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New examples of holomorphic foliations without algebraic leaves

by

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Abstract. We present a series of polynomial planar vector fields without algebraic invariant curves in $\mathbb{C}P^2$.

1. Introduction. In [1] Jouanolou has shown that the holomorphic foliation in $\mathbb{C}P^2$ defined in the affine plane by the vector field

$$\dot{x} = 1 - xy^s, \quad \dot{y} = x^s - y^{s+1}, \quad s \geq 2, \tag{1}$$

does not have algebraic leaves. This implies that the set of foliations without algebraic leaves is dense in the set of all foliations with fixed degree. Lins-Neto in [2] proved that this set is also open. (The degree of a foliation is defined as the number of its tangency points with a generic line.)

An important problem in the theory of analytic foliations is the problem of existence of strange limit sets, i.e. not containing algebraic leaves and singular points. There is a need for other explicit examples of algebraic foliations without algebraic leaves. The author thanks Prof. R. Moussu for asking this question.

In [3] the author presented his own proof of Jouanolou's theorem. (There are many of them in fact.) It turns out that the method of [3] works well for the following series of polynomial vector fields:

$$\dot{x} = y^a - x^{b+1}y^c, \quad \dot{y} = x^d - x^b y^{c+1}, \quad a \leq d \leq b + c. \tag{2}$$

THEOREM 1. *If the integer exponents a, b, c, d satisfy the assumptions* A.1–A.5 *stated below, then the system* (2) *does not have invariant algebraic curves in* $\mathbb{C}P^2$.

The systems (2) are very similar to the system (1) and probably will not be much more useful in the problem of limit sets. Anyway they are new.

Note also that the set of systems (2) is discrete in the space of all systems modulo the affine changes of variables and time. It would be very interest-

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ing to find an explicit 1-parameter family of polynomial systems without algebraic leaves.

The remaining part of the work is devoted to the proof of Theorem 1 which, as we have said, follows the proof from [3].

2. Symmetries. The Jouanolou system (1) admits a very large group of symmetries:

(i) the cyclic permutations of homogeneous coordinates, $\mathbb{Z}/3\mathbb{Z}$;

(ii) the complex conjugation $\varrho : (x, y) \to (\overline{x}, \overline{y})$, $\mathbb{Z}/2\mathbb{Z}$;

(iii) the cyclic group $\mathbb{Z}/N\mathbb{Z}$, $N = s^2 + s + 1$, with generator $\sigma : (x, y) \to (\zeta_N^{-s} x, \zeta_N y)$, $\zeta_N = e^{2\pi i/N}$.

For the system (2) the symmetry (i) breaks down (it was not used in [3] in fact), the conjugation (ii) holds and the cyclic symmetry (iii) is replaced by the cyclic group of order

$$N = (b + 1)(c + 1) - (a - c)(d - b).$$

The transformations from this group are of the form $\sigma^i : (x, y) \to (\mu x, \nu y)$, where $\mu^{b+1} = \nu^{a-c}$ and $\mu^{d-b} = \nu^{c+1}$. One can see that μ, ν are roots of unity of order N.

We also have $N = a(b + c - d) + c(d - a) + b + c + 1 > 0$.

3. Phase portrait

3.1. There are N singular points $P_0 = (1, 1) \in \mathbb{R}^2$ and $P_i = \sigma^i P_0$, $i = 1, \ldots, N - 1$, in $\mathbb{C}^2$ with the same phase portrait near them. The characteristic equation for the eigenvalues of P_0 is

$$\lambda^2 + (b + c + 2)\lambda + N = 0.$$

Our first assumption is that its discriminant is negative:

$$\text{(A.1)} \qquad (b + c + 2)^2 - 4N < 0.$$

In that case, the ratio of the eigenvalues at the point P_0 is non-real. Because the symmetries σ^i transform the phase portrait near P_0 to the phase portraits near P_i, also the ratios of the eigenvalues at P_i are the same and non-real. One also sees that the point P_0 is a stable focus. The points P_i are, in a sense, "complex" foci.

It is known that a singular point of the "complex" focus type has only two invariant local analytic curves passing through it (if $\dot{x} = \lambda_1 x$, $\dot{y} = \lambda_2 y$, then the phase curves are $x = 0$, $y = 0$ and the non-analytic curves $y = Cx^{-\lambda_2/\lambda_1}$).

Therefore, through each P_i only two invariant local analytic curves pass. Moreover, they intersect transversally.

3.2. If $a > 0$ and $d > 0$, then the point $Q = (0, 0)$ is also singular. The principal part of the vector field near Q is hamiltonian with Hamilton function $H = \frac{1}{a+1} y^{a+1} - \frac{1}{d+1} x^{d+1}$ and with separatrices $H = 0$. This picture also holds for the whole system.

One can see this using the blowing-up associated with the quasi-homogeneous filtration in the space of germs of vector fields: $d(x) = -d(\partial_x) = a+1$, $d(y) = -d(\partial_y) = d+1$. Then the hamiltonian part has degree $ad - 1$ and the terms $x^{b+1}y^c\partial_x$ and $x^b y^{c+1}\partial_y$ have greater degree $b(a + 1) + c(d + 1)$ (the difference is N). In this sense the system (2) is a small perturbation of the hamiltonian system and has also analytic separatrices $H + \ldots = 0$, where the dots mean terms of higher quasi-homogeneous degree.

We assume that there is exactly one local analytic irreducible invariant curve Γ_Q passing through Q:

$$\text{(A.2)} \qquad (a + 1, d + 1) = 1$$

(the greatest common divisor is 1).

The curve Γ_Q has a 1-dimensional real part.

3.3. The system (2) is chosen in such a way that the line at infinity is not invariant. The Jouanolou system does not have singularities at infinity but the system (2) can have some.

In the chart $z = 1/x$, $u = y/x$ we have the system

$$\text{(3)} \qquad \dot{z} = u^c - z^{b+c-a+1}u^a, \quad \dot{u} = z^{b+c-d} - z^{b+c-a}u^{a+1}$$

with singular point R: $z = u = 0$, existing for $d < b + c$ (recall that $a \le d$). Again the principal part of the system (3) is hamiltonian and we assume that only one irreducible analytic invariant curve Γ_R passes through R:

$$\text{(A.3)} \qquad (c + 1, b + c - d + 1) = 1.$$

In the chart $r = 1/y$, $v = x/y$ we get

$$\dot{r} = v^b - r^{b+c-d+1}v^d, \quad \dot{v} = r^{b+c-a} - r^{b+c-d}v^{d+1}$$

with singular point $S : r = v = 0$, existing for $a < b + c$. We assume that only one irreducible analytic invariant curve Γ_S passes through S:

$$\text{(A.4)} \qquad (b + 1, b + c - a + 1) = 1.$$

We see that all the other trajectories of the system (2) meeting the line at infinity $z = 0$ (or $r = 0$) cross it transversally.

3.4. Because the divergence of the vector field (2)

$$\text{(4)} \qquad \operatorname{div} = -(b + c + 2)x^b y^c$$

is negative for $x > 0, y > 0$, the whole quadrant $\Delta = \{x > 0, y > 0\} \subset \mathbb{R}^2$ is attracted by the focus P_0. (Otherwise, by the Poincaré–Bendixson theorem there should exist a limit cycle γ surrounding the stable focus P_0; by the Dulac criterion $\int_\gamma \mathrm{div} < 0$ it should be also stable.)

Because the curves $\Gamma_Q, \Gamma_R, \Gamma_S$ have some real parts in Δ, they cannot be algebraic (spirals near P_0).

Notice also that if the numbers b and c are both even, then $\mathrm{div} \le 0$ (see (4)) and the whole region $\mathbb{R}^2 \setminus \Gamma_Q \setminus \Gamma_R \setminus \Gamma_S$ is attracted by P_0 and/or other real focus P_i.

We can summarize the results of this section as follows.

LEMMA 3.5. *If the assumptions* (A.1)–(A.4) *hold, then*:

(i) *any invariant algebraic curve of the system* (2) *has singularities of the simplest kind, the double points*;

(ii) *it intersects the line at infinity transversally*;

(iii) *any two invariant algebraic curves intersect transversally and in the finite part of* $\mathbb{C}P^2$;

(iv) *there are no triple intersections of invariant algebraic curves.*

The statements (i), (iii) and (iv) follow from the fact that singular points and intersection points of invariant algebraic curves are also singular points of the holomorphic vector field. In particular, local branches of the invariant curves form analytic separatrices of the singular points. By 3.2–3.4 these singular points are not among Q, R, S. They can be among P_i's and we apply 3.1.

(ii) follows from 3.2.

4. Vector fields with given invariant algebraic curves. In [3] (Theorem 3, statement B) the following result was proved.

LEMMA 4.1. *Assume that irreducible algebraic curves* $C_1, \ldots, C_r$ *of degrees* $k_1, \ldots, k_r$ *respectively satisfy the conditions* (i)–(iv) *from Lemma* 3.5. *Let* $K(x, y)$ *be a polynomial of degree* k *vanishing at all the double points of these curves. If a polynomial vector field* V *of degree* $n < (\sum k_i) - k - 1$ *is tangent to all these curves, then* $V \equiv 0$.

REMARK. If we know that the curves are smooth, then the assertion of this lemma holds when $n < \sum k_i - 1$.

We shall apply this result to the vector field (2) using its first component as the function K. We shall strive to prove the inequality

$$2(b + c + 1) < \sum k_i - 1 \tag{5}$$

(under the assumption that some invariant algebraic curves exist).

5. Proof of the inequality (5). Assume that there is an invariant irreducible algebraic curve C_0. Then the curves $\sigma^i C_0$, $i = 1, \ldots, N-1$ (see Section 2), are also invariant. (It may happen that $\sigma^i C_0 = C_0$.) The curve $C = \bigcup \sigma^i C_0$ can be decomposed into irreducible components $C_1, \ldots, C_r$ with the properties (i)–(iv). The sum of their degrees is the degree of C and is equal to the number of intersection points of C with the line at infinity.

This number is $\ge N$:

If $a_0 = (x_0 : y_0 : 0) \in C_0$ and $\sigma^i a_0 = \sigma^i a_0$, $i \ne j$, then either $a_0 = (1 : 0 : 0) = R$ or $a_0 = (0 : 1 : 0) = S$. By 3.3 no invariant algebraic curve can pass through R and S.

Our last assumption

$$2(b + c + 1) < N - 1 \tag{A.5}$$

(equivalent to the inequality (5)) allows us to complete the proof of Theorem 1.

REMARK. One can easily see that the assumptions (A.1)–(A.5) are such that there is an infinite series of systems different from (1) which obey them.

6. Concluding remarks. Following our proof more carefully one can extend the list of systems without invariant algebraic curves.

In Section 3 (points 3.2 and 3.3) we assume that at most one irreducible analytic separatrix can pass through any of the points Q, R or S. In fact, we can allow two (but not more) invariant analytic curves passing through them. Then we have $(a + 1, d + 1) = 2$, etc.

We only need to ensure that, if these curves are singular and/or intersect non-transversally (i.e. some of the properties from Lemma 3.5 fail), then their real parts go to some real foci P_0 and $P_i = \sigma^i P_0$ (i.e. cannot be algebraic). For this an additional condition must be fulfilled: *at least two specific points from* $\{P_i\}$ *lie in* $\mathbb{R}^2$.

For example, if near Q we have $H = \frac{1}{4}x^4 - \frac{1}{6}y^6$ and $P_0 = (1, 1)$, $P_i = (-1, -1)$ are singular foci, then the two cusp curves $x^2/2 \pm y^3/\sqrt{6} + \ldots$ end in spirals around these foci.

We do not present the precise formulas.

Also we have not used the complex conjugation ϱ. Note that in Section 5 the curves $\varrho\sigma^i C_0 = \overline{\sigma^i C_0}$ are also invariant. So, the curve $\widetilde{C} = \bigcup \sigma^i C_0 \cup \bigcup \varrho\sigma^i C_0$ can have $\ge 2N$ intersection points with the line at infinity.

However, this holds when N is odd and we know that $\widetilde{C}$ has no real intersections with the line at infinity.

(We can represent $a_0 = (x_0 : 1 : 0)$ as the complex number $x_0 \ne 0$. We have $\sigma^i(a_0) = (x_i : 1 : 0)$, $x_i = \eta^i x_0$, where η is a root of unity. If $\{\bar{x}_0, \ldots, \bar{x}_{N-1}\} \cap \{x_0, \ldots, x_{N-1}\} \ne \emptyset$ and N is odd, then these two sets coincide and contain a real point.)

The absence of real points of C at infinity can be ensured when almost the whole $\mathbb{R}^2$ is attracted to the real foci P_i, i.e. when b and c are even (we use the formula (4) for divergence).

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Asymptotic stability in the Schauder fixed point theorem

by

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Abstract. This note presents a theorem which gives an answer to a conjecture which appears in the book *Matrix Norms and Their Applications* by Belitskiĭ and Lyubich and concerns the global asymptotic stability in the Schauder fixed point theorem. This is followed by a theorem which states a necessary and sufficient condition for the iterates of a holomorphic function with a fixed point to converge pointwise to this point.

The object of this note is to settle a conjecture raised by Belitskiĭ and Lyubich in 1984 concerning the global asymptotic stability in the Schauder fixed point theorem.

1. Conjecture of Belitskiĭ and Lyubich. Let E be a (real or complex) Banach space with a non-empty bounded convex open subset D, and let $f : \overline{D} \to \overline{D}$ ($\overline{D}$ stands for the closure of D) be a compact continuous map. The celebrated Schauder fixed point theorem [13], which is one of the fundamental theorems in nonlinear functional analysis, asserts that there exists a point $\widehat{x} \in \overline{D}$ such that $f(\widehat{x}) = \widehat{x}$. For $x \in E$, denote by $f'(x)$ the Fréchet derivative of f evaluated at x. For a bounded linear operator A on E, $r(A)$ stands for the spectral radius of A. Under the assumption that f is continuously Fréchet differentiable, Belitskiĭ and Lyubich ([1], p. 41) proposed the following conjecture in 1984 concerning the asymptotic behaviour of the fixed point in the Schauder fixed point theorem.

Conjecture of Belitskiĭ and Lyubich. *Let E be a (real or complex) Banach space with an open subset Ω and $f : \Omega \to E$ be compact and continuously Fréchet differentiable in Ω. Suppose D is a non-empty bounded convex open subset of E such that $f(\overline{D}) \subset \overline{D} \subset \Omega$ and $\sup_{x\in\overline{D}} r(f'(x)) < 1$. Then*

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