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Multipliers of Hardy spaces, quadratic integrals and Foiaş–Williams–Peller operators

by

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Abstract. We obtain a sufficient condition on a $B(H)$-valued function φ for the operator $f \mapsto \Gamma_\varphi f'(S)$ to be completely bounded on $H^\infty B(H)$; the Foiaş–Williams–Peller operator

$$R_\varphi = \begin{bmatrix} S^{\mathrm{t}} & \Gamma_\varphi \\ 0 & S \end{bmatrix}$$

is then similar to a contraction. We show that if $f : D \to B(H)$ is a bounded analytic function for which $(1-r)\|f'(re^{i\theta})\|^2_{B(H)} r\,dr\,d\theta$ and $(1-r)\|f''(re^{i\theta})\|_{B(H)} r\,dr\,d\theta$ are Carleson measures, then f multiplies $(H^1c^1)'$ to itself. Such f form an algebra $\mathcal{A}$, and when $\varphi' \in \mathrm{BMO}(B(H))$, the map $f \mapsto \Gamma_\varphi f'(S)$ is bounded $\mathcal{A} \to B(H^2(H), L^2(H) \ominus H^2(H))$. Thus we construct a functional calculus for operators of Foiaş–Williams–Peller type.

1. Introduction. Much work has been done to characterize those bounded linear operators T on Hilbert space H which are similar to contractions; that is, $T = SCS^{-1}$, where S is invertible and $\|C\|_{B(H)} \le 1$. The results of von Neumann [13, p. 3], Paulsen [10] and Pisier [14] may be summarized in the following:

THEOREM 1.1. *An operator T is similar to a contraction if and only if it is completely polynomially bounded, i.e. there is $C_T < \infty$ with*

$$\|[p_{jk}(T)]\|_{B(H)\check{\otimes} M_n} \le C_T \sup\{\|[p_{jk}(z)]\|_{M_n} \mid |z| \le 1\} \tag{1.1}$$

for all polynomials $[p_{jk}(z)]$ with $n \times n$ matrix coefficients, and all $n \ge 1$.

Further, it is not sufficient that T be polynomially bounded, where (1.1) *holds merely for all scalar-valued polynomials.*

An important test case in achieving this result was the operator consid-

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ered by Foiaş and Williams in [7], Peller [11] and others [3, 5, 14], namely

$$R_\varphi = \begin{bmatrix} S^t & \Gamma_\varphi \\ 0 & S \end{bmatrix}, \tag{1.2}$$

where $S : H^2 \to H^2$ is the right shift operator on Hardy space, with transpose S^t, and $\Gamma_\varphi : H^2 \to L^2 \ominus H^2$ is the Hankel operator represented by the matrix $[\widehat{\varphi}(-(m+n))]_{m,n\ge 0}$ with respect to the bases $(z^n)_{n\ge 0}, (\bar{z}^n)_{n>0}$. One can show that R_φ is bounded if and only if the symbol φ has $\|\varphi\|_{L^\infty/H^\infty} < \infty$, or equivalently, if $\sum_{n=0}^\infty \widehat{\varphi}(-n)\bar{z}^n$ is in BMO. Further, Aleksandrov and Peller have used multipliers of $(H^1)'$ to show that the functional calculus map $f \mapsto f(R_\varphi)$ is (completely) bounded $H^\infty \to B(L^2)$ if and only if $\varphi'(e^{i\theta})$ is in BMO; consequently, R_φ is similar to a contraction if and only if it is polynomially bounded [5, Theorem 4.8]. See also [13, Chapter 6].

Here we generalize some of these results to the case in which R_φ has an operator-valued symbol function; the example of [14] mentioned in Theorem 1.1 involves such a φ. We shall state our main Theorems 1.2 and 1.3 after introducing notation. Their proofs are in Sections 3 and 5 respectively. Sections 2 and 4 are concerned with closely related results on square functions and multipliers for $(H^1c^1)'$.

Henceforth we let φ be a strongly measurable $L^2(d\theta; B(H))$ function on the unit circle $\mathcal{T}$ with $\widehat{\varphi}(0) = 0$, extended to define a harmonic function on the disc $\varphi(z) = \int_{\mathcal{T}} P_z(\theta)\varphi(e^{i\theta})\,d\theta/(2\pi)$ by the Poisson kernel. For any Banach space X, we denote by $H^p(X)$ or H^pX the Hardy space of analytic functions $f : D \to X$ with norm $\|f\|_{H^p(X)} = \|f(re^{i\theta})\|_{L^\infty_r L^p_\theta X}$. The space $\mathrm{BMO}(X)$ consists of $L^2(X)$ functions ψ on the circle for which the norm

$$\|\psi\|_{\mathrm{BMO}(X)} = \|\widehat{\psi}(0)\|_X + \sup_{z\in D} \int_{\mathcal{T}} \|\psi(e^{i\theta}) - P_z\psi\|_X P_z(\theta)\,\frac{d\theta}{2\pi}$$

is finite. We write $\overline{\partial} = \partial/\partial\bar{z}$ and $\partial = \partial/\partial z$; and $A(dz)$ is planar Lebesgue measure. C is a positive constant taking possibly different values in successive expressions. We denote by c^1 the space of trace-class operators on H, and by c^2 the Hilbert–Schmidt ideal. The dual of c^1 is $B(H)$ under the bilinear pairing $\langle a, b\rangle = \mathrm{trace}(ab)$.

A positive Radon measure μ on the unit disc D is said to be a *Carleson measure* if there is a constant $C_*(\mu)$ such that $\mu(R(I)) \le C_*(\mu)|I|$ for each subinterval I of $[0, 2\pi]$, where $R(I)$ is the sector $R(I) = \{re^{i\theta} \in D \mid r \ge 1 - |I|,\ \theta \in I\}$ based upon I [8, p. 31].

THEOREM 1.2. *Let $\varphi \in L^2(d\theta; B(H))$ be as above, and suppose that*

$$Q_{\overline{\partial}\varphi}(dr d\theta) = (1-r)\|\overline{\partial}^2\varphi(re^{i\theta})\|^2_{B(H)}\, r\,dr\,d\theta \tag{1.3}$$

defines a Carleson measure on D. Then

(i) *R_φ is similar to a contraction on*

$$L^2(H) = (L^2(H) \ominus H^2(H)) \oplus H^2(H).$$

(ii) *The operator $W_\varphi : H^\infty B(H) \to B(L^2(H))$ is bounded, where*

$$W_\varphi : f \mapsto \begin{bmatrix} f(S^t) & \Gamma_\varphi f'(S) \\ 0 & f(S) \end{bmatrix} \quad (f \in H^\infty B(H)). \tag{1.4}$$

(iii) *Suppose further that $\frac{d}{d\theta}\varphi(e^{i\theta})$ is in $\mathrm{BMO}(B(H))$. Then W_φ extends to define a bounded linear operator $L^\infty B(H) \to B(L^2(H))$.*

When φ is scalar-valued and anti-analytic, $Q_{\overline{\partial}\varphi}$ is a Carleson measure if and only if $d\varphi/d\theta$ is in BMO [8, p. 240]. Thus Theorem 1.2 generalizes the sufficient condition for complete polynomial boundedness of R_φ from [3, Theorem 1; 11, p. 202].

For a $B(H)$-valued analytic function f, we introduce measures on D by

$$Q_f(dr d\theta) = (1-r)\|f'(re^{i\theta})\|^2_{B(H)}\, r\,dr\,d\theta, \tag{1.5}$$

$$\Delta_f(dr d\theta) = (1-r)\|f''(re^{i\theta})\|_{B(H)}\, r\,dr\,d\theta. \tag{1.6}$$

THEOREM 1.3. *Let $\mathcal{A}$ be the space of bounded analytic functions $f : D \to B(H)$ for which the norm*

$$\|f\|_{\mathcal{A}} = \|f\|_{H^\infty B(H)} + C_*(Q_f)^{1/2} + C_*(\Delta_f)/2 \tag{1.7}$$

is finite. Then W_φ defines a functional calculus map in $\mathcal{A}$:

(i) *$\mathcal{A}$ is a Banach algebra under pointwise multiplication;*

(ii) *$W_\varphi : \mathcal{A} \to B(L^2(H))$ is a bounded linear operator for each φ with $d\varphi/d\theta \in \mathrm{BMO}(B(H))$; and*

(iii) *W_φ is a homomorphism on the subalgebra $\mathcal{A}_\varphi$ of $\mathcal{A}$ given by*

$$\mathcal{A}_\varphi = \{f \in \mathcal{A} \mid f(z)\varphi(z) = \varphi(z)f(z),\ z \in D\}.$$

2. Square functions. For $e^{i\phi}$ on the unit circle, Ω_ϕ is the non-tangential approach region $\{z \in D \mid |z - e^{i\phi}| < 2(1 - |z|)\}$. For any Banach space X, and $1 \le p < \infty$, we let $G^p(X)$ be the Banach space of analytic functions $g : D \to X$ for which the norm

$$\|g\|_{G^p(X)} = \|g(0)\|_X + \bigg\{\int_{\mathcal{T}} \Big(\iint_{\Omega_\phi} \|g'(re^{i\theta})\|^2_X\, r\,dr\,d\theta\Big)^{p/2} \frac{d\phi}{2\pi}\bigg\}^{1/p} \tag{2.1}$$

is finite. When X is a Hilbert space, the norm of $G^p(X)$ is equivalent to the norm of $H^p(X)$ for $1 \le p < \infty$, by a theorem of Littlewood and Paley. This equivalence does not hold when X is the trace-class operators; nevertheless, due to the isomorphism $c^1 \sim H \widehat{\otimes} H$, we can apply the following principle of derivations for projective tensor products; cf. [1; 5, Remark 4.11].

PROPOSITION 2.1. *Let X and Y be Banach spaces for which:*

(i) *the multiplication map* $H^2(X)\widehat{\otimes} H^2(Y) \to H^1(X\widehat{\otimes} Y)$ *is surjective; and*

(ii) *the formal inclusions* $H^2(X) \to G^2(X)$ *and* $H^2(Y) \to G^2(Y)$ *are bounded.*

Then the formal inclusion $H^1(X\widehat{\otimes} Y) \to G^1(X\widehat{\otimes} Y)$ *is bounded.*

PROPOSITION 2.2. *Functions from* $H^\infty B(H)$ *multiply* $(H^1c^1)'$ *into* $(G^1c^1)'$; *that is, if* $f \in H^\infty B(H)$ *and* $g \in H^1c^1$, *then* k *belongs to* G^1c^1 *where*

$$k(z) = \int_{[0,z]} f(w)g'(w)\,dw \quad (z \in D). \tag{2.2}$$

Proof. We give a detailed proof, for the same method may be used to prove Proposition 2.1. By Sarason's factorization theorem [12, p. 62], we can write $g = h_1h_2$ where $h_j \in H^2c^2$ has $\|h_j\|^2_{H^2c^2} = \|g\|_{H^1c^1}$ for $j = 1, 2$. Then $fg' = fh_1'h_2 + fh_1h_2'$, so we can estimate the area integral of k by

$$S(k)(e^{i\phi})^2 = \iint_{\Omega_\phi} \|k'(re^{i\theta})\|^2_{c^1} r\,dr\,d\theta \tag{2.3}$$

$$\le 2\iint_{\Omega_\phi} \|f(re^{i\theta})h_1'(re^{i\theta})h_2(re^{i\theta})\|^2_{c^1} r\,dr\,d\theta + 2\iint_{\Omega_\phi} \|f(re^{i\theta})h_1(re^{i\theta})h_2'(re^{i\theta})\|^2_{c^1} r\,dr\,d\theta$$

$$\le 2\sup_{z\in D}\|f(z)\|^2_{B(H)} \sup_{z\in\Omega_\phi}\|h_2(z)\|^2_{c^2} S(h_1)(e^{i\phi})^2 + \text{similar term.} \tag{2.4}$$

Hence by the Cauchy–Schwarz inequality

$$\Big(\int_T S(k)(e^{i\phi})\frac{d\phi}{2\pi}\Big)^2 \le C\|f\|^2_{H^\infty B(H)} \int_T \sup_{z\in\Omega_\phi}\|h_2(z)\|^2 \frac{d\phi}{2\pi} \times \int_T S(h_1)(e^{i\phi})^2\frac{d\phi}{2\pi} + \text{similar term.} \tag{2.5}$$

By the Hardy–Littlewood Maximal Theorem [8, pp. 22, 24], the nontangential maximal function of $h_2 \in H^2c^2$ is square-integrable; while the area integral $S(h_1)(e^{i\phi})^2$ is integrable, as may be seen from the Littlewood–Paley identity [8, p. 236], which is valid for c^2-valued functions. The right-hand side of (2.5) is bounded by $C\|f\|^2_{H^\infty B(H)}\|g\|^2_{H^1c^1}$.

3. Proof of Theorem 1.2. (ii) Let us consider the top-right corner of the matrix of (1.4) representing $W_\varphi(f)$. When φ takes values in $B(H)$, one can view Γ_φ as an operator $H^2(H) \to L^2(H)\ominus H^2(H)$. With this interpretation, it is easy to see that

$$\|\Gamma_\varphi f'(S)\|_{B(H^2(H),L^2(H)\ominus H^2(H))} = \sup\Big\{\Re\int_T\langle\varphi(e^{i\theta})f'(e^{i\theta}), h_1(e^{i\theta})h_2(e^{i\theta})\rangle\frac{d\theta}{2\pi}\ \Big|\ \|h_1\|_{H^2c^2}, \|h_2\|_{H^2c^2} \le 1\Big\}$$

$$= \sup\Big\{\Re\int_T\langle\varphi(e^{i\theta})f'(e^{i\theta}), g(e^{i\theta})\rangle\frac{d\theta}{2\pi}\ \Big|\ \|g\|_{H^1c^1} \le 1\Big\}. \tag{3.1}$$

We can write $f'g = (fg)' - k'$, where $fg \in H^1c^1$ for $f \in H^\infty B(H)$ and $g \in H^1c^1$. As we shall see below, $k' = fg'$ need not be the derivative of an H^1c^1 function; nevertheless we can use the factorization $g = h_1h_2$, where $h_j \in H^2c^2$ have $\|h_j\|^2_{H^2c^2} = \|g\|_{H^1c^1}$, to write $k' = fh_1'h_2 + fh_1h_2'$. Integrating by parts and using the Littlewood–Paley identity, we see that

$$\int_T\langle\varphi(e^{i\theta}), k'(e^{i\theta})\rangle\frac{d\theta}{2\pi} = \int_T\Big\langle\frac{d}{d\theta}(ie^{-i\theta}\varphi(e^{i\theta})), k(e^{i\theta})\Big\rangle\frac{d\theta}{2\pi} \tag{3.2}$$

$$= \frac{2}{\pi}\iint_D\langle\overline{\partial}\overline{z}\overline{\partial}\overline{z}\varphi(z), f(z)h_1'(z)h_2(z) + f(z)h_1(z)h_2'(z)\rangle\log\frac{1}{|z|}A(dz). \tag{3.3}$$

We can bound a typical contribution to (3.3) by using the Cauchy–Schwarz inequality:

$$\frac{2}{\pi}\iint_D\|\overline{\partial}^2\varphi(z)\|_{B(H)}\|f(z)\|_{B(H)}\|h_1'(z)\|_{c^2}\|h_2(z)\|_{c^2}\log\frac{1}{|z|}A(dz) \tag{3.4}$$

$$\le \|f\|_{H^\infty B(H)}\Big(\frac{2}{\pi}\iint_D\|\overline{\partial}^2\varphi(z)\|^2_{B(H)}\|h_2(z)\|^2_{c^2}\log\frac{1}{|z|}A(dz)\Big)^{1/2} \times\Big(\frac{2}{\pi}\iint_D\|h_1'(z)\|^2_{c^2}\log\frac{1}{|z|}A(dz)\Big)^{1/2}.$$

The last integral may be bounded using the Littlewood–Paley identity, whereas the other integral on the right-hand side of (3.4) involves the Carleson measure $Q_{\overline{\partial}\varphi}$ of (1.3). An application of the Carleson Theorem [8, p. 33] to this factor leads to the following bound on (3.4):

$$C\|f\|_{H^\infty B(H)}C_*(Q_{\overline{\partial}\varphi})^{1/2}\|h_1\|_{H^2c^2}\|h_2\|_{H^2c^2} \le C\|f\|_{H^\infty B(H)}C_*(Q_{\overline{\partial}\varphi})^{1/2}\|g\|_{H^1c^1}. \tag{3.5}$$

A similar, but easier, argument shows that the same bound holds for $\int\langle\varphi,(fg)'\rangle$.

These give the following bound on the top-right entry of the matrix of (1.4):

$$\|\Gamma_\varphi f'(S)\|_{B(H^2(H),L^2(H)\ominus H^2(H))} \le CC_*(Q_{\bar\partial\varphi})^{1/2}\|f\|_{H^\infty B(H)}. \tag{3.6}$$

Since f is bounded, one can easily show that the other entries of the matrix (1.4) are bounded by

$$\|f(S)\|_{B(H^2(H))}, \|f(S^{\mathrm t})\|_{B(L^2(H)\ominus H^2(H))} \le \|f\|_{H^\infty B(H)}; \tag{3.7}$$

and so, combining this with (3.6), we have the desired bound

$$\|W_\varphi(f)\|_{B(L^2(H))} \le C(C_*(Q_{\bar\partial\varphi})^{1/2}+1)\|f\|_{H^\infty B(H)}. \tag{3.8}$$

(iii) For $f \in L^\infty B(H)$ we form the harmonic extension of f to the unit disc by the Poisson integral. For φ as above and such f, one can define a bounded bilinear form $T_\varphi(f)$ by $T_\varphi(f)(h_1,h_2) = (3.3)$ for each $h_1,h_2 \in H^2c^2$. Also, the map $f \mapsto T_\varphi(f)$ is bounded $L^\infty B(H) \to Bi(H^2c^2, H^2c^2)$ by the estimation which produced (3.6). There is a natural isometric isomorphism between $H_0^2(H)$ and $L^2(H)\ominus H^2(H)$ arising from the "flip" map $h(z)\mapsto h(\bar z)$ on formal power series. Thus $T_\varphi(f)$ gives rise to a bounded linear operator $\widetilde{T}_\varphi(f) : H^2(H) \to L^2(H)\ominus H^2(H)$ of norm not greater than $\|T_\varphi(f)\|_{Bi(H^2c^2,H^2c^2)}$. Here Bi denotes the bounded bilinear forms, with usual norm.

The contribution arising from $(fg)'$ requires more careful treatment, and we suppose additionally that $\varphi' \in \mathrm{BMO}(B(H))$. By the H^1-BMO duality theorem of [4, Corollary 16], we have, after integration by parts,

$$\Re\int_{\mathbb T}\langle\varphi(e^{i\theta}),(fg)'(e^{i\theta})\rangle\frac{d\theta}{2\pi} = \Re\int_{\mathbb T}\left\langle\frac{d}{d\theta}(ie^{-i\theta}\varphi(e^{i\theta})), f(e^{i\theta})g(e^{i\theta})\right\rangle\frac{d\theta}{2\pi}$$
$$\le C\left\|\frac{d}{d\theta}\varphi\right\|_{\mathrm{BMO}(B(H))}\|g\|_{H^1c^1}\|f\|_{H^\infty B(H)}. \tag{3.9}$$

The bounded linear functional on H^1c^1 associated with φ' by (3.9) may alternatively be obtained from some $\psi \in L^\infty B(H)$ by an application of Nehari's Theorem [9, p. 316]: precisely, there exists $\psi \in L^\infty B(H)$ with $\|\psi\|_{L^\infty B(H)} \le C\|\varphi'\|_{\mathrm{BMO}(B(H))}$ for which

$$V_\varphi(f)(h_1,h_2) = \int_{\mathbb T}\langle\psi(e^{i\theta}), f(e^{i\theta})h_1(e^{i\theta})h_2(e^{i\theta})\rangle\frac{d\theta}{2\pi} \tag{3.10}$$
$$= \int_{\mathbb T}\langle\varphi'(e^{i\theta}), f(e^{i\theta})h_1(e^{i\theta})h_2(e^{i\theta})\rangle\frac{d\theta}{2\pi} \quad (h_1,h_2\in H^2c^2)$$

is a bounded bilinear form on $H^2c^2\times H^2c^2$ for each $f\in H^\infty(B(H))$. The first integral also converges when $f\in L^\infty(B(H))$ and so defines a bounded bilinear form on $H^2c^2\times H^2c^2$. Thus, for each φ, (3.10) defines a bounded linear map $L^\infty B(H) \to Bi(H^2c^2, H^2c^2)$ and hence, using the flip map, a bounded linear map $\widetilde{V}_\varphi : L^\infty B(H) \to B(H^2(H), L^2(H)\ominus H^2(H))$.

One can extend W_φ to a bounded linear operator $L^\infty B(H) \to B(L^2(H))$ by using $\widetilde{T}_\varphi, \widetilde{V}_\varphi$ and the completely positive map

$$f \mapsto \sum_{n\ge0}\widehat{f}(n)S^n + \sum_{n<0}\widehat{f}(n)(S^*)^{-n}.$$

(i) The preceding estimates hold whether or not φ and f commute. With our conventions, $S^{\mathrm t}\bar z^m = \bar z^{m-1}$ for $m\ge 2$. When $\varphi(z)$ and $f(z)$ commute for all $z\in D$, one can deduce that $\Gamma_\varphi f(S) = f(S^{\mathrm t})\Gamma_\varphi$, which implies that W_φ is a homomorphism. Forming $[f_{jk}(R_\varphi)]$ for a matrix-valued polynomial $[f_{jk}(z)]\in H^\infty M_n$ amounts to having a symbol $\varphi_n = \varphi\otimes I_n$ and f of the form $I_H\otimes[f_{jk}(z)]$. In this case, the homomorphism $W_\varphi\otimes \mathrm{Id}_n : g\otimes a\mapsto W_\varphi(g)\otimes a$ is precisely the map $W_{\varphi_n} : H^\infty M_n \to B(L^2(H))\,\widetilde{\otimes}\, M_n$ and so, by (3.8), has operator norm

$$\|W_\varphi\otimes \mathrm{Id}_n\| \le C(C_*(Q_{\bar\partial\varphi})^{1/2}+1) \quad (n\ge1). \tag{3.11}$$

Hence R_φ is completely polynomially bounded and so, by Theorem 1.1, is similar to a contraction.

4. Multipliers of $(H^1c^1)'$.

A (*left*) *multiplier* of $(H^1c^1)'$ is an analytic function $f : D\to B(H)$ for which

$$k(z) = \int_{[0,z]} f(w)g'(w)\,dw \quad (z\in D) \tag{4.1}$$

belongs to H^1c^1 whenever $g\in H^1c^1$. Davidson and Paulsen have shown that there exists an $f\in H^\infty B(H)$ which is not a left multiplier of $(H^1c^1)'$ [5, Corollary 4.10]. Other properties of square functions on H^1c^1 are given in [1]. We recall the notation of (1.5) and (1.6).

PROPOSITION 4.1. *Let f belong to $H^\infty B(H)$, and suppose that Q_f and Δ_f define Carleson measures on the disc. Then f is a multiplier of $(H^1c^1)'$; and when $g\in H^1c^1$ and $k' = fg'$ with $k(0)=0$, we have*

$$\|k\|_{H^1c^1} \le C\|f\|_{\mathcal A}\|g\|_{H^1c^1}. \tag{4.2}$$

Proof. Once again we begin by taking a typical $g\in H^1c^1$ and factoring it as $g = h_1h_2$, where $h_j\in H^2c^2$ has $\|h_j\|^2_{H^2c^2} = \|g\|_{H^1c^1}$ for $j=1,2$. We see, on integrating by parts, that $k(z) = f(z)g(z) - f(0)g(0) - k_1(z)$, where $k_1' = f'g$. Since fg is clearly in H^1c^1, it suffices to show that so is k_1. This we do by duality: we recall that $(H^1c^1)^*$ may be identified with

$L^\infty B(H)/H_0^\infty B(H)$ [9, p. 316], so we let ψ be the harmonic extension to the disc of an $L^\infty B(H)$ function of unit norm, and consider

$$(4.3)\quad \int_T \langle \psi(e^{i\theta}), k_1(e^{i\theta})\rangle \frac{d\theta}{2\pi} = \frac{-2}{\pi}\iint_D \Big\langle \psi(re^{i\theta}), \Big(\frac{\partial^2}{\partial\theta^2} + 2i\frac{\partial}{\partial\theta} - 1\Big)k_1(re^{i\theta})\Big\rangle \log\frac{1}{r}\cdot r\,dr\,d\theta.$$

This last identity may be verified by considering Fourier series, and the most threatening terms in it arise from the second derivative, namely

$$(4.4)\quad \frac{2}{\pi}\iint_D \langle \psi(re^{i\theta}), f''(re^{i\theta})g(re^{i\theta})\rangle r^2 e^{2i\theta}\log\frac{1}{r}\cdot r\,dr\,d\theta$$

and two terms such as

$$(4.5)\quad \frac{2}{\pi}\iint_D \langle \psi(re^{i\theta}), f'(re^{i\theta})h_1'(re^{i\theta})h_2(re^{i\theta})\rangle r^2 e^{2i\theta}\log\frac{1}{r}\cdot r\,dr\,d\theta.$$

The term (4.4) is bounded in modulus by

$$(4.6)\quad C\|\psi\|_{L^\infty B(H)}\iint_D \|g(re^{i\theta})\|_{c^1}\|f''(re^{i\theta})\|_{B(H)}\log\frac{1}{r}\cdot r\,dr\,d\theta,$$

which involves the Carleson measure Δ_f. By Carleson's Theorem [8, p. 33], this is

$$\le CC_*(\Delta_f)\|\psi\|_{L^\infty B(H)}\int_T \sup_{0<r<1}\|g(re^{i\theta})\|_{c^1}\frac{d\theta}{2\pi}$$

$$(4.7)\quad \le CC_*(\Delta_f)\|\psi\|_{L^\infty B(H)}\|g\|_{H^1c^1},$$

where we have used Bourgain's maximal theorem [4, Corollary 16]. For expressions such as (4.5) we use the Cauchy–Schwarz inequality to achieve the bounds

$$(4.8)\quad C\|\psi\|_{L^\infty B(H)}\Big(\iint_D \|h_2(re^{i\theta})\|_{c^2}^2\|f'(re^{i\theta})\|_{B(H)}^2\log\frac{1}{r}\cdot r\,dr\,d\theta\Big)^{1/2} \times\Big(\iint_D \|h_1'(re^{i\theta})\|_{c^2}^2\log\frac{1}{r}\cdot r\,dr\,d\theta\Big)^{1/2}.$$

The last factor may be bounded using the Littlewood–Paley identity, while the other integral in (4.8) involves the Carleson measure Q_f. An application of the Hardy–Littlewood maximal theorem [8, pp. 22, 24] to this factor leads to a bound on (4.5) of

$$(4.9)\quad C\|\psi\|_{L^\infty B(H)}C_*(Q_f)^{1/2}\|h_2\|_{H^2c^2}\|h_1\|_{H^2c^2} \le CC_*(Q_f)^{1/2}\|g\|_{H^1c^1}.$$

Combining the estimate $\|fg\|_{H^1c^1} \le \|f\|_{H^\infty B(H)}\|g\|_{H^1c^1}$ with (4.9) and (4.7) gives the estimate (4.2).

REMARK. An example mentioned in [2, (6.22)] shows that $\mathcal{A}$ is a proper subset of $H^\infty B(H)$; this is also implied by [5, Corollary 4.10] and Proposition 4.1.

5. Proof of Theorem 1.3. (i) To check that $\mathcal{A}$ is an algebra, we take $f_1, f_2 \in \mathcal{A}$ and use the Leibniz rule and the Cauchy–Schwarz inequality to show:

$$(5.1)\quad C_*(Q_{f_1f_2})^{1/2} \le C_*(Q_{f_1})^{1/2}\|f_2\|_{H^\infty B(H)} + C_*(Q_{f_2})^{1/2}\|f_1\|_{H^\infty B(H)};$$

$$(5.2)\quad C_*(\Delta_{f_1f_2}) \le C_*(\Delta_{f_1})\|f_2\|_{H^\infty B(H)} + 2C_*(Q_{f_1})^{1/2}C_*(Q_{f_2})^{1/2} + \|f_1\|_{H^\infty B(H)}C_*(\Delta_{f_2}).$$

From these it follows by an elementary calculation that $f \mapsto \|f\|_{\mathcal{A}}$ is submultiplicative.

(ii) Now let φ and f be as in the Theorem, and $g \in H^1c^1$. On account of (3.1), to bound W_φ it suffices to bound the integral

$$(5.3)\quad \int_T \Big\langle \varphi(e^{i\theta})\frac{d}{d\theta}f(e^{i\theta}), g(e^{i\theta})\Big\rangle\frac{d\theta}{2\pi} = -\int_T \Big\langle \frac{d}{d\theta}\varphi(e^{i\theta}), f(e^{i\theta})g(e^{i\theta})\Big\rangle\frac{d\theta}{2\pi} + \int_T \Big\langle \frac{d}{d\theta}\varphi(e^{i\theta}), k(e^{i\theta})\Big\rangle\frac{d\theta}{2\pi},$$

where $k' = fg'$. Since f is a multiplier of $(H^1c^1)'$ by Proposition 4.1, k belongs to H^1c^1; clearly, fg belongs to H^1c^1, since f is bounded. By assumption, $\varphi' \in \mathrm{BMO}(B(H))$, and so by the H^1-BMO duality theorem of [4, Corollary 16], (5.3) is bounded in modulus by

$$(5.4)\quad C\|\varphi'\|_{\mathrm{BMO}(B(H))}\|f\|_{\mathcal{A}}\|g\|_{H^1c^1}.$$

One can conclude the proof of (ii) and (iii) using similar arguments to those of Section 3.

REMARK. The spaces H^1c^1 and $\mathcal{A}$ have a natural matricial structure associated with their presentation as spaces of operator-valued functions on Hilbert space. The space of bounded left multipliers of $(H^1c^1)'$ also forms a *matricially normed space* [6]. The proof of Theorem 1.3 shows that $(\varphi', f) \mapsto \Gamma_\varphi f'(S)$, $\mathrm{BMO}(B(H)) \times \mathcal{A} \to B(H^2(H), L^2(H) \ominus H^2(H))$, is a completely bounded bilinear map in the sense of Effros and Ruan [6].

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Multiplier transformations on H^p spaces

by

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Abstract. The authors obtain some multiplier theorems on H^p spaces analogous to the classical L^p multiplier theorems of de Leeuw. The main result is that a multiplier operator $(Tf)^\wedge(x) = \lambda(x)\widehat{f}(x)$ ($\lambda \in C(\mathbb{R}^n)$) is bounded on $H^p(\mathbb{R}^n)$ if and only if the restriction $\{\lambda(\varepsilon m)\}_{m\in\Lambda}$ is an $H^p(\mathbb{T}^n)$ bounded multiplier uniformly for $\varepsilon > 0$, where Λ is the integer lattice in $\mathbb{R}^n$.

1. Introduction. Consider the n-dimensional Euclidean space $\mathbb{R}^n$; let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz test functions on $\mathbb{R}^n$ and λ be any function on $\mathbb{R}^n$. *The multiplier operator* T associated with λ is defined by $(Tf)^\wedge(\xi) = \lambda(\xi)\widehat{f}(\xi)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. Let X, Y be two function spaces on $\mathbb{R}^n$ with norms $\|\ \|_X$ and $\|\ \|_Y$, respectively. If $\mathcal{S}(\mathbb{R}^n)$ is dense in both X and Y, and if there exists a constant C such that

$$\|Tf\|_Y \le C\|f\|_X$$

uniformly for $f \in \mathcal{S}(\mathbb{R}^n)$, then we say that T is a bounded operator from X to Y with finite norm

$$\|T\| = \sup_{\|f\|_X=1} \|Tf\|_Y \le C.$$

We denote this by writing $T \in (X, Y)$.

The n-torus $\mathbb{T}^n$ can be identified with $\mathbb{R}^n/\Lambda$, where Λ is the unit lattice which is the additive group of points in $\mathbb{R}^n$ having integral coordinates. The multiplier operator $\widetilde{T}_\varepsilon$ on $\mathbb{T}^n$ associated with a function λ on $\mathbb{R}^n$ is defined by

$$\widetilde{T}_\varepsilon \widetilde{f}(x) \sim \sum_{m\in\Lambda} \lambda(\varepsilon m) a_m e^{2\pi i m\cdot x}$$

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