STUDIA MATHEMATICA 131 (2) (1998)

# On multilinear mappings attaining their norms

by

MARIA D. ACOSTA (Granada)

**Abstract.** We show, for any Banach spaces $X$ and $Y$, the denseness of the set of bilinear forms on $X \times Y$ whose third Arens transpose attains its norm. We also prove the denseness of the set of norm attaining multilinear mappings in the class of multilinear mappings which are weakly continuous on bounded sets, under some additional assumptions on the Banach spaces, and give several examples of classical spaces satisfying these hypotheses.

**1. Introduction.** In 1995, R. Aron, C. Finet and E. Werner proved a positive result on the denseness of the set of norm attaining multilinear forms on a space with the Radon–Nikodym property [5]. Afterwards, Y. S. Choi and S. G. Kim also got some positive results on this topic [9]. The first counterexample of a Banach space $X$ for which the set of norm attaining bilinear forms is not dense in the space of all continuous bilinear forms was given by M. D. Acosta, F. Aguirre and R. Payá [1]. Y. S. Choi proved that the classical space $L_1[0,1]$ also fails this property [8]. The doctoral dissertation by F. Aguirre [2] contains several positive results in this line for multilinear forms or polynomials. For instance, if for some Banach space $X$, the set $\mathcal{A}(^{n+1}X)$ (the set of $(n+1)$-linear forms attaining their norms) is dense in $\mathcal{L}(^{n+1}X)$ (the set of $(n+1)$-linear and continuous forms on $X$), then the same also happens for $n$-linear forms. After that, it was proved that the situation really gets worse when the number of variables increases. In fact, M. Jiménez-Sevilla and R. Payá provided, for each natural number $n$, a Banach space $X$ such that the set of norm attaining $n$-linear forms on $X$ is dense in the space of all $n$-linear forms, but such that this does not hold for $(n+1)$-linear forms [16].

In this paper, we follow the idea, already used for the analogous problem with operators [19, 23], of going to the second dual by transposing the bilinear form and get a positive result without any restrictions on the Banach spaces involved. In this way, by using a perturbed optimization principle due

1991 *Mathematics Subject Classification*: Primary 46B20.
Research partially supported by D.G.I.C.Y.T., project no. PB96-1906.

to R. A. Poliquin and V. Zizler [22], we prove the denseness of the set of bilinear forms whose third Arens transpose attains its norm. This statement is the parallel version of Lindenstrauss' result for operators [19].

Also, for some spaces with "enough norm-one projections", we show that multilinear forms (resp. symmetric multilinear forms) which are weakly continuous on bounded sets can be approximated by norm attaining (resp. symmetric) multilinear forms of finite type. This can be applied, for instance, to spaces like $C(K)$ ($K$ a compact and Hausdorff topological space) and $L_1(\mu)$, for any measure $\mu$. Let us observe that under this assumption a general result for all multilinear forms does not hold (see [8]). Also, the result we get here extends a previous one [16, Proposition 2.10] for spaces with shrinking and monotone basis, since under the conditions assumed by these authors all multilinear forms are weakly continuous on bounded sets. By using similar arguments, we will get analogous results for those polynomials with weakly continuous restrictions to bounded sets. Finally, we will check that if a Banach space $X$ has the Dunford–Pettis property, $B_{X^{**}}$ is $w^*$-sequentially compact and $Y$ is reflexive, then any weakly continuous bilinear form on $X \times Y$ is such that its third Arens transpose attains its norm.

**2. The results.** Before showing the first result, let us recall the definition of the transpose of a bilinear mapping, due to R. Arens [3, 4]. For a bilinear continuous mapping

$$\phi : X \times Y \to Z$$

the first (right) *Arens transpose* of $\phi$, defined by $\phi^{\mathrm{t}} : Z^* \times X \to Y^*$, is the bilinear mapping given by

$$\phi^{\mathrm{t}}(z^*, x)(y) = z^*\phi(x, y).$$

Clearly, $\phi^{\mathrm{t}}$ is continuous, $\|\phi^{\mathrm{t}}\| = \|\phi\|$ and, by the Hahn–Banach theorem, if $\phi$ attains its norm then so does $\phi^{\mathrm{t}}$. Hence, the situation seems to be more promising when one increases the order of the transpose and the next result shows that it is so.

THEOREM 1. *Let $X$ and $Y$ be Banach spaces. Then the set of bilinear and continuous forms on $X \times Y$ such that the third Arens transpose attains its norm is dense in $\mathcal{L}(^2(X \times Y))$, the space of all continuous bilinear forms on $X \times Y$.*

Proof. Let $\phi$ be a bilinear form on $X \times Y$ such that $\|\phi\| = 1$ and $0 < \varepsilon < 1/2$. We define the mapping $f : B_{Y^{**}} \to \mathbb{R}$ by

$$f(y^{**}) = \|\phi^{\mathrm{tt}}(y^{**}, 1)\| \quad (y^{**} \in B_{Y^{**}}),$$

which is $w^*$-lower semicontinuous. So, by applying [22, Theorem 1], there is



an element $y^* \in Y^*$ with $\|y^*\| \le \varepsilon$ such that

$$\sup_{\|y^{**}\| \le 1} (f(y^{**}) + \operatorname{Re} y^{**}(y^*)) = f(y_0^{**}) + \operatorname{Re} y_0^{**}(y^*) \tag{1}$$

for some $y_0^{**} \in B_{Y^{**}}$ (Re will denote the real part). Since it is clearly true that $f(y^{**}) = f(\lambda y^{**})$ for any scalar $\lambda$ with $|\lambda| = 1$, it follows that in fact we get

$$\operatorname{Re} y_0^{**}(y^*) = |y_0^{**}(y^*)| \tag{2}$$

and by the same argument we also have

$$\sup_{\|y^{**}\| \le 1} (f(y^{**}) + \operatorname{Re} y^{**}(y^*)) = \sup_{\|y^{**}\| \le 1} f(y^{**}) + |y^{**}(y^*)|. \tag{3}$$

On the other hand, the choice of $\varepsilon$ and (1) give us $f(y_0^{**}) \ne 0$ and we set

$$x^* := \frac{\phi^{\mathrm{tt}}(y_0^{**}, 1)}{\|\phi^{\mathrm{tt}}(y_0^{**}, 1)\|} \in X^*.$$

Now define

$$\Psi(x, y) := \phi(x, y) + x^*(x) y^*(y) \quad ((x, y) \in X \times Y).$$

Then $\Psi$ is a continuous bilinear form on $X \times Y$ such that

$$\|\Psi - \phi\| = \|x^*\|\,\|y^*\| = \|y^*\| \le \varepsilon$$

and we just need to check that $\Psi^{\mathrm{ttt}}$ attains its norm. For this purpose, choose an element $x_0^{**} \in B_{X^{**}}$ such that

$$x_0^{**}(\phi^{\mathrm{tt}}(y_0^{**}, 1) + y_0^{**}(y^*) x^*) = \|\phi^{\mathrm{tt}}(y_0^{**}, 1) + y_0^{**}(y^*) x^*\|.$$

We will show that $\Psi^{\mathrm{ttt}}$ attains its norm at $(x_0^{**}, y_0^{**})$; on the one hand, the choice of $x_0^{**}$ gives us

$$\begin{aligned} \Psi^{\mathrm{ttt}}(x_0^{**}, y_0^{**}) &= x_0^{**}(\Psi^{\mathrm{tt}}(y_0^{**}, 1)) \\ &= x_0^{**}(\phi^{\mathrm{tt}}(y_0^{**}, 1) + y_0^{**}(y^*) x^*) = \|\phi^{\mathrm{tt}}(y_0^{**}, 1) + y_0^{**}(y^*) x^*\|. \end{aligned} \tag{4}$$

On the other hand, if $x^{**} \in B_{X^{**}}$ and $y^{**} \in B_{Y^{**}}$, again by using the definition of $\Psi$ and the Arens transpose we get

$$\begin{aligned} \Psi^{\mathrm{ttt}}(x^{**}, y^{**}) &= x^{**}(\Psi^{\mathrm{tt}}(y^{**}, 1)) = x^{**}(\phi^{\mathrm{tt}}(y^{**}, 1) + y^{**}(y^*) x^*) \\ &\le \|\phi^{\mathrm{tt}}(y^{**}, 1) + y^{**}(y^*) x^*\| \\ &\le \|\phi^{\mathrm{tt}}(y^{**}, 1)\| + |y^{**}(y^*)| \quad \text{(by (3), (1) and (2))} \\ &\le \|\phi^{\mathrm{tt}}(y_0^{**}, 1)\| + |y_0^{**}(y^*)| \\ &= \left\| \phi^{\mathrm{tt}}(y_0^{**}, 1) + |y_0^{**}(y^*)| \frac{\phi^{\mathrm{tt}}(y_0^{**}, 1))}{\|\phi^{\mathrm{tt}}(y_0^{**}, 1)\|} \right\| \\ &= \|\phi^{\mathrm{tt}}(y_0^{**}, 1) + y_0^{**}(y^*) x^*\| = \Psi^{\mathrm{ttt}}(x_0^{**}, y_0^{**}), \end{aligned}$$

so $\|\Psi^{\mathrm{ttt}}\| = \Psi^{\mathrm{ttt}}(x_0^{**}, y_0^{**})$ as we wanted to show. ∎

REMARK 2. Equation (4) gives us

$$\|\Psi\| = \Psi^{ttt}(x_0^{**}, y_0^{**}) = x_0^{**}(\Psi^{tt}(y_0^{**}, 1)) \le \|\Psi^{tt}(y_0^{**}, 1)\| \le \|\Psi\|.$$

So, $\Psi^{tt}$ attains its norm at $(y_0^{**}, 1)$ and by using the usual isometric identification of $\mathcal{L}(^2X \times Y)$ and $L(X, Y^*)$ (bounded linear operators from $X$ to $Y^*$) given by

$$\Psi(x, y) = T(x)(y),$$

the first Arens transpose of $\Psi$ is essentially the operator $T$. So, $\Psi^t(1, x)(y) = T(x)(y)$. Hence, if $\Psi^t$ attains its norm, so also does $T$. Since it is not true in general that the norm attaining operators from a Banach space $X$ to its dual form a dense set in the space $L(X, X^*)$ (see [1, Theorem 6]), the best we can say, for any Banach spaces, is that the set

$$\{\Psi \in \mathcal{L}(^2(X \times Y)) : \Psi^{tt} \text{ attains its norm}\}$$

is dense in $\mathcal{L}(^2(X \times Y))$.

The result we just showed is the version for bilinear forms of the theorem by J. Lindenstrauss who proved the denseness of the set of operators whose second adjoints attain their norms (see [19, Theorem 1] and also [23, Proposition 4]).

Now we will consider spaces with lots of norm one projections and the special class of multilinear mappings which are weakly continuous when restricted to bounded sets. If $X_1, \ldots, X_N, Y$ are Banach spaces, we will denote by $\mathcal{WL}(^N(X_1 \times \ldots \times X_N); Y)$ the space of $N$-linear mappings on $X_1 \times \ldots \times X_N$ with values in $Y$ that are weakly continuous on bounded sets, i.e., the images of bounded and weakly convergent nets are norm convergent; if all the spaces $X_i$ are equal to $X$, then $\mathcal{WL}_s(^NX; Y)$ will be the subset of symmetric mappings of $\mathcal{WL}(^NX; Y)$. We get a positive result for symmetric multilinear forms, a case in which very little is known.

THEOREM 3. *Let $N \ge 2$ be a natural number and for each $i \le N$, assume that $X_i$ is a Banach space satisfying the following condition: For every finite-dimensional space $F$, every operator $T : X_i \to F$ and $\varepsilon > 0$, there is a norm one projection $P : X_i \to X_i$ with finite-dimensional range such that $\|T - TP\| \le \varepsilon$. Then, for any Banach space $Y$, the set*

$$\{\phi \in \mathcal{WL}(^N(X_1 \times \ldots \times X_N); Y) : \phi \text{ attains its norm}\}$$

*is norm dense in the space $\mathcal{WL}(^N(X_1 \times \ldots \times X_N); Y)$ and the parallel assertion also holds for the subset of symmetric mappings in $\mathcal{WL}_s(^NX; Y)$.*

Proof. Fix a multilinear mapping $\phi : X_1 \times \ldots \times X_N \to Y$ and assume that $\phi$ is weakly continuous on bounded sets. Now, by using the proof of [6, Theorem 2.9] (or [12, Theorem 2.2.1] for multilinear forms), the associated operator

$$T_1 : X_1 \to \mathcal{WL}(^{N-1}(X_2 \times \ldots \times X_N); Y)$$

given by

$$T_1(x_1)(x_2, \ldots, x_N) = \phi(x_1, \ldots, x_N) \quad (x_i \in X_i,\ 1 \le i \le N)$$

is compact.



Now, the assumption on $X_1$ implies that $X_1^*$ has the 1-metric approximation property (see [17, Lemma 3.1] and for instance [20, Theorem 1.e.5]), and so, for any $\varepsilon > 0$, there is a finite rank operator $S_1$ such that

$$\|S_1 - T_1\| \le \frac{\varepsilon}{2N}.$$

By the hypothesis on $X_1$ again, there is a norm one projection $P_1$ on $X_1$ with finite rank so that

$$\|S_1 - S_1P_1\| \le \frac{\varepsilon}{2N}.$$

Thus far, we have a multilinear mapping (associated with $S_1$)

$$\phi_1 : X_1 \times \ldots \times X_N \to Y$$

given by

$$\phi_1(x_1, \ldots, x_N) = S_1(x_1)(x_2, \ldots, x_N) \quad (x_i \in X_i,\ i \le N).$$

Note also that, since $S_1$ is a finite rank operator whose values are weakly continuous mappings on bounded sets, it follows that $\phi_1$ is again weakly continuous on bounded sets. Now, by writing the conditions satisfied by the operator $S_1$ in terms of $\phi_1$, we get

$$\|\phi_1 - \phi\| \le \frac{\varepsilon}{2N} \tag{1}$$

and

$$\begin{aligned}\|\phi_1(x_1, \ldots, x_N) &- \phi_1(P_1x_1, x_2, \ldots, x_N)\| \\ &= \|(S_1x_1 - S_1P_1x_1)(x_2, \ldots, x_N)\| \le \frac{\varepsilon}{2N}\|x_1\| \ldots \|x_N\|.\end{aligned}$$

Now, we can repeat the same procedure using the multilinear mapping $\phi_1$ instead of $\phi$, and the compactness of the operator

$$T_2 : X_2 \to \mathcal{WL}(^{N-1}(X_1 \times X_3 \times \ldots \times X_N); Y)$$

associated with $\phi_1$, that is,

$$T_2(x_2)(x_1, x_3, \ldots, x_N) = \phi_1(x_1, x_2, x_3, \ldots, x_N) \quad (x_i \in X_i,\ 1 \le i \le N).$$

Now, after $N$ steps, we get, for any $i \le N$, a norm one projection $P_i : X_i \to X_i$ with finite-dimensional range and an $N$-linear mapping $\phi_i \in \mathcal{WL}(X_1 \times \ldots \times X_N; Y)$ such that

$$\|\phi_{i+1} - \phi_i\| \le \frac{\varepsilon}{2N} \quad (i \le N - 1) \tag{2}$$

and

$$\|\phi_j(x_1,\ldots,x_N)-\phi_j(x_1,\ldots,P_jx_j,\ldots,x_N)\| \le \frac{\varepsilon}{2N}\|x_1\|\ldots\|x_N\|. \tag{3}$$

Finally, the multilinear mapping given by

$$\Psi(x_1,\ldots,x_N)=\phi_N(P_1x_1,\ldots,P_Nx_N) \quad ((x_1,\ldots,x_N)\in X_1\times\ldots\times X_N)$$

attains its norm, since $P_i(B_{X_i})=B_{P_i(X_i)}$ for each $i\le N$ and each projection $P_i$ has finite rank.

Now, we will show that $\|\phi-\Psi\|\le\varepsilon$; if we fix $x_i\in B_{X_i}$ $(i\le N)$, we clearly have

$$\begin{aligned}
&\|(\phi-\Psi)(x_1,\ldots,x_N)\|\\
&\quad=\|\phi(x_1,\ldots,x_N)-\phi_N(P_1x_1,\ldots,P_Nx_N)\|\\
&\quad=\Big\|(\phi-\phi_1)(x_1,\ldots,x_N)\\
&\qquad+\sum_{i=1}^{N}\phi_i[(P_1x_1,\ldots,P_{i-1}x_{i-1},x_i,\ldots,x_N)\\
&\qquad-(P_1x_1,\ldots,P_ix_i,x_{i+1},\ldots,x_n)]\\
&\qquad+\sum_{i=1}^{N-1}(\phi_i-\phi_{i+1})(P_1x_1,\ldots,P_ix_i,x_{i+1},\ldots,x_N)\Big\|\\
&\quad=\|\phi-\phi_1\|+\sum_{i=1}^{N}\|\phi_i[(P_1x_1,\ldots,P_{i-1}x_{i-1},x_i,\ldots,x_N)\\
&\qquad-(P_1x_1,\ldots,P_ix_i,x_{i+1},\ldots,x_N)]\|\\
&\qquad+\sum_{i=1}^{N-1}\|\phi_i-\phi_{i+1}\|\\
&\quad\le\frac{\varepsilon}{2N}+\frac{\varepsilon}{2}+(N-1)\frac{\varepsilon}{2N} \quad \text{(by (1), (3) and (2))}\\
&\quad=\varepsilon,
\end{aligned}$$

as we wanted to show.

If the multilinear mapping $\phi$ fixed at the beginning of the proof is also symmetric we proceed as before and get the multilinear mapping $\Psi$ satisfying $\|\phi-\Psi\|\le\varepsilon$. Now, consider the partial operators $R_i:X\to\mathcal{WL}(^{N-1}X;Y)$ associated with $\Psi$, given by

$$R_i(x_i)(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N)=\Psi(x_1,\ldots,x_N) \quad (x_i\in X).$$

Since $R_iP_i=R_i$, the operator $R:X\to\bigoplus_{i=1}^{N}\mathcal{WL}(^{N-1}X,Y)$ given by $Rx=(R_1x,\ldots,R_Nx)$ is also a finite rank operator, hence there exists a norm one projection $P$ on $X$ with finite rank so that $\|R-RP\|\le\varepsilon/N$. Since we can consider the maximum norm in the finite sum, this clearly implies that

$$\|R_i-R_iP\|\le\varepsilon/N \quad (1\le i\le N).$$

Now, if we define

$$\Psi'(x_1,\ldots,x_N)=\Psi(Px_1,\ldots,Px_N),$$

then $\Psi'$ is weakly continuous on bounded sets. With the same kind of trick we did for the previous case, it is easily checked that

$$\|\Psi'-\Psi\|\le\varepsilon, \quad \text{so} \quad \|\phi-\Psi'\|\le 2\varepsilon,$$

and finally the symmetrization $\Psi'_{\rm s}$ of $\Psi'$ is the multilinear mapping we were looking for. It is clear that it attains its norm since

$$\Psi'_{\rm s}(x_1,\ldots,x_N)=\Psi'_{\rm s}(Px_1,\ldots,Px_N) \quad (x_i\in X). \ \blacksquare$$

Let us note that, as a consequence of the polarization identity, the symmetric $N$-linear mapping $\phi$ associated with a polynomial on $X$ with weakly continuous restrictions to bounded sets is also weakly continuous. So, by using the same proof as in the symmetric case we also get the analogous result for polynomials.

PROPOSITION 4. *Let $X$ be a Banach space such that for every finite-dimensional space $F$, every operator $T:X\to F$ and any $\varepsilon>0$, there is a finite rank and norm one projection $P$ on $X$ with $\|T-TP\|<\varepsilon$. Then, for any $N\ge 2$ and any Banach space $Y$, the set of norm attaining $N$-homogeneous polynomials which are weakly continuous on bounded sets is dense in the space of $N$-homogeneous polynomials from $X$ to $Y$ which are weakly continuous on bounded sets.*

Now, we will try to exhibit "classical" Banach spaces for which the previous results can be applied.

COROLLARY 5. *For $N\ge 2$ and $i\le N$, let $X_i$ be a Banach space of one of the following types*:

(a) *A space with a shrinking and monotone finite-dimensional decomposition.*

(b) *$C_0(L)$, for some locally compact and Hausdorff topological space $L$.*

(c) *$L_p(\mu)$, where $\mu$ is a finite measure and $1\le p\le\infty$.*

*Then the subset of the norm attaining multilinear mappings which are weakly continuous on bounded sets is dense in $\mathcal{WL}(^N(X_1\times\ldots\times X_N);Y)$, for any Banach space $Y$. If all the spaces $X_i$ coincide, also the set of symmetric multilinear mappings attaining their norms is dense in $\mathcal{WL}_{\rm s}(^NX;Y)$ and the analogous statement also holds for polynomials weakly continuous on bounded sets.*

Proof. If $X$ has a shrinking and monotone finite-dimensional decomposition $\{X_n\}$, then for any finite set $x_1^*, \ldots, x_n^*$ in $X^*$, and for any $\varepsilon > 0$, we can choose $m$ large enough so that

$$\|P_m^* x_i^* - x_i^*\| \le \varepsilon,$$

where $\{P_m\}$ is the sequence of projections associated with $\{X_n\}$. Monotonicity gives us $\|P_m\| \le 1$, so $X$ satisfies the assumption of Theorem 3 or Proposition 4 (see [17, Lemma 3.1]). Also $C_0(L)$ and $L_p(\mu)$ share the same property (see [17, Proposition 3.2]). ∎

As we mentioned in the introduction $L_1[0,1]$ is a space not satisfying the denseness of the norm attaining bilinear forms (or polynomials), so, for the spaces considered in Theorem 3 or Proposition 4, only results in this line for concrete classes of multilinear mappings or polynomials can be obtained.

Sometimes the space $\mathcal{WL}(^N(X_1 \times \ldots \times X_N); Y)$ coincides with the set of all continuous $N$-linear forms; this is the case for a space with property $S_p$ for some $p > N$ (see [13, Theorem 2.5] and [11, §2, Lemma]), so Corollary 5 generalizes Proposition 2.10 of [16]. More generally, if each space $X_i$ has separable dual, then there are some criteria which guarantee that the coincidence $\mathcal{WL} = \mathcal{L}$ holds, in terms of the Gonzalo–Jaramillo indices (see [11, §2, Lemma] and [13, Examples 1.3 and Theorem 2.5]). For instance, this happens for $X_1 = X_2 = c_0$, $X_3 = \ell_p$ for $1 < p < \infty$ and $Y = \mathbb{K}$ (the scalar field), so the set of norm attaining 3-linear mappings on $c_0 \times c_0 \times \ell_p$ is dense in $\mathcal{L}(^3(c_0 \times c_0 \times \ell_p))$. For a survey where examples of spaces for which the coincidence $\mathcal{WL} = \mathcal{L}$ holds see [15]. If $K$ is dispersed, A. Pełczyński proved that all the multilinear forms on $C(K)$ are weakly continuous [21, Corollary 4.4]. D. Leung provided an example of a space with a monotone shrinking basis and without the Dunford–Pettis property satisfying the above-mentioned coincidence; this space is also hereditarily $c_0$, so the known results cannot be applied to it [18, Example 13].

To finish, similar ideas to those used by J. M. Baker in [7] for norm attaining operators can also be applied to the questions we are dealing with. Certain isomorphic assumptions on the spaces allow us to strengthen the assertion of Theorem 1. Before stating this, let us recall that a Banach space $X$ has the *Dunford–Pettis property* if for every Banach space $Y$, any weakly compact operator $T : X \to Y$ maps weakly Cauchy sequences into strongly Cauchy sequences (see [14, Définition 1.2.2] and [10, Theorem, p. 177]). A. Grothendieck showed that the spaces $C(K)$ and $L_1(\mu)$ share this property [14, Théorème 1.3.1], so these results are also related to Corollary 5.

PROPOSITION 6. *Let $X$ be a Banach space for which $B_{X^{**}}$ is $w^*$-sequentially compact. Then*



(i) *If $Y$ is a Banach space, then any bilinear form $\phi$ on $X \times Y$ which is weakly continuous on bounded sets is such that $\phi^{ttt}$ attains its norm. The same result also holds for the Aron–Berner extension to $X^{**}$ of a polynomial on $X$ which is weakly continuous on bounded sets.*

(ii) *If $X$ has the Dunford–Pettis property and $Y$ is reflexive, then all the bilinear forms $\phi$ on $X \times Y$ are such that $\phi^{ttt}$ attains its norm on $X^{**} \times Y$.*

Proof. Let $\phi : X \times Y \to \mathbb{K}$ be a bilinear form and consider the corresponding operator $T : X \to Y^*$ given by

$$T(x)(y) = \phi(x, y) \quad (x \in X,\ y \in Y).$$

Choose a sequence $\{x_n\}$ in $B_X$ such that

$$\|Tx_n\| \to \|T\| = \|\phi\|.$$

Passing to a subsequence we can assume that $\{x_n\}$ converges in the $w^*$-topology of $X^{**}$ to an element $x^{**} \in B_{X^{**}}$. If we are assuming (i) then $\phi$ is weakly continuous on bounded sets, so $T$ is compact [6, Theorem 2.9]. Now, since $\{x_n\}$ is weakly Cauchy, either of the hypotheses implies that the image of a subsequence under $T$ is norm convergent to some $y^* \in Y^*$. Again we assume that this happens to $\{x_n\}$:

$$\{\|Tx_n - y^*\|\} \to 0.$$

On the other hand, since $\{x_n\}$ converges to $x^{**}$ in the $w^*$-topology of $X^{**}$, it follows that $\{Tx_n\}$ converges to $T^{**}x^{**}$ in the $w^*$-topology of $Y^{***}$, so $T^{**}x^{**} = y^*$. Now, we choose a sequence $\{y_n\}$ in $B_Y$ such that

$$\{Tx_n(y_n)\} \to \|\phi\|$$

and in view of the Banach–Alaoglu Theorem, there is a $w^*$-cluster point $y^{**}$ of $\{y_n\}$ in $Y^{**}$. Consider the inequality

$$\begin{aligned} &|Tx_n(y_n) - T^{**}x^{**}(y^{**})| \\ &\qquad \le |Tx_n(y_n) - T^{**}x^{**}(y_n)| + |T^{**}x^{**}(y_n) - T^{**}x^{**}(y^{**})| \\ &\qquad \le \|Tx_n - T^{**}x^{**}\| + |y^*(y_n - y^{**})|. \end{aligned}$$

By using the fact that $y^{**}(y^*)$ is a cluster point of $\{y^*(y_n)\}$ and the norm convergence of $\{Tx_n\}$ to $T^{**}x^{**}$, we see that also $\{Tx_n(y_n)\} \to T^{**}x^{**}(y^{**})$ and $|T^{**}x^{**}(y^{**})| = \|\phi\|$. The definition of Arens transpose is such that $T^{**}x^{**}(y^{**}) = \phi^{ttt}(x^{**}, y^{**})$, so $\phi^{ttt}$ attains its norm at $(x^{**}, y^{**})$.

The result for polynomials can be obtained with an identical argument, taking at the beginning $y_n = x_n$, for each $n$, and using the fact that multilinear forms which are weakly continuous on bounded sets extend $w^*$-continuously on bounded sets to mappings on the bidual [12, Theorem 2.2.1]. ∎

## References

[1] M. D. Acosta, F. J. Aguirre and R. Payá, *There is no bilinear Bishop–Phelps Theorem*, Israel J. Math. 93 (1996), 221–228.

[2] F. Aguirre, *Algunos problemas de optimización en dimensión infinita: aplicaciones lineales y multilineales que alcanzan su norma*, doctoral dissertation, University of Granada, 1996.

[3] R. Arens, *Operations induced in function classes*, Monatsh. Math. 55 (1951), 1–19.

[4] —, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. 2 (1951), 839–848.

[5] R. M. Aron, C. Finet and E. Werner, *Some remarks on norm-attaining $n$-linear forms*, in: Function Spaces, K. Jarosz (ed.), Lecture Notes in Pure and Appl. Math. 172, Marcel Dekker, New York, 1995, 19–28.

[6] R. M. Aron, C. Hervés and M. Valdivia, *Weakly continuous mappings on Banach spaces*, J. Funct. Anal. 52 (1983), 189–204.

[7] J. M. Baker, *A note on compact operators which attain their norm*, Pacific J. Math. 82 (1979), 319–321.

[8] Y. S. Choi, *Norm attaining bilinear forms on $L_1[0,1]$*, J. Math. Anal. Appl. 211 (1997), 295–300.

[9] Y. S. Choi and S. G. Kim, *Norm or numerical radius attaining multilinear mappings and polynomials*, J. London Math. Soc. 54 (1996), 135–147.

[10] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R.I., 1977.

[11] V. Dimant and I. Zalduendo, *Bases in spaces of multilinear forms over Banach spaces*, J. Math. Anal. Appl. 200 (1996), 548–566.

[12] T. W. Gamelin, *Analytic functions on Banach spaces*, in: Complex Potential Theory, P. M. Gauthier and G. Sabidussi (eds.), Kluwer, 1994, 187–233.

[13] R. Gonzalo and J. Jaramillo, *Compact polynomials between Banach spaces*, Extracta Math. 8 (1993), 42–48.

[14] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math. 5 (1953), 129–173.

[15] J. Gutierrez, J. Jaramillo and J. G. Llavona, *Polynomials and geometry of Banach spaces*, Extracta Math. 10 (1995), 79–114.

[16] M. Jiménez Sevilla and R. Payá, *Norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces*, Studia Math. 127 (1998), 99–112.

[17] J. Johnson and J. Wolfe, *Norm attaining operators*, ibid. 65 (1979), 7–19.

[18] D. Leung, *Uniform convergence of operators and Grothendieck spaces with the Dunford–Pettis property*, Math. Z. 197 (1988), 21–32.

[19] J. Lindenstrauss, *On operators which attain their norm*, Israel J. Math. 1 (1963), 139–148.

[20] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Springer, Berlin, 1977.

[21] A. Pełczyński, *A property of multilinear operations*, Studia Math. 16 (1957), 173–182.

[22] R. A. Poliquin and V. Zizler, *Optimization of convex functions on $w^*$-convex sets*, Manuscripta Math. 68 (1990), 249–270.

[23] V. Zizler, *On some extremal problems in Banach spaces*, Math. Scand. 32 (1973), 214–224.

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, Spain
E-mail: dacosta@goliat.ugr.es

*Received September 22, 1997* (3963)
*Revised version March 2, 1998*