{i=0}^{m} |\xi_{i,0}| + \sum_{j=1}^{m} |\xi_{0,j}| \le 3\|u\|_\varepsilon.\end{aligned} \tag{8}$$

Combining the inequalities (6)–(8) now gives the bound

$$\|R_{\ell^1(S)}(u)\|_1 \le 6\|R_{\ell^1_{\mathrm{r}}(S)}(u)\|_1 + \|u\|_1$$

for elements u of the form (5). This bound extends to the closure and so proves the theorem. ■

COROLLARY 3.4. *Let S be a countable semigroup with zero θ. Suppose that there is some $K \in \mathbb{N}$ such that for each non-zero $s \in S$ there are at most K elements $t \in S$ with $st \ne \theta$ and at most K elements $r \in S$ with $rs \ne \theta$. Then $\ell^1(S)$ is injective and $\|R_{\ell^1(S)}\| \le 6K + 1$.*

We remark that the above results do not provide a characterisation of the semigroups with zero such that $\ell^1(S)$ is injective. Consider the semigroup $S = \{\theta, e_1, e_2, \ldots\}$ with product $e_ie_j = e_i$ $(i, j \in \mathbb{N})$. Clearly, S satisfies the conclusions of Propositions 2.2 and 2.4, while not the hypotheses of Corollary 3.4.

To conclude this section we invite the reader to compare Corollary 3.4 with Proposition 2.3.

4. The weighted case and an application. Some of what is described above can be extended to cover the weighted case: if S is a semigroup with zero θ, we say that $\omega : S\backslash\{\theta\} \to (0, \infty)$ is an *algebra weight* if

$$\omega(st) \le \omega(s)\omega(t) \quad (s, t, st \in S\backslash\{\theta\}).$$

The weighted reduced semigroup algebra $\ell^1_{\mathrm{r}}(S, \omega)$ is then defined analogously to $\ell^1_{\mathrm{r}}(S)$ as the completion of $\mathbb{C}_r[S]$ with respect to the norm

$$\Big\|\sum_{s\in S\backslash\{\theta\}} \xi_s s\Big\|_\omega := \sum_{s\in S\backslash\{\theta\}} |\xi_s|\omega(s).$$

In particular, the following version of Varopoulos's Lemma holds, the proof again being an increment on that in [12].

LEMMA 4.1. *Let $S = \{\theta, e_1, e_2, \ldots\}$ be a countable semigroup with zero θ, ω an algebra weight and suppose that*

$$u = \sum_{i,j=1}^{m} \xi_{i,j} e_i \otimes e_j \in \ell^1_{\mathrm{r}}(S) \otimes_\varepsilon \ell^1_{\mathrm{r}}(S).$$

Then for any permutation σ on $\{1,\ldots,m\}$,

$$\sum_{i=1}^{m}|\xi_{i,\sigma(i)}|\omega(e_i)\omega(e_{\sigma(i)}) \le \|u\|_\varepsilon.$$

The point of passing to the weighted case is that a sufficiently rapid rate of decrease in the weight can play the role that finiteness does in the unweighted case.

Let $e_{i,j}$ denote the infinite matrix with one as the i,j-th entry and zeros elsewhere, and θ the infinite matrix of zeros. Then with the usual matrix multiplication the set

$$S := \{e_{i,j} : 1 \le i < j\} \cup \{\theta\}$$

is a semigroup with zero. Define a weight ω on $S\backslash\{\theta\}$ by

$$\omega(i,j) = 2^{-(j-i)^2} \quad (1 \le i < j).$$

To see that this is an algebra weight note that

$$\omega(i,j)\omega(j,k) = 2^{-(j-i)^2-(k-j)^2} = 2^{2(j-i)(k-j)}\omega(i,k)$$

and so, by a short calculation,

$$\omega(i,k) \le 2^{-2(j-i)(k-j)}\omega(i,j)\omega(j,k) \le 2^{-2(k-i-1)}\omega(i,j)\omega(j,k).$$

PROPOSITION 4.2. *With S and ω defined as above, the Banach algebra $A = \ell^1_{\mathrm{r}}(S,\omega)$ is injective.*

Proof. Suppose that $u \in A \otimes_\varepsilon A$ is of the form

$$u = \sum_{i<j,\,k<l} \xi_{i,j,k,l}\, e_{i,j} \otimes e_{k,l}$$

where only finitely many of the $\xi_{i,j,k,l}$ are non-zero. Then

$$R_A(u) = \sum_{i<j<l} \xi_{i,j,j,l}e_{i,l} = \sum_{m=2}^{\infty}\sum_{i<j<i+m} \xi_{i,j,j,i+m}e_{i,i+m}$$

so that

$$(9)\qquad \|R_A(u)\|_\omega = \sum_{m=2}^{\infty}\sum_{i<j<i+m}|\xi_{i,j,j,i+m}|\omega(i,i+m)$$
$$\le \sum_{m=2}^{\infty}\sum_{i<j<i+m}|\xi_{i,j,j,i+m}|2^{-2(m-1)}\omega(i,j)\omega(j,i+m)$$
$$= \sum_{m=2}^{\infty}2^{-2(m-1)}\sum_{i<j<i+m}|\xi_{i,j,j,i+m}|\omega(i,j)\omega(j,i+m).$$

Now, with m fixed, for each pair (i,j) there is exactly one pair (k,l) such that $\xi_{i,j,k,l}$ occurs in the inner sum of (9). Thus, by a suitable relabelling of the semigroup elements $e_{i,j}$, we can apply Lemma 4.1 to obtain

$$\sum_{i<j<i+m}|\xi_{i,j,j,i+m}|\omega(i,j)\omega(j,i+m) \le \|u\|_\varepsilon \quad (m = 2,3,\ldots)$$

and so, from (9),

$$R_A(u) \le \sum_{m=2}^{\infty}2^{-2(m-1)}\|u\|_\varepsilon = \frac{1}{3}\|u\|_\varepsilon.$$

The result now follows since elements of the form u (i.e. those with finite support) are dense in $A \otimes_\varepsilon A$. ■

We find the injectivity of this example to be of interest for the following reason. In [1] Aristov shows that a C^*-algebra is injective if and only if it is *subhomogeneous*, i.e. if there is some uniform bound on the dimensions of its continuous irreducible representations. It is well known that a semisimple Banach algebra is subhomogeneous if and only if it satisfies a polynomial identity [8, Prop. 6.1], so it is natural to ask whether these three properties coincide for Banach algebras more general than C^*-algebras. The fact that there are commutative semisimple Banach algebras which are not Arens regular (for example $\ell^1(\mathbb{Z})$) gives a negative answer in one direction, while the above proposition gives a partial negative answer in the opposite direction once we note that A does not satisfy a polynomial identity. If an algebra (not even necessarily normed) satisfies a polynomial identity then it satisfies a homogeneous multilinear identity of no greater degree [6, Lemma 6.2.4], so it suffices to show that A does not satisfy an identity of the form

$$p(X_1,\ldots,X_n) := X_1\ldots X_n + \sum_{\sigma\ne 1}\lambda_\sigma X_{\sigma(1)}\ldots X_{\sigma(n)}$$

where the summation is over all non-trivial permutations on $\{1,\ldots,n\}$. But this is obvious since A contains half of "Kaplansky's staircase"

$$p(e_{1,2},e_{2,3},\ldots,e_{n,n+1}) = e_{1,n+2} \ne 0.$$

We conclude by mentioning that A is a radical Banach algebra and so trivially subhomogeneous. Thus it does provide an answer to the more difficult question as to whether there is a *semisimple* injective Banach algebra that does not satisfy a polynomial identity, or equivalently is not subhomogeneous.

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162 Nottingham St.
Pitsmoor
Sheffield, UK
E-mail: jjg1@cise.npl.co.uk

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L^q-spectrum of the Bernoulli convolution associated with the golden ratio

by

KA-SING LAU (Hong Kong and Pittsburgh, Penn.) and
SZE-MAN NGAI (Ithaca, N.Y.)

Abstract. Based on a set of higher order self-similar identities for the Bernoulli convolution measure for $(\sqrt{5}-1)/2$ given by Strichartz *et al.*, we derive a formula for the L^q-spectrum, $q > 0$, of the measure. This formula is the first obtained in the case where the open set condition does not hold.

1. Introduction. Let μ be a positive bounded regular Borel measure on $\mathbb{R}^d$ with compact support. For $h > 0$ and $q > 0$, we define the L^q-(*moment*) *spectrum* of μ by

$$\tau(q) = \varliminf_{h\to 0+} \frac{\ln \sum_i \mu(Q_i(h))^q}{\ln h}, \tag{1.1}$$

where $\{Q_i(h)\}_i$ is the family of h-mesh cubes

$$[n_1h, (n_1+1)h) \times \ldots \times [n_dh, (n_d+1)h), \quad (n_1, \ldots, n_d) \in \mathbb{Z}^d.$$

We also define the (*lower*) L^q-*dimension* of μ by

$$\underline{\dim}_q(\mu) = \tau(q)/(q-1), \quad q > 1.$$

These notions were first used by Rényi [Ré] to extend the entropy dimension (corresponding to $q = 1$). Some variants of these definitions and the basic properties of $\tau(q)$ can be found in [LN1], [St]. We prefer to use $\varliminf$ rather than $\varlimsup$ because the $\tau(q)$ defined by using $\varliminf$ is concave.

Recently there are a large number of papers in the mathematics and physics literature investigating the relationship of the L^q-spectrum and the local dimension of the measures that arise from dynamical systems (the multifractal formalism) (e.g., Frisch and Parisi [FP], Halsey *et al.* [H], Collet *et al.* [CLP], Lopes [Lo], Rand [R], Cawley and Mauldin [CM], Edgar and

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