This completes the proof.

Neither the sufficient condition nor the necessary condition are valid in the real case, as the following one-dimensional examples show, respectively.

EXAMPLE 1. $f(x) = 4x^3(1-x^2)$, $|x| < 1$.

EXAMPLE 2. $f(x) = x - x^3$, $|x| < 1$.

Let us remark that the analogue of Theorem 2 for differential equations was earlier proved by Yu. I. Lyubich [11] and the same method of proof is applicable to iterations. The proof given in Theorem 2 is somewhat different and, formally, Theorem 2 is an infinite-dimensional version.

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The uniform zero-two law for positive operators in Banach lattices

by

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Dedicated to Shaul Foguel upon his retirement

Abstract. Let T be a positive power-bounded operator on a Banach lattice. We prove: (i) If $\inf_n \|T^n(I-T)\| < 2$, then there is a $k \geq 1$ such that $\lim_{n\to\infty} \|T^n(I-T^k)\| = 0$. (ii) $\lim_{n\to\infty} \|T^n(I-T)\| = 0$ if (and only if) $\inf_n \|T^n(I-T)\| < \sqrt{3}$.

In their ground breaking paper [OSu], Ornstein and Sucheston proved that if T is a positive contraction of L_1, then $\sup_{\|f\|_1 \leq 1} \lim_n \|T^n(I-T)f\|$ is 0 or 2, and coined the term *zero-two* law. Using their method, Foguel [F] proved that if T is a positive contraction of L_1, then $\lim_n \|T^n(I-T)\|$ is 0 or 2 (the *uniform* zero-two law). This easily implies that if T is a positive contraction of $C(K)$ with K compact Hausdorff, then $\lim_n \|T^n(I-T)\|$ is 0 or 2.

Using the regular norm (the norm of the modulus), Zaharopol [Z_1] restated [F] as

$$(*) \qquad \inf_n \|T^{n+1} - T^n\|_r < 2 \Rightarrow \lim_n \|T^{n+1} - T^n\| = 0.$$

He proved $(*)$ for positive contractions of L_p spaces $(1 < p < \infty)$, $p \neq 2$. Katznelson and Tzafriri [KT] removed the restriction $p \neq 2$ of [Z_1], and proved $(*)$ for a larger class of Banach lattices. Finally, Schaefer [S_2] proved $(*)$ for a positive contraction T in any Banach lattice.

The reverse implication in $(*)$ is false: a positive contraction in L_p can satisfy $\lim_n \|T^{n+1} - T^n\| = 0$ and $\inf_n \|T^{n+1} - T^n\|_r = 2$ (see [W_2]). For certain Banach lattices, a stronger version of $(*)$, in which the conclusion is $\lim_n \|T^{n+1} - T^n\|_r = 0$, was later proved in [$W_2$], [$Z_2$], [Sc].

In this note we prove that for a power-bounded positive operator T in a Banach lattice, $\inf_n \|T^n(I-T)\| < \sqrt{3}$ implies $\lim_{n\to\infty} \|T^n(I-T)\| = 0$. For contractions in L_p this follows from [W_1] (see also [M]).

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Recall that for T power-bounded, the spectral radius $r(T)$ is at most 1. It was known [S_1] that the asymptotic behaviour of the powers of a *positive* operator with $r(T)=1$ depends upon the study of $\sigma(T)\cap\Gamma$, where $\Gamma=\{\lambda\in\mathbb{C}:|\lambda|=1\}$. For their result, Katznelson and Tzafriri proved the following general theorem, which is a major contribution to operator ergodic theory. (See [AR] for its equivalence to an old result of Gelfand. The survey [Ze] contains several generalizations of Gelfand's result, and has a comprehensive bibliography.)

THEOREM A. *Let T be a power-bounded operator on a complex Banach space. Then*

$$\lim_{n\to\infty}\|T^n(I-T)\|=0\Leftrightarrow\sigma(T)\cap\{\lambda\in\mathbb{C}:|\lambda|=1\}\subset\{1\}.$$

PROPOSITION 1. *Let T be a positive power-bounded operator on a Banach lattice. If $\sigma(T)\cap\Gamma\neq\Gamma$, then it is a finite set, and there is $k\geq 1$ such that $\sigma(T^k)\cap\Gamma\subset\{1\}$.*

Proof. If $r(T)<1$, the result is obvious, with $k=1$. Assume $r(T)=1$. Since T is positive, the peripheral spectrum $\sigma(T)\cap\Gamma$ is cyclic [S_1, p. 327] (i.e., if $\alpha\in\sigma(T)$ with $|\alpha|=1$, then all powers α^j are in $\sigma(T)$). Since the peripheral spectrum is not Γ, it does not contain any irrational rotation, so any spectral point on the unit circle is a root of unity. Let $1\neq\alpha\in\sigma(T)\cap\Gamma$, and let d be the smallest positive integer with $\alpha^d=1$. Since $\sigma(T)\cap\Gamma$ is cyclic, it contains the group of *all* dth roots of unity. Since the peripheral spectrum is closed, and is not all of Γ, it consists of only finitely many such groups, of orders $d_1,\ldots,d_m$. We now take $k=\prod_{j=1}^m d_j$, and use the spectral mapping theorem. ∎

PROPOSITION 2. *Let T be a positive power-bounded operator on a Banach lattice. Then $\sigma(T)\cap\Gamma\neq\Gamma$ if and only if there is $k\geq 1$ such that*

$$\lim_{n\to\infty}\|T^n(I-T^k)\|=0.$$

Proof. (i) If $\sigma(T)\cap\Gamma\neq\Gamma$, then by Proposition 1 there is $k\geq 1$ such that $\sigma(T^k)\cap\Gamma\subset\{1\}$. By Theorem A, $\lim_{n\to\infty}\|T^{kn}(I-T^k)\|=0$, which proves the claim.

(ii) If $\lim_{n\to\infty}\|T^n(I-T^k)\|=0$, then $\sigma(T^k)\cap\Gamma\subset\{1\}$, so by the spectral mapping theorem $\sigma(T)\cap\Gamma\neq\Gamma$. ∎

THEOREM 3. *Let T be a positive power-bounded operator on a Banach lattice. If $\inf_n\|T^n(I-T)\|<2$, then there is $k\geq 1$ such that*

$$\lim_{n\to\infty}\|T^n(I-T^k)\|=0.$$

Proof. By the spectral mapping theorem, for some n we have

$$\sup\{|\lambda^n|\cdot|1-\lambda|:\lambda\in\sigma(T)\}=r(T^n(I-T))\leq\|T^n(I-T)\|<2.$$

Hence -1 is not in $\sigma(T)$. We can now apply Proposition 2. ∎

REMARKS. 1. If T is a contraction, then we have $\inf_n\|T^n(I-T)\|=\lim_n\|T^n(I-T)\|$.

2. The simple example of the contraction $T(a,b)=(b,a)$ on $\mathbb{R}^2$ with the Euclidean norm shows that the condition $\inf_n\|T^n(I-T)\|<2$ is not necessary.

3. The example in [W_1] of a positive contraction on L_2 satisfying $\lim_n\|T^n(I-T)\|=\sqrt{3}$ shows that k in the theorem may be different from 1.

COROLLARY 4. *Let T be a positive power-bounded operator on a Banach lattice. Then the following are equivalent*:

(i) $\Gamma\subset\sigma(T)$.
(ii) $\|T^n(I-T^k)\|\geq 2$ *for every* $n,k\geq 1$.

Proof. (i)⇒(ii). Assume (ii) fails for k' and some n. By the proof of the theorem, -1 is not in $\sigma(T^{k'})$. By the spectral mapping theorem, (i) fails—a contradiction.

(ii)⇒(i) by Proposition 2. ∎

REMARKS. 1. If T is a contraction, then (ii) becomes $\|T^n(I-T^k)\|=2$ for every $n,k\geq 1$.

2. See [Sc, Corollary 5] for a similar relation between the order spectrum and the regular norm.

THEOREM 5. *Let T be a positive power-bounded operator on a Banach lattice. Then the following are equivalent*:

(i) $\inf_n\|T^n(I-T)\|<\sqrt{3}$.
(ii) $\lim_{n\to\infty}\|T^n(I-T)\|=0$.
(iii) $\sigma(T)\cap\Gamma\subset\{1\}$.

Proof. The equivalence of (ii) and (iii) is Theorem A [KT], and obviously (ii)⇒(i).

(i)⇒(iii). We have to prove the implication only for $r(T)=1$. The proof of the previous corollary shows $\sigma(T)\cap\Gamma\neq\Gamma$. The proof of Proposition 1 shows that the peripheral spectrum is the union of finitely many groups of roots of unity. Let $\alpha\in\sigma(T)\cap\Gamma$ be a primitive root of unity, of order d, so $\alpha^j\in\sigma(T)$ for $0\leq j<d$. Since for any $\lambda\in\sigma(T)\cap\Gamma$ we have

$$|1-\lambda|=|\lambda^n|\cdot|1-\lambda|\leq r(T^n(I-T))\leq\|T^n(I-T)\|,$$

condition (i) implies that $|1-\alpha^j|<\sqrt{3}$ for $0\leq j<d$, so the arc $\{z\in\Gamma:2\pi/3\leq\arg z\leq 4\pi/3\}$ does not contain any α^j. Hence $d=1$, and (iii) holds. ∎

REMARKS. 1. For any Banach lattice E, Theorem 5 yields $c_E \geq \sqrt{3}$, where

$$c_E = \sup\{c : \lim \|T^n(I-T)\| < c \text{ for a positive contraction } T \Rightarrow \lim \|T^n(I-T)\| = 0\}.$$

In fact, c_E is a maximum, and $c_E \leq 2$. Since $c_{L_2} = \sqrt{3}$ (see [W1]), $\sqrt{3}$ is the best bound.

2. For E an abstract L_p space ($1 \leq p < \infty$), $c_E \geq \sqrt{3}$ was proved by Martinez [M] as an application of a certain representation he obtained for such spaces. However, for $E = L_p$ with $1 < p < \infty$, $c_E > \sqrt{3}$ for $p \neq 2$ is already proved in [W1], with c_E computed in [B].

3. The proof in [M] uses part (ii) of the following corollary, which is proved there by an argument similar to ours in Theorem 5, and an application of Gelfand's theorem.

COROLLARY 6. *Let T be a positive power-bounded operator on a Banach lattice.*

(i) *If $r(I-T) < \sqrt{3}$, then all the equivalent conditions of Theorem* 5 *hold.*

(ii) *If T is an isometry, then $r(I-T) < \sqrt{3}$ if and only if $T = I$.*

Proof. (i) implies that $-1 \notin \sigma(T)$, so the result follows from the proof of (i)⇒(iii).

(ii) If $r(I-T) < \sqrt{3}$, we have $\|T^n(I-T)\| \to 0$ by (i), so when T is an isometry, $T = I$. ■

REMARKS. 1. The condition $r(I-T) < \sqrt{3}$ is not necessary for the conditions of Theorem 5 to hold: Let S be the operator induced on $L_p(\Gamma)$ (with Lebesgue measure) by an irrational rotation, and $T = \varepsilon I + (1-\varepsilon)S$ for $0 < \varepsilon < 1$. By [FWe], $\lim_n \|T^n(I-T)\| = 0$. Since $\sigma(S) = \Gamma$, we have $r(I-T) = (1-\varepsilon)r(I-S) = 2(1-\varepsilon)$.

2. The condition $r(I-T) < 2$, which implies the equivalent conditions of Proposition 2, is necessary, but not sufficient, for the conditions of Theorem 5 to hold.

PROBLEM. Is there a Banach lattice E, different from L_2, with $c_E = \sqrt{3}$?

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