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The higher order Riesz transform for Gaussian measure need not be of weak type (1,1)

by

LILIANA FORZANI (Minneapolis, Minn.) and
ROBERTO SCOTTO (Madrid)

Abstract. The purpose of this paper is to prove that the higher order Riesz transform for Gaussian measure associated with the Ornstein–Uhlenbeck differential operator $L := d^2/dx^2 - 2xd/dx$, $x \in \mathbb{R}$, need not be of weak type $(1,1)$. A function in $L^1(d\gamma)$, where $d\gamma$ is the Gaussian measure, is given such that the distribution function of the higher order Riesz transform decays more slowly than C/λ.

Introduction. Let $f \in L^1(d\gamma)$ be given, and let α be a natural number. Then the *Riesz transform of order* α for the Gaussian measure is defined as

$$\mathcal{K}_\alpha f(y) = \text{p.v.}\int_{\mathbb{R}} k_\alpha(y,z) f(z)\,dz,$$

where

$$k_\alpha(y,z) = \int_0^1 \varphi_\alpha(r) h_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)\frac{e^{-(z-ry)^2/(1-r^2)}}{(1-r^2)^{3/2}}\,dr,$$

$$\varphi_\alpha(r) = C_\alpha r^{\alpha-1}\left(\frac{-\log r}{1-r^2}\right)^{(\alpha-2)/2},$$

and h_α is the Hermite polynomial of degree α.

It is known that $\mathcal{K}_\alpha$ maps $L^p(d\gamma)$ continuously into itself for $1 < p < \infty$ (see [M], [U] and [G-S-T]). Furthermore, for $\alpha = 1$, $\mathcal{K}_\alpha$ maps $L^1(d\gamma)$ into $L^{1,\infty}(d\gamma)$ (see [M]). The purpose of this paper is to show that even though for $\alpha = 2$ the operator is still of weak type $(1,1)$ the result breaks down for $\alpha > 2$. In fact, a counterexample will be shown.

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Indeed $\mathcal{K}_\alpha f(y)$ can be written as the sum of two operators, one of which satisfies the weak type inequality, while the distribution function associated with the other decays more slowly than C/λ as long as $\alpha \geq 3$; i.e.

$$\mathcal{K}_\alpha f(y) = \mathcal{K}_{\alpha,w} f(y) + \mathcal{K}_{\alpha,l} f(y),$$

where $\mathcal{K}_{\alpha,w}$ is of weak type and there exists a function $f \in L^1(d\gamma)$ such that for λ large

$$\gamma\{y \in \mathbb{R} : \mathcal{K}_{\alpha,l} f(y) > \lambda\} \geq C \frac{(\log \lambda)^{(\alpha-2-\delta)/2}}{\lambda}.$$

The paper is organized as follows. In Section 1 it is proved that $\mathcal{K}_\alpha$ need not be of weak type $(1,1)$ for $\alpha \geq 3$. In Section 2 it is proved that $\mathcal{K}_2$ maps $L^1(d\gamma)$ into $L^{1,\infty}(d\gamma)$.

1. It is enough to work with $y > 0$ and large, say $y > R$, where R is to be chosen. Define

$$N_R^y = \{z \in \mathbb{R} : |z - y| \leq R/y\}, \qquad \mathbb{R} \setminus N_R^y = \bigcup_{i=1}^5 D_i^y$$

where $D_1^y = (-\infty, 0)$, $D_2^y = [0, (1-\beta)y)$, $D_3^y = [(1-\beta)y,\, y - R/y)$, $D_4^y = (y + R/y, y/(1-\beta)]$ and $D_5^y = (y/(1-\beta), \infty)$.

Clearly

$$\mathcal{K}_\alpha f(y) = \text{p.v.} \int_{N_R^y} k_\alpha(y,z) f(z)\,dz + \sum_{i=1}^5 \int_{D_i^y} k_\alpha(y,z) f(z)\,dz$$

$$= \mathcal{K}_{\alpha,N_R} f(y) + \sum_{i=1}^5 \mathcal{K}_{\alpha,i} f(y).$$

Set $\mathcal{K}_{\alpha,w} = \mathcal{K}_{\alpha,N_R} + \mathcal{K}_{\alpha,2} + \mathcal{K}_{\alpha,3}$. In Lemma 1 it will be proved that $\mathcal{K}_{\alpha,N_R}$ is of weak type $(1,1)$; in Lemma 2 it will be proved that $\mathcal{K}_{\alpha,i}$, $i = 2, 3$, are of strong type $(1,1)$.

For $0 < \delta < 1$, let

$$f(z) = \frac{e^{z^2}}{z^{1+\delta}} \chi_{(1,\infty)}(z) \in L^1(d\gamma).$$

It is easy to see that $\mathcal{K}_{\alpha,1} f(y) = 0$ for all $y > R$. In Lemma 3 it will be shown that $\mathcal{K}_{\alpha,4} f(y)$ is bounded below by $C y^{\alpha-1-\delta} e^{y^2}$. The same procedure shows that $\mathcal{K}_{\alpha,5} f(y) \geq 0$.

On the other hand, the distribution function of $y^k e^{y^2}$ for $y > R$, i.e., $E_k(\lambda) = \gamma\{y > R : y^k e^{y^2} > \lambda\} \geq C(\log \lambda)^{(k-1)/2}/\lambda$, $\lambda > 0$, which decays more slowly than C/λ for $k > 1$ and λ large.

Thus for $\alpha \geq 3$, if $\mathcal{K}_\alpha$ were a weak type $(1,1)$ operator then $\mathcal{K}_{\alpha,1} + \mathcal{K}_{\alpha,4} + \mathcal{K}_{\alpha,5}$ would be as well. And this is a contradiction with the above statement since $\alpha - 1 - \delta > 1$.

Let $f \in L^1(d\gamma)$ be given, and $\mathcal{H}$ be a *Calderón–Zygmund kernel*, that is,

(i) $\mathcal{H} \in C^1(\mathbb{R} - \{0\})$,
(ii) $|\widehat{\mathcal{H}}(\xi)| \leq C$,
(iii) $|d\mathcal{H}/dy| \leq C/|y|^2$.

We define

$$Tf(y) = \text{p.v.} \int_{N_R^y} \mathcal{H}(y - z) f(z)\,dz.$$

This operator turns out to be of weak type $(1,1)$ with respect to the Gaussian measure. This result is standard; a proof can be found in [M], [Sj], [U], or [F-G-S].

LEMMA 1. *$\mathcal{K}_{\alpha,N_R}$ satisfies the weak type $(1,1)$ inequality with respect to the Gaussian measure.*

Proof. The procedure is to split $k_\alpha(y,z)$ into the sum of two kernels

$$k_\alpha(y,z) = \mathcal{H}_\alpha(y - z) + k_\alpha^1(y,z)$$

where

$$\mathcal{H}_\alpha(x) = \int_0^1 \varphi_\alpha(r) h_\alpha\Big(\frac{x}{(1-r^2)^{1/2}}\Big) \frac{e^{-x^2/(1-r^2)}}{(1-r^2)^{3/2}}\,dr.$$

The weak type $(1,1)$ inequality follows once it is proved that $\mathcal{H}_\alpha$ is a Calderón–Zygmund kernel, and the operator $\int_{N_R^y} k_\alpha^1(y,z) f(z)\,dz$ is bounded on $L^1(d\gamma)$.

We have

$$\widehat{\mathcal{H}}_\alpha(\xi) = \int_{-\infty}^{\infty} e^{i\xi z} \mathcal{H}_\alpha(z)\,dz$$

$$= \int_0^1 \frac{\varphi_\alpha(r)}{1-r^2} \int_{-\infty}^{\infty} e^{i\sqrt{1-r^2}\xi z} h_\alpha(z) e^{-z^2}\,dz\,dr$$

$$= \int_0^1 \frac{\varphi_\alpha(r)}{1-r^2} (-1)^\alpha \int_{-\infty}^{\infty} e^{i\sqrt{1-r^2}\xi z} \frac{d^\alpha}{dz^\alpha}(e^{-z^2})\,dz\,dr.$$

After an integration by parts,

$$\widehat{\mathcal{H}}_\alpha(\xi) = (i\xi)^\alpha \int_0^1 \frac{\varphi_\alpha(r)}{1-r^2} (1-r^2)^{\alpha/2} (\widehat{e^{-|z|^2}})(\sqrt{1-r^2}\,\xi)\,dr$$

$$= (i\xi)^\alpha \int_0^1 \varphi_\alpha(r)(1-r^2)^{(\alpha-2)/2} e^{-(1-r^2)\xi^2/4}\,dr.$$

Thus

$$|\widehat{\mathcal{H}}_\alpha(\xi)| \le C\int_0^1 |\xi|^\alpha (1-r)^{(\alpha-2)/2} e^{-(1-r)\xi^2/c}\,dr \le C\int_0^1 \frac{|\xi|}{(1-r)^{1/2}} e^{-(1-r)\xi^2/c}\,dr \le C,$$

and

$$\frac{d}{dx}\mathcal{H}_\alpha(x) = \int_0^1 \varphi_\alpha(r)\bigg[\alpha(1-r^2)^{1/2} h_{\alpha-1}\bigg(\frac{x}{(1-r^2)^{1/2}}\bigg) + 2xh_\alpha\bigg(\frac{x}{(1-r^2)^{1/2}}\bigg)\bigg]\frac{e^{-x^2/(1-r^2)}}{(1-r^2)^{5/2}}\,dr.$$

Using the fact that any Hermite polynomial h_α can be bounded in absolute value by $C\sum_{1\le j\le\alpha}|x|^j$, we get

$$\bigg|\frac{d}{dx}\mathcal{H}_\alpha(x)\bigg| \le C\bigg(\sum_{j=1}^{\alpha-1}|x|^j\int_0^1 \frac{e^{-|x|^2/(1-r^2)}}{(1-r)^{(j+4)/2}}\,dr + \sum_{j=1}^{\alpha}|x|^{j+1}\int_0^1 \frac{e^{-|x|^2/(1-r^2)}}{(1-r)^{(j+5)/2}}\,dr\bigg) \le \frac{C}{|x|^2}.$$

Now define

$$\psi(t) = h_\alpha\bigg(\frac{z-ty}{(1-r^2)^{1/2}}\bigg),\qquad \theta(t) = e^{-|ty-z|^2/(1-r^2)},\qquad 0\le r\le t\le 1.$$

Then

$$|\psi'(t)| \le C\frac{|y|}{(1-r)^{1/2}}\sum_{0\le j\le\alpha-1}\ \sum_{0\le k\le j}\bigg(\frac{|z-y|}{(1-r)^{1/2}}\bigg)^k ((1-r)^{1/2}|y|)^{j-k},$$

$$|\theta'(t)| \le C\frac{|y|}{(1-r)^{1/2}} e^{-c|y-z|^2/(1-r)}\qquad \text{on } N_R^y.$$

Thus

$$\begin{aligned}|k_\alpha^1(y,z)| &\le \bigg|\int_0^1 \frac{\varphi_\alpha(r)}{(1-r^2)^{3/2}}\bigg[h_\alpha\bigg(\frac{z-ry}{(1-r^2)^{1/2}}\bigg)e^{-|ry-z|^2/(1-r^2)} - h_\alpha\bigg(\frac{z-y}{(1-r^2)^{1/2}}\bigg)e^{|y-z|^2/(1-r^2)}\bigg]\,dr\bigg|\\ &\le C\int_0^1 \frac{1}{(1-r^2)^{3/2}}|\psi(r)-\psi(1)|e^{-|ry-z|^2/(1-r^2)}\,dr\\ &\quad + C\int_0^1 \frac{1}{(1-r^2)^{3/2}}\bigg|h_\alpha\bigg(\frac{z-y}{(1-r^2)^{1/2}}\bigg)\bigg||\theta(r)-\theta(1)|\,dr\\ &= k_\alpha^{1,1}(y,z) + k_\alpha^{1,2}(y,z).\end{aligned}$$

On N_R^y the following estimates for $k_\alpha^{1,1}$ and $k_\alpha^{1,2}$ hold:

$$\begin{aligned}|k_\alpha^{1,1}(y,z)| &\le C\int_0^1 \frac{|y|}{1-r}\sum_{\substack{j,k\\ 0\le j\le\alpha-1\\ j-k\ge 0}}\bigg(\frac{|z-y|}{(1-r)^{1/2}}\bigg)^k e^{-|y-z|^2/(c(1-r))}\\ &\quad\times ((1-r)^{1/2}|y|)^{j-k}e^{-(1-r)|y|^2/c}\,dr\\ &\le C|y|\int_0^1 \frac{e^{-c|y-z|^2/(1-r)}}{1-r}\,dr \le C(1+|y|)\log\frac{C}{|y|\cdot|y-z|}\end{aligned}$$

and

$$\begin{aligned}|k_\alpha^{1,2}(y,z)| &\le C\sum_{j=0}^{\alpha}\int_0^1 \frac{|z-y|^j}{(1-r)^{(j+3)/2}}|y|(1-r)^{1/2}e^{-c|y-z|^2/(1-r)}\,dr\\ &\le C|y|\sum_{j=0}^{\alpha}|z-y|^j\int_0^1 \frac{e^{-|y-z|^2/(c(1-r))}}{(1-r)^{(j+2)/2}}\,dr\\ &\le C(1+|y|)\log\frac{C}{|y|\cdot|y-z|}.\end{aligned}$$

Therefore on N_R^y we have $|k_\alpha^1(y,z)| \le C(1+|y|)\log(C/(|y|\cdot|y-z|))$ and hence $\int_{N_R^y} k_\alpha^1(y,z)f(z)\,dz$ is of strong type $(1,1)$ with respect to the Gaussian measure.

LEMMA 2. *The operators $\mathcal{K}_{\alpha,2}$ and $\mathcal{K}_{\alpha,3}$ are continuous on $L^1(d\gamma)$.*

Proof. First of all we prove the strong type $(1,1)$ for $\mathcal{K}_{\alpha,2}$. Since

$$\bigg|h_\alpha\bigg(\frac{z-ry}{(1-r^2)^{1/2}}\bigg)e^{-(z-ry)^2/(1-r^2)}\bigg| \le Ce^{-c(z-ry)^2/(1-r^2)},$$

it follows that

$$|k_{\alpha,2}(y,z)| \le C\int_0^1 \frac{e^{-c(z-ry)^2/(1-r^2)}}{(1-r^2)^{3/2}}\,dr.$$

This last integral is bounded by a constant as was proved in [Sc]. From these estimates the strong type $(1,1)$ of $\mathcal{K}_{\alpha,2}$ follows.

Let us prove the strong type $(1,1)$ for $\mathcal{K}_{\alpha,3}$.

If $z \in D_3^y$ then $(1-\beta)y \le z < y - R/y$. We split the integral in r into the sum of integrals over

$$E_1 = \left[0, 1 - \frac{1-\beta}{\beta}\cdot\frac{y-z}{y}\right],$$
$$E_2 = \left[1 - \frac{1-\beta}{\beta}\cdot\frac{y-z}{y}, 1-\beta\frac{y-z}{y}\right] \quad \text{and}$$
$$E_3 = \left[1-\beta\frac{y-z}{y}, 1\right].$$

Then

$$k_{\alpha,3}(y,z) = \sum_{i=1}^{3}\int_{E_i}\varphi_\alpha(r)h_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)\frac{e^{-(z-ry)^2/(1-r^2)}}{(1-r^2)^{3/2}}\,dr$$
$$= \sum_{i=1}^{3} k^i_{\alpha,3}(y,z).$$

On E_1 we use

$$\left(\frac{1-3\beta}{1-\beta}\right)(1-r)y < z - ry < \left(\frac{1+\beta}{1-\beta}\right)(1-r)y$$

to get

$$|k^1_{\alpha,3}(y,z)| \le C\sum_{j=0}^{\alpha}\int_0^{1-\frac{1-\beta}{\beta}\cdot\frac{y-z}{y}} y^j(1-r)^{(j-3)/2}e^{-c(1-r)y^2}\,dr$$
$$\le C\sum_{j=0}^{\alpha} y \int_{\frac{1-\beta}{2\beta}(y-z)y}^{y^2} e^{-s}s^{(j-3)/2}\,ds \le Cy,$$

since $(y-z)y > C$.

On E_3, since $0 < 1-r < \beta(y-z)/y$, we have $(1-\beta)(y-z) < ry - z < y - z$ and

$$|k^3_{\alpha,3}(y,z)| \le C\sum_{j=0}^{\alpha}\int_{1-\beta(y-z)/y}^{1}\frac{(y-z)^j}{(1-r)^{(j+3)/2}}e^{-c(y-z)^2/(1-r)}\,dr$$
$$\le \frac{C}{y-z} \le Cy.$$

On E_2 we use the fact that $1-r \sim (y-z)/y$ and we write $e^{-(z-ry)^2/(1-r^2)} = e^{y^2}e^{-(rz-y)^2/(1-r^2)}e^{-z^2}$, therefore

$$|k^2_{\alpha,3}(y,z)| \le Ce^{y^2}\sum_{j=0}^{\alpha}\left(\frac{y}{y-z}\right)^{(j+3)/2}(y-z)^j\int_{E_2} e^{-(y-rz)^2/(1-r^2)}\,dr\, e^{-z^2}$$
$$\le Ce^{y^2}\sum_{j=0}^{\alpha}(y(y-z))^{(j+2)/2}e^{-cy(y-z)}\frac{1}{y^{1/2}(y-z)^{3/2}}e^{-z^2}$$
$$\le Ce^{y^2}\frac{1}{y^{1/2}(y-z)^{3/2}}e^{-z^2}.$$

From these estimates, it is easy to see that $\mathcal{K}^i_{\alpha,3}$ for $i = 1,2,3$ are of strong type $(1,1)$ with respect to the Gaussian measure.

LEMMA 3. *For* $f(z) = (e^{z^2}/z^{1+\delta})\chi_{(1,\infty)}(z)$ *we have*

$$\gamma\{y > R : \mathcal{K}_{\alpha,4}f(y) > \lambda\} \ge \frac{C}{\lambda}(\log\lambda)^{(\alpha-1-\delta)/2}.$$

We will prove that for $y > R$, $\mathcal{K}_{\alpha,4}f(y)$ can be bounded below by $y^{\alpha-1-\delta}e^{y^2}$. Hence

$$\gamma\{y > R : \mathcal{K}_{\alpha,4}f(y) > \lambda\} \ge \gamma\{y > R : y^{\alpha-1-\delta}e^{y^2} > C\lambda\}$$
$$\ge \frac{C}{\lambda}(\log\lambda)^{(\alpha-2-\delta)/2}.$$

Thus if we take $\alpha - 2 - \delta > 0$, i.e. $\alpha > 2+\delta$, then the operator $\mathcal{K}_{\alpha,4}$ is not of weak type $(1,1)$.

If $z \in D_4^y$ then $y + R/y < z \le y/(1-\beta)$ and

$$\frac{z-ry}{(1-r^2)^{1/2}} \ge C((z-y)y)^{1/2} \ge CR^{1/2}.$$

Pick R large enough so that $C\sqrt{R} \gg \max(\text{roots of } h_\alpha)$, where h_α is the Hermite polynomial of order α. Thus

$$h_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right) \ge C_\alpha\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)^\alpha.$$

Therefore

$$\mathcal{K}_{\alpha,4}f(y) \ge C_\alpha\int_{1-2(z-y)/z}^{1-(z-y)/(2z)}\varphi_\alpha(r)\left(\frac{z-ry}{(1-r^2)^{1/2}}\right)^\alpha\frac{e^{-(z-ry)^2/(1-r^2)}}{(1-r^2)^{3/2}}\,dr$$
$$\ge C_\alpha(\beta)\int_{1-2(z-y)/z}^{1-(z-y)/(2z)}\frac{(z-ry)^\alpha}{(1-r)^{(\alpha+3)/2}}\cdot\frac{e^{-(rz-y)^2/(1-r^2)}}{(1-r^2)^{3/2}}\,dr\,e^{y^2-z^2}.$$

The second inequality is true since

$$1 - 2\frac{z-y}{z} \ge 1 - \frac{2\beta}{1-\beta} = \frac{1-3\beta}{1-\beta}$$

and $\varphi_\alpha(r)$ is bounded below on that interval by a positive constant.

We now use the fact that $1-r \sim (z-y)/z$, $z(z-y) \geq (1-r)y$, $z(z-y) \geq R$ and $z \sim y$ to get

$$k_{\alpha,4}(y,z) \geq C\Big(\int_{1-2(z-y)/z}^{1-(z-y)/(2z)} (1-r)^{(\alpha-3)/2} e^{-cz(rz-y)^2/(z-y)}\,dr\Big) y^\alpha e^{y^2-z^2}$$
$$\geq C\Big(\int_{1-2(z-y)/z}^{1-(z-y)/2z} e^{-cz(rz-y)^2/(z-y)}\,dr\Big)\Big(\frac{z-y}{z}\Big)^{(\alpha-3)/2} y^\alpha e^{y^2-z^2}$$
$$\geq C\Big(\int_{-c\sqrt{z(z-y)}}^{c\sqrt{z(z-y)}} e^{-u^2}\,du\Big)\frac{(z-y)^{1/2}}{z^{3/2}}\Big(\frac{z-y}{z}\Big)^{(\alpha-3)/2} y^\alpha e^{y^2-z^2}$$
$$\geq C\Big(\int_{-c\sqrt{z(z-y)}}^{c\sqrt{z(z-y)}} e^{-u^2}\,du\Big)(y(z-y))^{(\alpha-2)/2} z e^{y^2-z^2}$$
$$\geq C(y(z-y))^{(\alpha-2)/2} z e^{y^2-z^2}.$$

Therefore

$$\mathcal{K}_{\alpha,4} f(y) \geq \int_{y+R/y}^{y/(1-\beta)} k_{\alpha,4}(y,z) f(z)\,dz$$
$$\geq C y^{(\alpha-2)/2} e^{y^2} \int_{y+R/y}^{y/(1-\beta)} (z-y)^{(\alpha-2)/2} z e^{-z^2} \frac{e^{z^2}}{z^{1+\delta}}\,dz$$
$$\geq C y^{((\alpha-2)/2)-\delta} e^{y^2} \int_{y+R/y}^{y/(1-\beta)} (z-y)^{(\alpha-2)/2}\,dz$$
$$\geq C y^{((\alpha-2)/2)-\delta} e^{y^2} y^{\alpha/2} = C y^{\alpha-1-\delta} e^{y^2}.$$

2. In this section the weak type $(1,1)$ inequality for $\mathcal{K}_2$ will be proved. Due to the symmetry of $k_2(y,z)$, it is enough to work with $y>0$. N_R^y is now defined as $\{z \in \mathbb{R} : |z-y| < R \wedge R^2/y\}$. The split of $\mathcal{K}_2 f(y)$ into the sum of six operators is carried out in the same fashion as before. Just a few modifications are necessary in the definition of D_i^y's:

$$D_1^y = (-\infty, 0 \wedge (y-R)], \quad D_2^y = (0, (1-\beta)y \wedge (y-R^2/y)],$$
$$D_3^y = ((1-\beta)y, (y-R^2/y)], \quad D_4^y = [(y+R^2/y, y/(1-\beta)),$$
$$D_5^y = [((y+R) \wedge (y+R^2/y) \wedge y/(1-\beta), \infty).$$

These regions are the ones into which $\mathbb{R} \setminus N_R^y$ is divided and they are essentially the same as Muckenhoupt's in [M]. So the estimates on $k_2(y,z)$ on $\mathbb{R} \setminus N_R^y$ are quite the same. Once the right estimates for $k_2(y,z)$ are obtained, the weak type $(1,1)$ of $\mathcal{K}_2$ follows by means of standard procedures thoroughly detailed in [M].

$\mathcal{K}_{\alpha,N_R}$ and $\mathcal{K}_{\alpha,i}$, $i=2,3$, were already proved to be operators of weak type $(1,1)$ for all α. Moreover, the equivalence between $k_{\alpha,4}(y,z)$ and the kernel $y(z-y))^{(\alpha-2)/2} z e^{y^2-z^2}$ follows as well since the inequalities obtained there can be reversed.

Thus if $\alpha = 2$, then $k_{2,4}(y,z) \leq C y e^{y^2-z^2}$ on D_4^y. It remains to prove that $k_{2,1}(y,z) \leq C e^{-z^2}$ on D_1^y, and $k_{2,5}(y,z) \leq C(1 \vee y) e^{y^2-z^2}$ on D_5^y.

On D_1^y, $y.z \leq 0$, $y-z > R$, and $y-z \geq y$. We split $k_{2,1}(y,z)$ into the sum of two integrals: $k_{2,1}^1(y,z)$ over $[0,3/4]$, and $k_{2,1}^2(y,z)$ over $[3/4,1]$. Thus

$$k_{2,1}^1(y,z) \leq C\Big[1 + \int_0^{3/4} r(ry)^2 e^{-(ry)^2}\,dr + \int_0^{3/4} rz^2 e^{-(rz)^2}\,dr\Big] e^{-z^2} \leq C e^{-z^2},$$

and

$$k_{2,1}^2(y,z) \leq C \int_0^{1/4} \Big[1 + \frac{(z-y+ry)^2}{(2-r)r}\Big] e^{-\frac{(z-y)^2}{(2-r)r}} e^{\frac{2y(y-z)}{(2-r)r}} \frac{1-r}{((2-r)r)^{3/2}}\,dr$$
$$\leq C(y-z)^2 \Big(\int_0^{1/4} e^{-\frac{(z-y)^2}{(2-r)r}} \frac{1-r}{((2-r)r)^{3/2}}\,dr\Big) e^{\frac{8}{7}y(y-z)}.$$

If $u = (z-y)/((2-r)r)^{1/2}$, then

$$k_{2,1}^2(y,z) \leq \frac{C}{y-z}\Big(y-z+\frac{1}{y-z}\Big)^{-\frac{16}{7}(z-y)^2} e^{\frac{8}{7}y(y-z)} \leq C e^{-z^2}.$$

On D_5^y, $z-y > \beta z$, and $z-y > CR$. We split

$$k_{2,5}(y,z) = C\Big(\int_0^1 r\Big(4\frac{(ry-z)^2}{1-r^2} - 2\Big)\frac{e^{-(y-rz)^2/(1-r^2)}}{(1-r^2)^{3/2}}\,dr\Big) e^{y^2-z^2}$$

into the sum of two integrals over $[0, 1-(z-y)/(2z))$ and $[1-(z-y)/(2z), 1]$. On the first interval $1-r > 1-\beta/2$. Replacing $1-r$ by the appropriate bound, and using the fact that $r(z-ry) = |z-ry| r(z-ry) \leq z(|rz-y| + (1-r^2)y)$, we bound this part by

$$C\Big(\int_0^1 z(1\vee y) e^{-(rz-y)^2/C}\,dr\Big) e^{y^2-z^2} \leq C(1 \vee y) e^{y^2-z^2}.$$

On the second interval $(z-ry)^2 \leq C(z-y)^2$ and $|y-rz| \geq \frac{1}{2}(z-y)$. By an appropriate substitution this part is then bounded by $Ce^{y^2-z^2}$.

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School of Mathematics
University of Minnesota
Minneapolis, Minnesota 55455
U.S.A.
E-mail: forzani@intec.unl.edu.ar

Departamento de Matemáticas
Universidad Autónoma de Madrid
Ciudad Universitaria de Canto Blanco
28049 Madrid, Spain

Current adress:
Departamento de Matemáticas
Facultad de Ciencias Exactas
Universidad Nacional Salta
Bs. As. 177
4400 Salta, Argentina
E-mail: scotto@ciunsa.unsa.edu.ar

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Injective semigroup-algebras

by

J. J. GREEN (Sheffield)

Abstract. Semigroups S for which the Banach algebra $\ell^1(S)$ is injective are investigated and an application to the work of O. Yu. Aristov is described.

1. Introduction. Injective Banach algebras were introduced by Varopoulos in [12] and have continued to attract investigation some 25 years later. In this note we make some progress towards a structural description of the semigroups S for which the Banach algebra $\ell^1(S)$ is injective.

To introduce our notation suppose that A and B are Banach algebras. We write $A \otimes B$ for the algebraic tensor product over $\mathbb{C}$, and $A \otimes_\varepsilon B$ for $A \otimes B$ equipped with (but not completed in) the injective tensor norm

$$\Big\| \sum_{i=1}^{n} a_i \otimes b_i \Big\|_\varepsilon := \sup \Big\{ \Big| \sum_{i=1}^{n} f(a_i) g(b_i) \Big| : f \in A_1^*,\ g \in B_1^* \Big\}.$$

Tensor products and tensor norms are given a detailed treatment in [5], while [3, §42] provides an introduction. Following Varopoulos we will say that a Banach algebra A is *injective* if the mapping

$$R_A : A \otimes_\varepsilon B \to A, \qquad \sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i b_i,$$

often called the *product morphism*, is bounded.

If S is a semigroup we write $\mathbb{C}[S]$ for the algebra of formal sums

$$x = \sum_{s \in S} \xi_s s \qquad (1)$$

for which only finitely many of the $\xi_s \in \mathbb{C}$ are non-zero. When equipped with the ℓ^1 norm

$$\|x\|_1 := \sum_{s \in S} |\xi_s|,$$

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