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Continuity of the Drazin inverse II

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Abstract. We study the continuity of the generalized Drazin inverse for elements of Banach algebras and bounded linear operators on Banach spaces. This work extends the results obtained by the second author on the conventional Drazin inverse.

1. Introduction and preliminaries. The continuity properties of the conventional Drazin inverse were investigated by Campbell and Meyer [1, 2] for matrices, and by Rakočević [13] for bounded linear operators and elements of Banach algebras. In the present paper we investigate the continuity of the generalized Drazin inverse introduced recently by Koliha [9]. We start in the Banach algebra setting, exploring analogies between the continuity of the ordinary inverse and the continuity of the (generalized) Drazin inverse, then move on to bounded linear operators.

We denote by A a unital Banach algebra with unit e. For an element $a \in A$ we denote by $\varrho(a)$, $\sigma(a)$ and $r(a)$ the resolvent set, the spectrum and the spectral radius of a, respectively. The set of all isolated and accumulation spectral points of a is denoted by $\operatorname{iso}\sigma(a)$ and $\operatorname{acc}\sigma(a)$, respectively. If $\lambda \in \varrho(a)$, then $R(\lambda; a) = (\lambda e - a)^{-1}$ is the resolvent of a. By $\operatorname{Inv}(A)$, $\operatorname{qNil}(A)$ and $\operatorname{Idem}(A)$ we denote the sets of all invertible, quasinilpotent and idempotent elements of A, respectively. For basic facts about Banach algebras see [12, 15].

DEFINITION 1.1. Let $a \in A$. Following [9], we say that a is *Drazin invertible* if there exists $b \in A$ such that

$$ab = ba, \quad ab^2 = b, \quad a^2b - a \in \operatorname{qNil}(A). \tag{1.1}$$

Such a b, if it exists, is unique [9]; it is called the *generalized Drazin inverse* of a, and will be denoted by a^{D}. If $a^2b - a$ is in fact nilpotent, then a^{D}

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is the conventional Drazin inverse of a (see [2, 3, 4, 8]). The *Drazin index* $i(a)$ of a is equal to 0 if $a \in \mathrm{Inv}(A)$, to k if $a^2b - a$ is nilpotent of index k and $a \notin \mathrm{Inv}(A)$, otherwise $i(a) = \infty$. From this point on we use the term "Drazin inverse" instead of "generalized Drazin inverse". This extension of the Drazin inverse was anticipated by Harte in [6].

The basic existence results for Drazin inverse are summarized in the following lemma ([9, Theorem 4.2], [10, Theorem 1.2]).

LEMMA 1.2. *Let $a \in A$. The following conditions on a are equivalent.*

(i) *a is Drazin invertible.*

(ii) $0 \notin \operatorname{acc} \sigma(a)$.

(iii) *There is $p \in \mathrm{Idem}(A)$ commuting with a such that $ap \in \mathrm{qNil}(A)$ and $a + p \in \mathrm{Inv}(A)$.*

If (iii) *is satisfied, then the Drazin inverse of a is given by*

$$a^{\mathrm{D}} = (a+p)^{-1}(e-p). \tag{1.2}$$

The element p from the preceding lemma is given by

$$p = e - a^{\mathrm{D}}a. \tag{1.3}$$

It follows that $p = 0$ if $0 \in \varrho(a)$; if $0 \in \operatorname{iso} \sigma(a)$, then p is the spectral idempotent of a corresponding to 0. From (1.2) we can deduce a representation in terms of the holomorphic calculus for a (see [9, Theorem 4.4]), that is,

$$a^{\mathrm{D}} = f(a), \tag{1.4}$$

where f is a function holomorphic in an open neighbourhood of $\sigma(a)$ such that $f(\lambda) = \lambda^{-1}$ in a neighbourhood of $\sigma(a)\backslash\{0\}$, and $f(\lambda) = 0$ in a neighbourhood of 0.

The following result will be needed in the next section. For $\mu \in \mathbb{C}$ and $K \subset \mathbb{C}$ we define $d(\mu, K) = \inf\{|\lambda - \mu| : \lambda \in K\}$ if $K \neq \emptyset$ and $d(\mu, \emptyset) = \infty$.

LEMMA 1.3. *Let a be a Drazin invertible element of the Banach algebra A and let $r(a) > 0$. Then*

$$d(0, \sigma(a)\backslash\{0\}) = (r(a^{\mathrm{D}}))^{-1}. \tag{1.5}$$

Proof. In accordance with (1.4) write $a^{\mathrm{D}} = f(a)$ for a suitable holomorphic function f. For each $\lambda \in \sigma(a)\backslash\{0\}$,

$$|\lambda^{-1}| = |f(\lambda)| \le r(f(a)) = r(a^{\mathrm{D}}),$$

that is, $|\lambda| \ge (r(a^{\mathrm{D}}))^{-1}$, and $d(0, \sigma(a)\backslash\{0\}) \ge (r(a^{\mathrm{D}}))^{-1}$. By the spectral mapping theorem and by the compactness of $\sigma(a)\backslash\{0\}$ there is $\mu \in \sigma(a)\backslash\{0\}$ such that $|\mu|^{-1} = r(a^{\mathrm{D}})$, that is, $|\mu| = (r(a^{\mathrm{D}}))^{-1}$. This proves (1.5).

2. The continuity results for the Drazin inverse in Banach algebras. We start by reviewing two standard results on the continuity of the ordinary inverse in a Banach algebra, formulated below as Theorems A and B.

THEOREM A. *If a is an invertible element of the Banach algebra A and if $a_n \to a$, then a_n are invertible for all sufficiently large n, and $a_n^{-1} \to a^{-1}$.*

An analogue of Theorem A is not available for the Drazin inverse, mainly because the set of all Drazin invertible elements in a Banach algebra need not be an open set [9, Example 8.4]; in such a situation we have to assume not only the Drazin invertibility of a, but of the a_n as well. Even under this stronger assumption, the mere convergence $a_n \to a$ is not sufficient to enforce $a_n^{\mathrm{D}} \to a^{\mathrm{D}}$.

EXAMPLE 2.1. Let A be the Banach algebra of all complex-valued functions continuous on the set $[0,1] \cup [2,3]$, equipped with the supremum norm. Define a and a_n by $a(t) = 0$ if $t \in [0,1]$, $a(t) = t$ if $t \in [2,3]$, $a_n(t) = t/n$ if $t \in [0,1]$ and $a_n(t) = t$ if $t \in [2,3]$, $n = 1, 2, \ldots$ Then a is Drazin invertible with a^{D} defined by $a^{\mathrm{D}}(t) = 0$ if $t \in [0,1]$, $a^{\mathrm{D}}(t) = 1/t$ if $t \in [2,3]$. However, none of the a_n is Drazin invertible as $\sigma(a_n) = [0, 1/n]$ and $0 \in \operatorname{acc} \sigma(a_n)$ (see Lemma 1.2).

EXAMPLE 2.2. In a Banach algebra A let a be nilpotent of index 3, and therefore Drazin invertible with $a^{\mathrm{D}} = 0$. Each $a_n = a + e/n$ is Drazin invertible with $a_n^{\mathrm{D}} = (a + e/n)^{-1} = ne - n^2a + n^3a^2$. We have $a_n \to a$, but $a_n^{\mathrm{D}} \not\to a^{\mathrm{D}}$ as the sequence $\|a_n^{\mathrm{D}}\|$ is unbounded. This phenomenon is already in evidence for matrices (see [2, Example 10.7.4]).

THEOREM B. *If a_n are invertible elements of the Banach algebra A such that $a_n \to a$ and that the norms $\|a_n^{-1}\|$ are bounded, then a is invertible and $a_n^{-1} \to a^{-1}$.*

Since the set of all Drazin invertible elements need not be open, we also have to assume the Drazin invertibility of a in the setting of Theorem B. This is demonstrated in the following example.

EXAMPLE 2.3 (based on Rickart [15, p. 282]). There is a sequence (a_n) of Drazin invertible elements of a Banach algebra A such that $a_n \to a$, (a_n^{D}) converges, yet the limit a is not Drazin invertible: Let A be the Banach algebra of all bounded linear operators on the space ℓ^2. We denote by $e_1, e_2, \ldots$ the standard Schauder basis of ℓ^2. Define $\alpha_m = \exp(-k_m)$, where $m = 2^{k_m}(2l_m + 1)$ is the (unique) factorization of the positive integer m. If k is an arbitrary positive integer, set

$$Te_m = \alpha_m e_{m+1}, \quad T_k e_m = Te_m \text{ if } k \neq k_m, \quad T_k e_m = 0 \text{ if } k = k_m$$

$(m = 1, 2, \ldots)$; this defines elements $T, T_k \in A$ for $k = 1, 2, \ldots$ such that $\|T_k - T\| \to 0$. Each T_k is nilpotent, and consequently Drazin invertible with $T_k^{\mathrm{D}} = 0$; hence, T_k^{D} converges to 0. However, 0 is an accumulation point of $\sigma(T)$, which means that T is not Drazin invertible (Lemma 1.2).

The main result on the continuity of the Drazin inverse in a Banach algebra is expressed in the following analogue of Theorem B. In the case of the conventional Drazin inverse the equivalence of conditions (2.1), (2.2) and (2.6) was proved by Rakočević [13]. The proof given here uses only Banach algebra arguments while the proofs in [13] are based on operator theory techniques, in particular on the concept of the gap between subspaces of X.

For notational convenience we will write

$$\Delta_r = \{\lambda : 0 < |\lambda| < r\}$$

for the punctured disc with centre 0 and radius $r > 0$.

THEOREM 2.4. *Let a_n and a be Drazin invertible elements of the Banach algebra A such that $a_n \to a$, and let p_n and p be the spectral projections of a_n and a corresponding to 0. Then the following conditions are equivalent*:

(2.1) $a_n^{\mathrm{D}} \to a^{\mathrm{D}}$,

(2.2) $\sup_n \|a_n^{\mathrm{D}}\| < \infty$,

(2.3) $\sup_n r(a_n^{\mathrm{D}}) < \infty$,

(2.4) $\inf_n d(0, \sigma(a_n)\backslash\{0\}) > 0$,

(2.5) *there is $r > 0$ such that* $\Delta_r \subset \varrho(a) \cap \bigcap_{n=1}^{\infty} \varrho(a_n)$,

(2.6) $a_n^{\mathrm{D}} a_n \to a^{\mathrm{D}} a$,

(2.7) $p_n \to p$.

Proof. The implications (2.1)⇒(2.2)⇒(2.3)⇒(2.4)⇒(2.5) follow easily when we take into account Lemma 1.3, remembering that $d(0, \sigma(a_n)\backslash\{0\}) = \infty$ if $r(a_n) = 0$.

(2.5)⇒(2.7). Let $\Omega_0 = \{\lambda : |\lambda| < \frac{1}{3}r\}$, $\Omega_1 = \{\lambda : |\lambda| > \frac{2}{3}r\}$, and let f be the holomorphic function on $\Omega_0 \cup \Omega_1$ such that $f(\lambda) = 1$ on Ω_0 and $f(\lambda) = 0$ on Ω_1. Since Ω_1 contains the sets $\sigma(a)\backslash\{0\}$ and $\sigma(a_n)\backslash\{0\}$ for all n, we have $f(a) = p$ and $f(a_n) = p_n$ for all n. By [12, Theorem 3.3.7(i)], $f(a_n) \to f(a)$, and (2.7) is proved.

(2.7)⇔(2.6) since $p_n = e - a_n^{\mathrm{D}} a_n$ and $p = e - a^{\mathrm{D}} a$.

(2.7)⇒(2.1). If (2.7) holds, then $a_n + p_n \to a + p$ with $a + p \in \mathrm{Inv}(A)$ (see Lemma 1.2), and hence $(a_n + p_n)^{-1} \to (a + p)^{-1}$ by Theorem A. By Lemma 1.2, $a_n^{\mathrm{D}} = (a_n + p_n)^{-1}(e - p_n) \to (a + p)^{-1}(e - p) = a^{\mathrm{D}}$ as $n \to \infty$, and (2.1) holds.

3. Preliminaries on bounded linear operators. We denote by $L(X)$ the Banach algebra of all bounded linear operators acting on the complex Banach space X with the usual operator norm $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$. The null space and the range of $T \in L(X)$ will be denoted by $N(T)$ and $R(T)$, respectively. Following Kato [7, p. 197], for any closed subspaces M, N of X we define the *gap* (or *opening*) *between M and N* by

$$\mathrm{gap}(M, N) = \max\{\delta(M, N), \delta(N, M)\},$$

where

$$\delta(M, N) = \sup\{\mathrm{dist}(u, N) : u \in M,\ \|u\| = 1\}.$$

Alternative notation for the gap is $\widehat{\delta}(M, N)$ (see [7]) or $\theta(M, N)$ (see [5, 11]).

The following lemma, needed in the sequel, generalizes results obtained by Rakočević [14, Lemma 2.1 and 2.2] for Hermitian projections.

LEMMA 3.1. *Let $P, Q \in L(X)$ be projections. Then*

$$\mathrm{gap}(R(Q), R(P)) \le \max\{\|(I - Q)P\|, \|(I - P)Q\|\}, \tag{3.1}$$

$$\mathrm{gap}(N(Q), N(P)) \le \max\{\|Q(I - P)\|, \|P(I - Q)\|\}, \tag{3.2}$$

$$\|(I - P)Q\| \le \|I - P\|\|Q\|\mathrm{gap}(R(Q), R(P)), \tag{3.3}$$

$$\|P(I - Q)\| \le \|P\|\|I - Q\|\mathrm{gap}(N(Q), N(P)). \tag{3.4}$$

Proof. To obtain (3.1), let $u \in R(P)$ with $\|u\| = 1$. Then

$$\begin{aligned}\mathrm{dist}(u, R(Q)) &= \inf_{x \in X} \|u - Qx\| \le \|u - Qu\| = \|(I - Q)u\| \\ &= \|(I - Q)Pu\| \le \|(I - Q)P\|.\end{aligned}$$

Consequently, $\delta(R(P), R(Q)) \le \|(I - Q)P\|$. A symmetrical argument shows that $\delta(R(Q), R(P)) \le \|(I - P)Q\|$, and (3.1) follows. Inequality (3.2) is obtained from (3.1) on observing that $N(P) = R(I - P)$ and $N(Q) = R(I - Q)$.

Let $u \in R(P)$ with $\|u\| = 1$. Then, for every $x \in X$, $(I - P)Qx = (I - P)(Qx - u)$, which implies

$$\begin{aligned}\|(I - P)Qx\| &\le \|(I - P)\|\mathrm{dist}(Qx, R(P)) \\ &\le \|I - P\|\mathrm{gap}(R(Q), R(P))\|Qx\| \\ &\le \|I - P\|\mathrm{gap}(R(Q), R(P))\|Q\|\|x\|;\end{aligned}$$

hence (3.3) follows. Observing that $N(P) = R(I - P)$ and $N(Q) = R(I - Q)$, we deduce (3.4) from (3.3).

COROLLARY 3.2. *Let $P, Q \in L(X)$ be commuting projections. Then*

$$\|P - Q\| \le \alpha(P, Q)\mathrm{gap}(R(Q), R(P)), \tag{3.5}$$

$$\|P - Q\| \le \alpha(P, Q)\mathrm{gap}(N(Q), N(P)), \tag{3.6}$$

where $\alpha(P, Q) = \|I - P\|\|Q\| + \|P\|\|I - Q\|$.

Proof. By [7, Theorem IV.2.9] we deduce that $\operatorname{gap}(N(T^*), N(S^*)) = \operatorname{gap}(R(T), R(S))$ for any $T, S \in L(X)$. (See also [11, p. 269].) Hence we can replace P, Q in (3.4) by P^*, Q^* to obtain

$$\|(I-Q)P\| = \|P^*(I-Q^*)\| \le \|P\|\|I-Q\|\operatorname{gap}(R(Q), R(P)). \tag{3.7}$$

If $PQ = QP$, then $\|P-Q\| \le \|(I-P)Q\| + \|(I-Q)P\|$, and (3.5) follows from (3.3) and (3.7). Inequality (3.6) follows from (3.5) by duality.

The following result characterizes the convergence of projections in $L(X)$ in terms of the gap convergence of null spaces and ranges.

LEMMA 3.3. *If P and P_n are projections in $L(X)$, then the following conditions are equivalent*:

(i) $\|P_n - P\| \to 0$,
(ii) $\operatorname{gap}(R(P_n), R(P)) \to 0$ *and* $\operatorname{gap}(N(P_n), N(P)) \to 0$.

If $P_nP = PP_n$ for all sufficiently large n, then either of the conditions in (ii) *implies* (i).

Proof. (i)⇒(ii) follows from (3.1) and (3.2) with $Q = P_n$.

(ii)⇒(i). From $\|P_n - P\| \le \|(I-P)P_n\| + \|P(I-P_n)\|$, (3.3) and (3.4) with $Q = P_n$ we obtain

$$\begin{aligned}\|P_n - P\| &\\ &\le \mu(P_n)\{\|I-P\|\operatorname{gap}(R(P_n), R(P)) + \|P\|\operatorname{gap}(N(P_n), N(P))\},\end{aligned} \tag{3.8}$$

where $\mu(P_n) = \max\{\|P_n\|, \|I-P_n\|\}$. The result will follow when we show that $\mu(P_n)$ is bounded.

Since $\mu(P_n) \le \|P_n\|+1$, we can rewrite (3.8) as $\|P_n - P\| \le (\|P_n\|+1)\varepsilon_n$, where $\varepsilon_n \ge 0$ and $\varepsilon_n \to 0$. Then

$$\begin{aligned}\|P_n\| &\le \|P\| + \|P_n - P\| \le \|P\| + (\|P_n\|+1)\varepsilon_n,\\ (1-\varepsilon_n)\|P_n\| &\le \|P\| + \varepsilon_n,\\ \|P_n\| &\le (1-\varepsilon_n)^{-1}(\|P\|+\varepsilon_n) \le 2(\|P\|+1) \quad \text{if } 0 \le \varepsilon_n < 1/2.\end{aligned}$$

The statement about commuting projections follows from Corollary 3.2.

The noncommutative part of the preceding lemma is implicit in the proof of [11, Theorem 3] under the additional assumption that the projections are of finite rank.

If $T \in L(X)$, we define the *reduced minimum modulus* $\gamma(T)$ of T by $\gamma(T) = \infty$ if $T = 0$ and

$$\gamma(T) = \inf\{\|Tu\|/\operatorname{dist}(u, N(T)) : \operatorname{dist}(u, N(T)) > 0\} \tag{3.9}$$

if $T \ne 0$. We recall that $\gamma(T) > 0$ if and only if $R(T)$ is closed [7, p. 251]. An operator $T \in L(X)$ is *upper semi-Fredholm* if $\gamma(T) > 0$ and the nullity $\alpha(T) = \dim N(T)$ is finite.

For convenience of the reader we recall the following relations between the reduced minimum modulus and the gap.

LEMMA 3.4 (Markus [11, pp. 268–269]). *The following inequalities are true for any closed range operators $A, B \in L(X)$ provided* $\operatorname{gap}(N(A), N(B)) < 1/2$ *in* (3.10) *and* $\operatorname{gap}(R(A), R(B)) < 1/2$ *in* (3.11):

$$|\gamma(A) - \gamma(B)| \le \frac{3\|A-B\|}{1 - 2\operatorname{gap}(N(A), N(B))}, \tag{3.10}$$

$$|\gamma(A) - \gamma(B)| \le \frac{3\|A-B\|}{1 - 2\operatorname{gap}(R(A), R(B))}, \tag{3.11}$$

$$\operatorname{gap}(N(A), N(B)) \le \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\}\|A-B\|, \tag{3.12}$$

$$\operatorname{gap}(R(A), R(B)) \le \max\left\{\frac{1}{\gamma(A)}, \frac{1}{\gamma(B)}\right\}\|A-B\|. \tag{3.13}$$

This leads to the following result on convergence of operators.

LEMMA 3.5 (Markus [11, Theorem 2 and Remark 1]). *Suppose that C_n and C are closed range operators in $L(X)$ such that $C_n \to C$. The following conditions are equivalent*:

$$\inf_n \gamma(C_n) > 0, \tag{3.14}$$

$$\lim_{n\to\infty} \gamma(C_n) = \gamma(C), \tag{3.15}$$

$$\operatorname{gap}(R(C_n), R(C)) \to 0, \tag{3.16}$$

$$\operatorname{gap}(N(C_n), N(C)) \to 0. \tag{3.17}$$

If C_n and C are upper semi-Fredholm, we have an additional equivalent condition

$$(\exists n_0)(\forall n \ge n_0) \quad \alpha(C_n) = \alpha(C). \tag{3.18}$$

4. Drazin inverse for bounded linear operators. After we review some facts about the core-quasinilpotent decomposition for a Drazin invertible operator [9, Theorem 6.4], we will be ready to present the main result of this section concerning the continuity of the Drazin inverse for bounded linear operators. Let $A \in L(X)$ be Drazin invertible. Then

$$A = C + Q,$$

where $i(C) \le 1$, Q is quasinilpotent, and $CQ = QC = 0$. We call C the *core operator* of the Drazin invertible operator A. Since Q is quasinilpotent and commutes with A and C, $\lambda I - A$ is invertible if and only if $\lambda I - C$ is invertible; hence

$$\sigma(A) = \sigma(C).$$

Further, by [9, Theorem 5.7], $C^{\mathrm{D}} = A^{\mathrm{D}}$. The spectral projection P of A corresponding to 0 is also the spectral projection of C corresponding to 0, and

$$N(C) = R(P), \quad R(C) = N(P). \tag{4.1}$$

Hence C is a closed range operator. From [13, Lemma 2.1] we deduce that

$$\gamma(C) \le d(0, \sigma(A)\backslash\{0\}). \tag{4.2}$$

THEOREM 4.1. *Let $A_n, A \in L(X)$ be Drazin invertible operators such that $A_n \to A$, and let P_n and P be the spectral projections of A_n and A corresponding to* 0. *In addition, let C_n and C be the core operators of A_n and A. Then the following conditions are equivalent*:

(4.3) $A_n^{\mathrm{D}} \to A^{\mathrm{D}}$,

(4.4) $\sup_n \|A_n^{\mathrm{D}}\| < \infty$,

(4.5) $\sup_n r(A_n^{\mathrm{D}}) < \infty$,

(4.6) $\inf_n d(0, \sigma(A_n^{\mathrm{D}})\backslash\{0\}) > 0$,

(4.7) *there is $r > 0$ such that* $\Delta_r \subset \varrho(A) \cap \bigcap_{n=1}^{\infty} \varrho(A_n)$,

(4.8) $A_n^{\mathrm{D}} A_n \to A^{\mathrm{D}} A$,

(4.9) $P_n \to P$,

(4.10) $\mathrm{gap}(R(P_n), R(P)) \to 0$ *and* $\mathrm{gap}(N(P_n), N(P)) \to 0$,

(4.11) $\mathrm{gap}(R(C_n), R(C)) \to 0$ *and* $\mathrm{gap}(N(C_n), N(C)) \to 0$,

(4.12) $C_n \to C$ *and* $\mathrm{gap}(R(C_n), R(C)) \to 0$,

(4.13) $C_n \to C$ *and* $\mathrm{gap}(N(C_n), N(C)) \to 0$,

(4.14) $\gamma(C_n) \to \gamma(C)$,

(4.15) $\inf_n \gamma(C_n) > 0$.

Proof. (4.3)–(4.9) are equivalent by Theorem 2.4.

(4.9)⇔(4.10) is a consequence of Lemma 3.3.

(4.10)⇔(4.11) follows from (4.1).

(4.11)⇒(4.12). By the preceding arguments, (4.11) is equivalent to (4.9). Therefore $C_n = A_n(I - P_n) \to A(I - P) = C$.

(4.12)⇔(4.13) holds in view of Lemma 3.5.

(4.13)⇒(4.14) holds also by Lemma 3.5.

(4.14)⇒(4.15) is clear.

(4.15)⇒(4.6) follows in view of (4.2).

EXAMPLE 4.2. Condition (4.12) or (4.13) in the preceding theorem cannot be reduced to $C_n \to C$ alone. For a counterexample set

$$A_n = C_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/n & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $A_n \to A$, $C_n \to C$, but

$$A_n^{\mathrm{D}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{bmatrix} \not\to A^{\mathrm{D}} = A.$$

NOTE 4.3. From Lemma 3.4 it would appear that, in general, (4.12) cannot be reduced to $\mathrm{gap}(R(C_n), R(C)) \to 0$ alone. Similarly, we may expect that (4.13) cannot be reduced to $\mathrm{gap}(N(C_n), N(C)) \to 0$ alone. However, at this stage this remains an open problem. Let us observe that this reduction is successful in several special cases:

(i) If $A_n A = A A_n$ for all sufficiently large n—see the commutative part of Lemma 3.3.

(ii) If 0 is a simple pole for A_n and A, then $C_n = A_n$, $C = A$, and $C_n \to C$ follows automatically from $A_n \to A$.

(iii) If the indices $i(A_n)$ are finite and uniformly bounded—see [13, Corollary 3.3].

(iv) If the space X is finite dimensional—a special case of (iii).

If Theorem 4.1 is specialized to operators A_n, A of finite Drazin index, we recover [13, Theorem 2.2]. Note that (4.5), (4.6), and (4.7) are new conditions, even in the case of finite index operators; (4.7) is already implicit in [13].

5. The finite index Drazin inverse. In this section we focus our attention on one aspect of the continuity of the finite index Drazin inverse, namely on a generalization of a theorem due to Campbell and Meyer [2, Theorem 10.7.1]. Rakočević [13, Corollary 3.4] gave a somewhat different generalization of this result.

THEOREM 5.1. *Let $A_n, A \in L(X)$ be Drazin invertible operators such that $A_n \to A$, the indices $i(A_n)$ are bounded, and the spectral projections P_n, P of A_n, A corresponding to 0 are of finite rank. Then $A_n^{\mathrm{D}} \to A^{\mathrm{D}}$ if and only if there exists n_0 such that* $\mathrm{rank}\, P_n = \mathrm{rank}\, P$ *for all $n \ge n_0$.*

Proof. By hypothesis there is an integer p such that $p \ge i(A_n)$ for all n and $p \ge i(A)$.

First suppose $A_n^{\mathrm{D}} \to A^{\mathrm{D}}$. According to (4.9), $P_n \to P$. Then by Lemma 3.3, $\mathrm{gap}(R(P_n), R(P)) \to 0$. By [7, Corollary IV.2.6], $\dim R(P_n) = \dim R(P)$ whenever $\mathrm{gap}(R(P_n), R(P)) < 1$; the result then follows.

Conversely, assume that there exists n_0 such that $\operatorname{rank} P_n = \operatorname{rank} P$ for all $n \geq n_0$. Let $A_n = C_n + N_n$ and $A = C + N$ be the core-nilpotent decompositions. Then $A_n^p = (C_n + N_n)^p = C_n^p + N_n^p = C_n^p$ and $A^p = (C+N)^p = C^p + N^p = C^p$. Hence $C_n^p \to C^p$. By [13, Lemma 3.1] applied to C_n, C, we obtain $C_n \to C$. Since $i(C_n) \leq 1$ and $i(C) \leq 1$, $N(C_n^k) = N(C_n)$ and $N(C^k) = N(C)$ for all k. From (4.1) we get $\alpha(C_n) = \operatorname{rank} P_n = \operatorname{rank} P = \alpha(C)$ for all $n \geq n_0$, and by Lemma 3.5 we conclude that $\operatorname{gap}(N(C_n), N(C)) \to 0$. The result then follows from Theorem 4.1.

The preceding theorem provides an alternative route to [13, Corollary 3.4]. More importantly, it implies the main result of Campbell and Meyer on the continuity of the Drazin inverse for matrices.

THEOREM 5.2 (Campbell and Meyer [2, Theorem 10.7.1]). *Let A_n and A be complex $d \times d$ matrices such that $A_n \to A$. Then $A_n^{\mathrm{D}} \to A^{\mathrm{D}}$ if and only if there exists n_0 such that* $\operatorname{rank} C_n = \operatorname{rank} C$ *for all $n \geq n_0$, where C_n and C are the core matrices of A_n and A, respectively.*

Proof. Follows from the preceding theorem when we observe that the index of matrices is bounded by the dimension d and that $\operatorname{rank} C_n = d - \operatorname{rank} P_n$ and $\operatorname{rank} C = d - \operatorname{rank} P$, where P_n and P are the eigenprojections of A_n and A at 0.

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