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B^q for parabolic measures

by

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Abstract. If Ω is a Lip(1,1/2) domain, μ a doubling measure on $\partial_{\mathrm{p}}\Omega$, $\partial/\partial t - L_i$, $i = 0,1$, are two parabolic-type operators with coefficients bounded and measurable, $2 \le q < \infty$, then the associated measures ω_0, ω_1 have the property that $\omega_0 \in B^q(\mu)$ implies ω_1 is absolutely continuous with respect to ω_0 whenever a certain Carleson-type condition holds on the difference function of the coefficients of L_1 and L_0. Also $\omega_0 \in B^q(\mu)$ implies $\omega_1 \in B^q(\mu)$ whenever both measures are center-doubling measures. This is B. Dahlberg's result for elliptic measures extended to parabolic-type measures on time-varying domains. The method of proof is that of Fefferman, Kenig and Pipher.

A result of B. Dahlberg on two elliptic measures satisfying a $B^q(\mu)$ condition for μ a doubling measure is extended to parabolic-type measures on time-varying domains. The $B^q(\mu)$ condition for ω on $\partial\Omega$ is

$$\left(\frac{1}{\mu(\triangle_r(Q,s))}\int_{\Psi_r(Q,s)\cap\Omega}\left(\frac{d\omega}{d\mu}(\widehat{Q},\widehat{s})\right)^q d\mu(\widehat{Q},\widehat{s})\right)^{1/q} \le \frac{C}{\mu(\triangle_r)}\int_{\Psi_r\cap\Omega}\frac{d\omega}{d\mu}\,d\mu.$$

Here C is independent of (Q,s), $\triangle_r$ is a boundary cube in $\partial\Omega$, $\Psi_r(Q,s)$ is a cylinder of dimension r centered at (Q,s), and r is any real number with $0<r<r_0$.

Dahlberg [D] proved that if one elliptic measure ω_0 is in $B^q(\mu)$ and if a certain Carleson-type condition holds for the difference function of the coefficients of two elliptic operators L_0, L_1 on a domain D with respect to a doubling measure μ on ∂D, then the second measure ω_1 is also in $B^q(\mu)$.

The main result of this paper is to obtain the preservation of the B^q condition for parabolic-type operators on Lip(1,1/2) domains in $\mathbb{R}^{n+1}$. This result has been proved independently by Professor Kaj Nystrom [N].

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Specifically, let

$$\Big(\frac{\partial}{\partial t}-L_0\Big)u_0=0 \quad \text{in } \Omega, \quad u_0|_{\partial_p\Omega}=f\in L^p(\partial_p\Omega),$$
$$\Big(\frac{\partial}{\partial t}-L_1\Big)u_1=0 \quad \text{in } \Omega, \quad u_1|_{\partial_p\Omega}=f,$$

where

$$L_0=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial}{\partial x_j}\Big), \qquad L_1=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big(b_{ij}(x,t)\frac{\partial}{\partial x_j}\Big),$$

$a_{ij}(x,t)$ and $b_{ij}(x,t)$ are bounded and measurable. Then given a Carleson-type condition similar to the one in [FKP], Theorem 2.20, the following inequality will be proved:

$$\|N(u_1)\|_{L^p(\partial_p\Omega,d\mu)}\le c\|f\|_{L^p(\partial_p\Omega,d\mu)}$$

assuming $\omega_0\in B^q(d\mu)$. If ω_1 is a center-doubling measure, then also $\omega_1\in B^q(d\mu)$. The method of proof is an adaptation of the proof of Theorem 2.18 in [FKP].

The paper is organized as follows: Section 1 contains the basic set-up and definitions to be used in later sections; Section 2 gives some standard estimates for parabolic measures and solutions on Lip(1, 1/2) domains, these estimates are used in the proof of Theorem 4; Section 3 contains Theorem 4 and its proof. The main part of the proof is establishing the good-λ inequality in Lemma 6. Section 4 has a brief discussion on extensions of absolute continuity results to degenerate operator measures.

In addition to [FKP] the chief sources for the material presented here are [RB] and [FGS].

1. A bounded domain $\Omega\subseteq\mathbb{R}^{n+1}=\mathbb{R}^{n-1}\times\mathbb{R}^1\times\mathbb{R}^1$ is a Lip(1, 1/2) *domain* if its lateral boundary can be given in local coordinates as the graph of a function φ which is Lipschitz in the space variable and Lip $\frac{1}{2}$ in time. Specifically, $\partial\Omega$ can be covered by finitely many cylinders of the form

$$\Psi_r(Q,s)=\{(x',x_n,t):|x_i-Q_i|<r,\ i=1,\ldots,n-1,\ |t-s|<r^2,\ |x_n-Q_n|<2nMr\},$$

where $r>0$, $Q\in\mathbb{R}^n$ and $s\in\mathbb{R}$. In local coordinates $\partial\Omega\cap\Psi_r(Q,s)=\{(x',x_n,t):|x_i|<r,\ |t|^{1/2}<r,\ x_n=\varphi(x',t)\}$ where we have set $(Q,s)=\vec{0}$ and $\varphi(Q,s)=0$. Moreover,

$$\Omega\cap\Psi_r(Q,s)=\{(x',x_n,t):2nMr>x_n>\varphi(x',t),\ |x_i|<r,\ |t|^{1/2}<r\}$$

and

$$|\varphi(x',t)-\varphi(y',s)|\le M(|x'-y'|+|t-s|^{1/2}).$$

The *parabolic metric* is

$$\delta(\vec{x},t;\vec{y},s)=(|x'-y'|^2+|x_n-y_n|^2)^{1/2}+|t-s|^{1/2}$$

so that $\delta(\vec{x},t;E)=\inf_{(y,s)\in E}\delta(\vec{x},t;\vec{y},s)$. Also the notation

$$\delta(x,t)=\delta(x,t;\partial_p\Omega)$$

is used below.

The lateral parabolic boundary of Ω is defined as in [Do]: $\partial_p\Omega=\{(x,t)\in\partial\Omega: t>0$ and there is a path $\gamma\subseteq\Omega$ whose initial point is (x,t) and whose time coordinate function is strictly increasing with time$\}$.

A *surface cube* $\Delta_r(Q,s)\subseteq\partial_p\Omega$ is given by $\Delta_r(Q,s)=\Psi_r(Q,s)\cap\partial\Omega=\{(x',x_n,t):|x_i-Q_i|<r,\ i=1,\ldots,n-1,\ x_n=\varphi(x',t),\ |t-s|^{1/2}<r\}$ for any $(Q,s)\in\partial_p\Omega$.

The points

$$A_r(Q,s)=(Q',Q_n+8nMr,s),$$
$$\overline{A}_r(Q,s)=(Q',Q_n+8nMr,s+2r^2),$$
$$\underline{A}_r(Q,s)=(Q',Q_n+8nMr,s-2r^2)$$

are used for estimates in §2.

The nontangential approach regions, which will also be called "cones",

$$\Gamma_\alpha(Q,s)=\{(\vec{x},t)\in\Omega:\delta(\vec{x},t;Q,s)<(1+\alpha)\delta(x,t)\}$$

for $(Q,s)\in\partial_p\Omega$ are used in maximal functions and the Lusin area integral

$$N_\alpha(u)(Q,s)=\sup_{(x,t)\in\Gamma_\alpha(Q,s)}|u(x,t)|,$$

and

$$\widetilde{N}_\alpha(F)(Q,s)=\sup_{(x,t)\in\Gamma_\alpha(Q,s)}\Big(\frac{1}{|\widehat{\Psi}_{\delta/4}(x,t)|}\int_{\widehat{\Psi}_{\delta/4}(x,t)}|F(y,t)|^2\,dy\,dt\Big)^{1/2}$$

(where $\delta=\delta(x,t;\partial_p\Omega)$) is an averaged maximal function. Here

$$\widehat{\Psi}_r(y,s)=\{(x,t):|x-y|<r,\ |t-s|<r^2\}.$$

As in [FKP], if $u(y,t)$ is a solution to $(\partial/\partial t-L)u=0$ in Ω then

$$N_\alpha(u)(Q,s)\le c_1\widetilde{N}_\beta(u)(Q,s)\le c_2N_\gamma(u)(Q,s)$$

where $\alpha<\beta<\gamma$ and c_1, c_2 depend on the "cone" openings. These inequalities are valid for solutions since Harnack's inequality holds with a time lag.

We will also use

$$S_\alpha(u)(Q,s)=\Big(\int_{\Gamma_\alpha(Q,s)}|\nabla u(x,t)|^2\delta^{-n}(x,t)\,dx\,dt\Big)^{1/2}$$

and the Hardy–Littlewood maximal function with respect to a boundary measure μ:

$$M_\mu(f)(Q,s)=\sup_{\substack{\triangle_r(Q,s)\\ r>0}}\frac{1}{\mu(\triangle_r)}\int\limits_{\triangle_r}|f(P,t)|\,d\mu(P,t).$$

Let $\Gamma(Q,s)=\Gamma_1(Q,s)$, $N(F)=N_1(F)$ etc.

u_0 and u_1 will be called *solutions to the Dirichlet Problem* (DP) on Ω if $(\partial/\partial t-L_i)u_i=0$ in Ω in the weak sense, i.e.

$$\int\limits_{\Omega}\varphi(y,s)\frac{\partial u_i}{\partial s}(y,s)+(\nabla\varphi\cdot[a_{ij}]\nabla u_i)(y,s)\,dy\,ds=0$$

for all $\varphi\in C_0^\infty(\Omega)$ and $u_i|_{\partial_{\rm p}\Omega}=f$. We then have $u(x,t)\in W^{1,1}_{2,\rm loc}(\Omega)$; $u(x,t)$ is a global solution if $u\in W_2^{1,0}(\overline{\Omega})$ in addition.

For such solutions we have $u_i(x,t)=\int_{\partial_{\rm p}\Omega}f(Q,s)\,d\omega_i^{(x,t)}(Q,s)$ where $\omega_i^{(x,t)}(\cdot)$ is the parabolic measure on $\partial_{\rm p}\Omega$ associated with $\partial/\partial t-L_i$. Moreover, $\omega_i^{(x,t)}(E)$ is the solution to (DP) which has boundary values $\chi_E(Q,s)$ for E any Borel subset of $\partial_{\rm p}\Omega$.

The existence of such solutions will be assumed for $f\in L^p(d\mu)$. (See the literature for solutions to (DP), in particular [LSU], [K], [YH], [HL], [LM] for domains which are Lip$(1,1/2)$ or slightly more regular—on these latter domains one can take $d\mu=$ surface measure.)

For Lip$(1,1/2)$ domains, ω_0 will always be assumed to be a center-doubling measure (see Section 4) and $f\in L^p(d\omega_0)\Rightarrow f\in L^1(d\omega_0)$ (so that Kemper's results hold if $\Delta=L_0$).

$\Gamma_i(x,t;y,s)$ is the fundamental solution for $\partial/\partial t-L_i$ on $\mathbb{R}^{n+1}$ and $G_i(x,t;y,s)$ is the Green's function for $\partial/\partial t-L_i$ on Ω.

One other construction, a saw-tooth domain over $E\subseteq\partial_{\rm p}\Omega$, will be defined in Section 3.

For $L_0=\frac{\partial}{\partial x_i}\big(a^{ij}(x,t)\frac{\partial}{\partial x_j}\big)$ and $L_1=\frac{\partial}{\partial x_i}\big(b^{ij}(x,t)\frac{\partial}{\partial x_j}\big)$ define

$$\varepsilon_{ij}(y,s)=b_{ij}(y,s)-a_{ij}(y,s),$$
$$|[\varepsilon_{ij}(y,s)]|=\sup_{i,j}|\varepsilon_{ij}(y,s)|\equiv\varepsilon(y,s),$$
$$a(x,t)=\sup_{(y,s)\in\widehat{\Psi}_{\delta(x,t)/2}(x,t)}|\varepsilon(y,s)|,$$

so that

$$\Big(\frac{1}{|\widehat{\Psi}_{\delta/4}|}\int\limits_{\widehat{\Psi}_{\delta/4}}|\varepsilon(y,s)|^2\,dy\,ds\Big)^{1/2}\le a(x,t)$$
$$\lesssim\Big(\frac{1}{|\widehat{\Psi}_{\delta/4}|}\int\limits_{\widehat{\Psi}_{\delta/4}}|a(y,s)|^2\,dy\,ds\Big)^{1/2}\quad\text{for }(x,t)\in\Omega.$$

If u_0, u_1 are solutions to (DP) on Ω as above let

$$F(x,t)=u_1(x,t)-u_0(x,t),$$
$$\|u\|_{L^p(\partial_{\rm p}\Omega,d\mu)}\equiv\Big(\int\limits_{\partial_{\rm p}\Omega}|u(x,s)|^p\,d\mu(x,s)\Big)^{1/p}.$$

2. Assume that Ω is a Lip$(1,1/2)$ domain and

$$L_0=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\frac{\partial}{\partial x_j}\Big)$$

where there are constants $\mu,\lambda>0$ such that

$$\lambda|\xi|^2\le\sum_{i,j=1}^n\xi_ia_{ij}(x,t)\xi_j\le\mu|\xi|^2\quad\text{for all }(x,t)\in\Omega.$$

$G_0(x,t;y,s)$ is the Green's function for $\partial/\partial t-L_0$ on Ω and $\omega_0^{(x,t)}(E)$ is the parabolic measure associated with $\partial/\partial t-L_0$ of E, where E is any Borel subset in $\partial_{\rm p}\Omega$.

The interior estimates for nonnegative weak solutions u of $(\partial/\partial t-L_0)u=0$ on Ω such as Harnack's inequality (with a time-lag), Hölder continuity, and the energy estimate are all valid by classical proofs from Moser, Nash, Aronson etc. since they are independent of the kind of boundary Ω has. Other results such as the maximum principle follow easily and can also be found in the literature. The standard estimates given below for parabolic measure and solutions vanishing on $\partial_{\rm p}\Omega$ are valid at the boundary of any Lip$(1,1/2)$ domain and are easily proved by methods in [CFMS], [FGS], [S], [RB]. Such estimates have been proved on Lip$(1,1/2)$ domains for parabolic functions in [N], [YH] and for solutions to the heat equation in [FGSII]. The proofs are therefore only briefly indicated. These results will be used to prove the main theorem in Section 3.

LEMMA 1. *There is a constant* $c=c(\lambda,n,m,r_0)$ *so that*

$$\omega^{A_r(Q,s)}(\triangle_r(Q,s))\ge c>0$$

for all $(Q,s)\in\partial_{\rm p}\Omega$, $0<r<r_0$ *and* $A_r(Q,s)\in\Omega$.

Proof. Let ω_0' be the parabolic measure of $\Psi_r(Q,s)$ evaluated at the point $\underline{A}_{r/8}(Q,s)$. Then ω_0' is associated to $\partial/\partial t-L_0'$ where L_0' is the div form operator on $\Psi_r(Q,s)$ obtained by extending L_0 across $\partial_{\rm p}\Omega$ (say to be equal to Δ) in $\Psi_r(Q,s)\cap\Omega^c$. Let $B_r(Q,s)=\Psi_r(Q,s)\cap\{x_n=-2nMr\}$. Then $\omega_0'(B_r(Q,s))\ge c$ by a result of Salsa ([S], proof of Lemma 4.2). The max principle gives $\omega_0^{(x,t)}(\triangle_r(Q,s))\ge\omega_0'^{(x,t)}(B_r(Q,s))$ for all $(x,t)\in\Psi_r(Q,s)\cap\Omega$. Now Harnack's inequality gives $\omega_0^{A_r(Q,s)}(\triangle_r(Q,s))\ge c>0$.

LEMMA 2. *Suppose $u(x,t)$ is any nonnegative solution to $(\partial/\partial t - L_0)u = 0$ in Ω and $u(x,t)$ vanishes continuously for all $(x,t) \in \triangle_{2r}(Q_0,s_0)$. Then there is a constant $c = c(\lambda, n, M, r_0) > 0$ such that $u(x,t) \le cu(\overline{A}_r(Q,s))$ for all $(x,t) \in \Psi_{r/4}(Q,s) \cap \Omega$.*

Proof. The method of Salsa (in the proof of Theorem 3.1 of [S]) can be used here. There is a Whitney type decomposition of $\Omega \cap \Psi_r(Q,s)$ into dyadic parabolic "cubes" whose dimension compares with their distance from $\partial_p \Omega$, so $\Psi_r(Q,s) \cap \Omega = Q_{k,h,j}$ where

$$Q_{k,h,j} = \{(x', x_n, t) : c_1(M)r/2^k \le x_n - \varphi(x',t) \le c_2(M)r/2^{k-1},\ hr/2^{k-1} < |x_i - Q_i| \le (h+1)r/2^{k-1},\ i = 1,\dots,n-1,\ -r^2 + jr^2/4^{k+2} < |t - s_i| \le (j+1)r^2/4^{k+2} - r^2\}$$

for $k = 1,2,\dots$; $h = -2^{k-1}, -2^{k-1}+1, \dots, 2^{k-1}-1$; $j = 0,1,2,\dots, 2\cdot 4^{k+2}-1$. $(Q_i, 0, s_i)$ are parabolic dyadic lattice points.

Since odd reflection across a $\mathrm{Lip}(1,1/2)$ boundary brings in a drift term whose coefficient can be unbounded, an internal estimate on u must be used in place of the role of $\operatorname{osc} u$ in Salsa's proof. The following estimate for solutions vanishing on $\triangle_{2r}(Q_0,s_0)$ gives such a result and it can also be used to prove Hölder continuity at the boundary.

Assume $\sup_{\Psi_{r/2}(Q_0,s_0)\cap\Omega} u(x,t) = 1$. Let $\omega^{(x,t)}_{\Psi_{r/2}}(\partial_p \Psi_{r/2} \cap \Omega^c)$ be the parabolic measure on $\Psi_{r/2}(Q_0,s_0)$ of the part of $\partial_p \Psi_{r/2}(Q_0,s_0)$ external to Ω. Now $u(x,t) \le 1 - \omega^{(x,t)}_{\Psi_{r/2}}(\partial_p \Psi_{r/2} \cap \Omega^c)$ for $(x,t) \in \partial_p(\Omega \cap \Psi_{r/2})$ so by the max principle the estimate holds for $(x,t) \in \Omega \cap \Psi_{r/2}(Q_0,s_0)$ also.

By Lemma 1 there is a constant $c > 0$ so that $\omega^{(x,t)}_{\Psi_{r/2}}(\partial_p \Psi_{r/2} \cap \Omega^c) \ge c > 0$ if $(x,t) \in \Psi_{r/4}(Q_0,s_0)$. Then $u(x,t) \le 1 - \omega^{(x,t)}_{\Psi_{r/2}}(\partial_p \Psi_{r/2} \cap \Omega^c) \le 1 - \varepsilon = (1-\varepsilon)\sup_{\Psi_r(Q_0,s_0)} u$ whenever $(x,t) \in \Psi_{r/4}(Q_0,s_0) \cap \Omega$. Iteration gives Hölder continuity for $u(x,t)$, and the estimate

$$\sup_{F^N_{p_0}} u(x,t) \ge \frac{1}{(1-\varepsilon)^N} \sup_{F_{p_0}} u$$

can be used to demonstrate the existence of a sequence $\{P_l\}_{l=1}^{\infty} \subseteq \Psi_r(Q,s) \cap \Omega$ such that $u(P_l) \ge H^{c(M)l}$ but $\lim_{l\to\infty} \delta(P_l, \partial_p \Omega) = 0$; H is a constant > 1, and $c(M) > 0$, as Salsa does in his proof. Here $F^N_{p_0}$ and F_{p_0} are the analogues of $E^N_{p_0}$ and E_{p_0} for a $\mathrm{Lip}(1,1/2)$ domain. This contradicts u vanishing continuously on $\triangle_{2r}(Q,s)$.

Lemmas 1 and 2 can be used to prove a standard comparison of the Green's function with parabolic measure, the fact that two solutions vanishing on $\partial_p \Omega$ vanish at the same rate. The proofs of these results are basically the same as the proofs on a cylinder domain and can be found in [FGS]:

LEMMA 3. *There is a constant $c = c(\lambda, n, M, r_0)$ so that for all $(x,t) \in \Omega$, if $t > s + 3r^2$, then*

$$\frac{1}{c} r^n G_0(x,t;\overline{A}_r(Q,s)) \le \omega_0^{(x,t)}(\triangle_r(Q,s)) \le cr^n G_0(x,t',\underline{A}_r(Q,s)).$$

Proof. The argument in the proof of Lemma 4.8 in [JK] can be used, with some minor changes needed to deal with the operator $\partial/\partial t - L_0$ and its solutions, to show that

$$(*)\quad \omega_0^{(x,t)}(\triangle_r(Q,s)) \le \int_{\partial_p \Omega} \varphi(P,\tau)\, d\omega_0^{(x,t)}(P,\tau) \le \int_{\Psi_{2r}(Q,s)} \Big(G_0(x,t;y,s) \frac{\partial \varphi}{\partial s}(y,s) + \nabla_y G_0(x,t;y,s) \cdot [a_{ij}(y,s)] \nabla_y \varphi(y,s) \Big)\, dy\, ds$$

where $1 < \alpha < 2$, $\operatorname{supp} \varphi \subseteq \Psi_{\alpha r}(Q,s)$, $\varphi \equiv 1$ on $\Psi_r(Q,s)$ and $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. Since $L_0 = \Delta$ on Ω^c and $G_0(x,t;y,s)$ has been extended to equal 0 outside Ω, it follows that G_0 is a subsolution on $\mathbb{R}^{n+1} \setminus \{(x,t)\}$. The representation

$$G_0(x,t;y,s) = c(n)\Big[\Gamma(x,t;y,s) - \int_{\partial_p \Omega} \Gamma(P,\tau;y,s)\, d\omega_0^{(x,t)}(P,\tau)\Big]$$

has been used to avoid having to use surface measure on $\partial_p \Omega$ (which may not be finite).

From $(*)$ the proof of Theorem 1.4 in [FGS] shows that the result of Lemma 3 is valid.

Now the local and global comparison theorems for solutions vanishing on $\triangle_{2r}(Q,s)$, resp. $\partial_p \Omega$, follow on $\mathrm{Lip}(1,1/2)$ domains by the methods of proof of Theorems 1.6 and 1.7 in [FGS] and [FGSII].

3. Let μ be a doubling measure on $\partial_p \Omega$.

As in [FKP], [CS] the difference function $F(x,t) = u_1(x,t) - u_0(x,t)$, defined in Section 1, has an integral expression over Ω. For $f \in C(\partial_p \Omega)$,

$$(3.1)\qquad F(x,t) = \int_{\Omega} \nabla_y G_0(x,t;y,s) \cdot [\varepsilon_{ij}(y,s)] \nabla_y u_1(y,s)\, dy\, ds$$

by using Green's identity for smooth operators $L_0^m = \frac{\partial}{\partial x_i}\big(a^{ij} * \varphi_m(x,t) \frac{\partial}{\partial x_j}\big)$ and smooth functions u_0^m, u_1^m, F^m etc. on the domain $\Omega_t = \{(\vec{y}, s) : \vec{y} = (y', y_n),\ s \equiv t; (\vec{y}, s) \in \Omega\}$ in $\mathbb{R}^n$. Then proceeding as in Doob [Do] one can show

$$F^m(x,t) = \int_{\Omega} G_0^m(x,t;y,s) \Big(\Big[\frac{\partial}{\partial t} - L_0^m \Big] F^m(y,s) \Big)\, dy\, ds.$$

By elementary manipulations, (3.1) follows for smooth solutions. Now from the fact that $G_0^m(x,t;y,s)$ is a solution to $\partial/\partial s + L_0^m$ for $(y,s)\in\Omega$, $s<t$, and that $u_i^m \to u_i$, $G_0^m \to G_0$ in $W^{1,1}_{2,\mathrm{loc}}(\Omega)$ and $u_i^m \to u_i$ pointwise by Hölder continuity of the solutions on the interior of Ω, (3.1) holds for rough coefficient operators in the weak sense.

Two inequalities will be used in the proof of the theorem:

(iii) $$\|N(u_0)\|_{L^p(d\mu)} \le c\|f\|_{L^p(d\mu)},$$

(iv) $$\|S(u_0)\|_{L^p(d\mu)} \le c'\|f\|_{L^p(d\mu)}.$$

(iii) follows from the comparison of $N(u_0)$ with $M_{\omega_0}(f)$ and a standard maximal theorem for $1<p<\infty$; (iv) is easy to obtain when $p\le 2$, but for $p>2$ a more subtle argument is needed. For $1<p\le 2$, using Green's theorem on $u(x,t)\in C^2(\overline{\Omega})$ if $f(Q,s)\ge 0$, $u(x,t)\ge 0$ we get

$$\int_{\partial\Omega} f(Q,s)^p\,d\omega_0(Q,s) = u(x_0,T_0)^p - \int_{\Omega} p(p-1)u(x,t)^{p-2} \times \frac{\partial u}{\partial x_i}(x,t)a^{ij}(x,t)\frac{\partial u}{\partial x_i}(x,t)G_0(x_0,T_0;x,t)\,dx\,dt.$$

Hence

$$\int_{\partial\Omega} f^p\,d\omega_0 \ge C\Big|\int_{\partial\Omega}\Big(\int_{\Gamma(Q,s)} p(p-1)u^{p-2}|\nabla u|^2\Big)\,d\omega_0\Big|.$$

Let

$$E=\{(Q,s): S_\alpha(u)(Q,s)\le N_\beta(u)(Q,s)\},\qquad \beta\gg\alpha,$$
$$E^c=\{(Q,s): N_\beta(u)(Q,s)\le S_\alpha(u)(Q,s)\}.$$

Then for $(Q,s)\in E^c$,

$$\Big(\frac{1}{|R_j\cap\Gamma_\beta\cap S|}\int_{R_j\cap\Gamma_\beta\cap S}|u(y,\tau)|^2\,dy\,d\tau\Big)^{1/2} \lesssim S_\alpha u(Q,s)$$

by definition of $N(u)(Q,s)$ and Harnack's inequality if $\alpha<\beta$. Here $R_j=\{(x,t)\in\Omega: d_p(x,t;\partial_p\Omega)\sim 2^{-j}\}$, and S is any subset of R_j.

Now

$$\int_E S_\alpha(u)^p\,d\omega_0 \le \int_E N_\beta(u)^p\,d\omega_0 \le \int_{\partial\Omega} f^p\,d\omega_0$$

by (iii), whereas

$$\int_{E^c} S_\alpha(u)^p\,d\omega_0 = \int_{E^c} S_\alpha(u)^{p-2}\Big(\sum_j\int_{\Gamma_\alpha(Q,s)\cap R_j}|\nabla u|^2\delta^{-n}\Big)\,d\omega_0$$
$$\le \int_{E^c}\sum_j\Big(\frac{1}{|\Gamma_\beta\cap S_j|}\int_{\Gamma_\beta\cap S_j}u^2\Big)^{(p-2)/2}\Big(\int_{\Gamma_\beta\cap R_j}|\nabla u|^2\delta^{-n}\Big)\,d\omega_0$$

where $p-2\le 0$ and $S(u)(Q,s)\ge (|S_j|^{-1}\int_{S_j}u^2)^{1/2}$ for $(Q,s)\in E^c$ have been used for

$$S_j = R_j\cap\Gamma_\beta\setminus\Gamma_{b\alpha}\cap\{(x,t): t\ge \max s: (y,s)\in\Gamma_\alpha,\ y_n=x_n\}$$

where $b>1$.

Now

$$\int_{E^c}\sum_j\Big(\frac{1}{|S_j|}\int_{S_j}u^2\Big)^{(p-2)/2}\Big(\int_{\Gamma_\beta\cap R_j}|\nabla u|^2\delta^{-n}\Big)\,d\omega_0$$
$$\lesssim \int_{E^c}\sum_j\int_{\Gamma_\beta\cap R_j}u^{p-2}|\nabla u|^2\delta^{-n}\,d\omega_0$$
$$\lesssim \int_{E^c}\int_{\Gamma(Q,s)} p(p-1)u^{p-2}\frac{\partial u}{\partial x_i}a^{ij}\frac{\partial u}{\partial x_j}\delta^{-n}\,d\omega_0$$
$$\le \int_{\Omega} p(p-1)u^{p-2}\frac{\partial u}{\partial x_i}a^{ij}\frac{\partial u}{\partial x_j}G_0\,dx\,dt \le c\int_{\partial\Omega} f^p\,d\omega_0.$$

Harnack was used again to obtain the first inequality.

Altogether,

$$\int_{\partial\Omega}(S(u)(Q,s))^p\,d\omega_0(Q,s)\le \int_E + \int_{E^c} \lesssim \int_{\partial\Omega} f^p\,d\omega_0$$

when $1<p\le 2$.

Now let $f_n\to f\in L^p$, $f_n\in C^\infty(\partial\Omega)$ to obtain the inequality for $f\in L^p$, $u\in W^{1,2}(\Omega)$.

If (iv) holds for solutions of $(\partial/\partial t - L_0)u_0=0$ in Ω, $u_0|_{\partial_p\Omega}=f$, then Theorem 4 is true for $1<p<\infty$ by the same arguments shown below for the case $p\le 2$. For ω_1 a center doubling measure, (iv) holds if $p>2$. This result is proved in Nystrom's paper [N] using Russell Brown's proof of the area integral theorem for solutions to the heat equation on $\mathrm{Lip}(1,1/2)$ domains [RB], and by using (iii).

For $\partial/\partial t - L_i$, u_i, $\omega_i(x,t)$ being parabolic type operators, solutions and measures on Ω as in Section 1, $i=0,1$, the following theorem can be proved:

THEOREM 4. *If $\omega_0\in B^q(d\mu)$, $2\le q$, $1/p+1/q=1$ and if for every $(Q,s)\in\partial_p\Omega$ and $r<r_0$,*

(Cc) $$\int_{\Delta_r(Q,s)}\Big(\int_{\Gamma(Q,s)}\frac{a(y,s)^2}{\delta(y,s)^{n+2}}\,dy\,ds\Big)\frac{d\mu(Q,s)}{\mu(\Delta_r)}\le C\varepsilon(r)$$

with c independent of r and $\varepsilon(r)\to 0$ as $r\to 0$, then $\|N(u_1)\|_{L^p(\partial_p\Omega,d\mu)}\le c\|f\|_{L^p(\partial_p\Omega,d\mu)}$, and ω_1 is absolutely continuous with respect to ω_0. If ω_1 is a center-doubling measure then $\omega_1\in B^q(d\mu)$ (1).

(1) If $\mu(\Delta_r)\sim r^{n+1}$ for all $\Delta_r\subseteq\partial_p\Omega$, condition (Cc) can be replaced by $\frac{a(y,s)^2}{\delta(y,s)^2}\,dy\,ds$ being Carleson of vanishing trace with respect to μ (see Theorem 2.18 of [FKP]).

See Theorem 2.18 of [FKP].

In the following lemmas $\Gamma_\beta(Q,s)$ is a nontangential approach region of larger aperture than $\Gamma(Q,s)$.

LEMMA 5. *Given the hypotheses of the theorem there are constants C_1 and C_2 depending on λ, n, M and r_0 so that*

(i) $$\widetilde{N}(F)(Q,s) \le C_1\varepsilon_0 M_{\omega_0}(S_\beta(u_1))(Q,s),$$

(ii) $$\|N(\delta\nabla F)\|_{L^p(\partial_p\Omega,d\mu)} \le C_2\varepsilon_0(\|S(u_1)\|_{L^p(\partial_p\Omega,d\mu)} + \|f\|_{L^p(\partial_p\Omega,d\mu)}).$$

LEMMA 6. *There is a constant $C_3 = C(\lambda,n,M,r_0,\beta)$ such that*

$$\|S(F)\|^p_{L^p(\partial_p\Omega,d\mu)} \le C_3(\|\widetilde{N}_\beta(F)\|^p_{L^p(\partial_p\Omega,d\mu)} + \|\widetilde{N}_\beta(\delta\nabla F)\|^p_{L^p(\partial_p\Omega,d\mu)} + \|f\|^p_{L^p(\partial_p\Omega,d\mu)}).$$

To prove Lemma 6 the good-λ inequality of Lemma 7 is needed:

LEMMA 7. *There are constants $c,\eta > 0$ so that for any $\gamma < 1$,*

$$\mu(E) \le c\eta\gamma\mu(SF > \lambda)$$

where

$$\begin{aligned}E = \{(Q,s)\in\partial_p\Omega : S(F) > 2\lambda,\ &\widetilde{N}_\beta(F)(Q,s) \le \gamma\lambda,\\ &\widetilde{N}_\beta(\delta\nabla F)(Q,s) \le \gamma\lambda,\ N_\beta(u_0)(Q,s) \le \gamma\lambda,\\ &N_\beta(u_0)S_\beta(u_1)(Q,s) \le (\gamma\lambda)^2,\\ &\widetilde{N}_\beta(F)(Q,s)S_\beta(u_1)(Q,s) \le (\gamma\lambda)^2,\\ &\widetilde{N}_\beta(\delta\nabla F)S_\beta(u_1)(Q,s) \le (\gamma\lambda)^2\}.\end{aligned}$$

Given the lemma, the theorem follows as in [FKP]. In particular, the condition (Cc) gives a Carleson-type condition for

$$\frac{G_0(X_0,T_0;y,s)a(y,s)^2}{\delta(y,s)^2}\,dy\,ds$$

with respect to ω_0, and reducing the theorem to the case $L_0 \equiv L_1$ on $\Omega_{\delta_0} = \{(\vec{x},t)\in\Omega : \delta(\vec{x},t;\partial_p\Omega) > \delta_0\}$ allows one to take the ε_0 in Lemma 5 as small as necessary, given the vanishing trace condition in (Cc).

In fact, ε_0 sufficiently small gives the estimate

$$(\|\widetilde{N}(F)\|^p_{L^p(\partial_p\Omega,d\mu)} + \|\widetilde{N}(\delta\nabla F)\|^p_{L^p(\partial_p\Omega,d\mu)}) \le c\|f\|^p_{L^p(\partial_p\Omega,d\mu)}$$

by using Lemmas 5 and 6:

$$\begin{aligned}\int_{\partial_p\Omega}(\widetilde{N}(F)^p + \widetilde{N}(\delta\nabla F)^p)\,d\mu &\overset{\text{L.5}}{\lesssim} \varepsilon_0\int_{\partial_p\Omega} M_{\omega_0}(S_\beta u_1)^p d\mu + \int_{\partial_p\Omega}|f|^p\,d\mu\\ &\overset{\text{(i)}}{\lesssim} \varepsilon_0\int_{\partial_p\Omega} S_\gamma(u_1)^p\,d\mu + \int_{\partial_p\Omega}|f|^p\,d\mu\\ &\lesssim \varepsilon_0\int_{\partial_p\Omega}(S_\gamma(F)^p + S(u_0)^p + |f|^p)\,d\mu\\ &\overset{\text{L.6, (iv)}}{\lesssim} \varepsilon_0\int_{\partial_p\Omega}(\widetilde{N}(F)^p + \widetilde{N}(\delta\nabla F)^p + |f|^p)\,d\mu\end{aligned}$$

so for ε_0 sufficiently small,

$$(1-\varepsilon_0)\int_{\partial_p\Omega}(\widetilde{N}(F)^p + \widetilde{N}(\delta\nabla F)^p)\,d\mu \le c\int_{\partial_p\Omega}|f|^p\,d\mu.$$

Here γ is smaller than β and smaller than the opening of $\Gamma(Q,s)$. (i) uses a standard maximal function result. For the change in cone aperture see below.

Now

$$\|N(u_1)\|^p_{L^p(d\mu)} \le \|N_\alpha(u_1)\|^p_{L^p(d\mu)} \lesssim \|\widetilde{N}_\alpha(F) + N_\alpha(u_0)\|^p_{L^p(d\mu)} \le c\|f\|^p_{L^p(d\mu)},$$

and $N(u_1) \sim M_{\omega_1}f$ for ω_1 a center-doubling measure (see [FGS] and Section 4) gives that $\omega_1 \in B^q(d\mu)$.

The places where there are some differences in the proof of the theorem from [FKP] are in the lemma proofs. These are outlined below; they are mainly due to $\partial_p\Omega$ being a $\mathrm{Lip}(1,1/2)$ domain.

First to show that Lemma 7 gives Lemma 6 is a standard good-λ inequality argument:

$$\begin{aligned}\int_{\partial_p\Omega} S(F)^p\,d\mu &= \int_0^\infty p\lambda^{p-1}\mu\{S(F) > \lambda\}\,d\lambda = \int_0^\infty p(2\lambda)^{p-1}\mu\{S(F) > 2\lambda\}d(2\lambda)\\ &\le \int_0^\infty p2^p\lambda^{p-1}\mu(E)\,d\lambda\\ &\quad+ \int_0^\infty p2^p\lambda^{p-1}\cdot[\mu\{\widetilde{N}_\beta(F) > \gamma\lambda\} + \mu\{N_\beta(u_0) > \gamma\lambda\}\\ &\quad+ \mu\{\widetilde{N}_\beta(\delta\nabla F) > \gamma\lambda\} + \mu\{\widetilde{N}_\beta(F)S_\beta(u_1) > (\gamma\lambda)^2\}\\ &\quad+ \mu\{\widetilde{N}_\beta(\delta\nabla F)S_\beta(u_1) > (\gamma\lambda)^2\}\\ &\quad+ \mu\{N_\beta(u_0)S_\beta(u_1) > (\gamma\lambda)^2\}]\,d\lambda\\ &\le c\eta\gamma\int_0^\infty p\lambda^{p-1}\mu\{S(F) > \lambda\}\,d\lambda\\ &\quad+ c(p)[\|\widetilde{N}_\beta(F)\|^p_p + \|\widetilde{N}(\delta\nabla F)\|^p_p + \|N_\beta(u_0)\|^p_p\\ &\quad+ \|\widetilde{N}_\beta(F)\|^{p/2}_p\|S_\beta(u_1)\|^{p/2}_p + \|\widetilde{N}_\beta(\delta\nabla F)\|^{p/2}_p\|S_\beta(u_1)\|^{p/2}_p\\ &\quad+ \|N_\beta(u_0)\|^{p/2}_p\|S_\beta(u_1)\|^{p/2}_p]\\ &\le c\eta\gamma\|S(F)\|^p_p + \ldots\end{aligned}$$

Hence for γ sufficiently small (depending on p)

$$(1-c\eta\gamma)\|S(F)\|_p^p \le c[\|\widetilde{N}_\beta(F)\|_p^p + \|N_\beta(u_0)\|_p^p + \|\widetilde{N}(\delta\nabla F)\|_p^p + \|\widetilde{N}_\beta(F)\|_p^{p/2}(\|S(F)\|_p^{p/2} + \|S(u_0)\|_p^{p/2}) + \ldots].$$

If $\|S(F)\|_p \le \|\widetilde{N}_\beta(F)\|_p$, $\|\widetilde{N}(\delta\nabla F)\|_p$ or $\|f\|_p$ we are done, if not the right hand side of the last inequality is less than or equal to

$$c\|S(F)\|_p^{p/2}\{\|\widetilde{N}_\beta(F)\|_p^{p/2} + \|\widetilde{N}(\delta\nabla F)\|_p^{p/2} + \|f\|_p^{p/2}\}$$

and dividing by $\|S(F)\|_p^{p/2}$ and taking $(p/2)$th roots gives Lemma 6.

Any integral of $S(u_i)$ with respect to a doubling measure μ can be written (up to a harmless constant) as an integral of $S_\beta(u_i)$ for a "cone" of different aperture (as long as $\beta \ge \beta_0$ = a minimal constant depending on n, Γ and r_0 such that $|\Gamma_{\beta_0}(Q,s)| \ge \delta_0 > 0$, for some fixed δ_0, see remark after Lemma 3.1 of [RB]). This fact allows norm estimates over uniform approach regions to be used in proving the theorem, although the estimates in the lemmas require increasing "cone" apertures.

Proof of Lemma 5. The argument in [FKP] to prove Lemma 2.9 is used. The estimates taken over $P_{\delta(x,t)/2}(x,t)$ are exactly as in the proof of Lemma 1 of [CS] since this region is well inside Ω, and they are not affected by the $\mathrm{Lip}(1,1/2)$ boundary. To prove the stopping time argument on Ω, the only new ingredient is that the dyadic decomposition of $\partial_p D_T$ and the regions R_j in D_T, whose dimension compares with that of $\mathrm{Proj}_{\partial_p\Omega} R_j = I_j$ = parabolic cube of dimension $2^{-j}\delta(x,t)$ in $\partial_p D_T$, must be defined to fit the time-varying boundary of Ω.

In the following argument (x,t) is a fixed point in $\Gamma(Q,s)$ and (x^*,t^*) is its projection onto $\partial_p\Omega$. Break $\triangle_0 = \triangle_{\delta(x,t)}(x^*,t^*) = \partial\Omega \cap \Psi_{\delta(x,t)}(x^*,t^*)$ into "dyadic" subsets $\varphi(I_j)$ where I_j is a dyadic parabolic cube of dimension $2^{-j}r$, $I_j \subseteq \varphi^{-1}(\triangle_0)$; for example $I_j = \{(x',0,t) : |x_i| < 2^{-j}r,\ i = 1,\ldots,n-1,\ t^{1/2} < 2^{-j}r\}$. Now $\varphi(\bar{I}_j)$ form a disjoint cover of $\triangle_0$ (up to boundaries of φ-cubes; taking I_j to be half-open cubes gives a cover of $\triangle_0$ which is disjoint). In fact, $\varphi(I_j) = \triangle_j(Q_j,\varphi(Q_j,s_j),s_j)$ if $(Q_j,0,s_j)$ is the center of I_j.

Now set $R_j = \{(x',x_n,t) : |x_i - Q_i| < 2^{-j}r,\ i = 1,\ldots,n-1,\ |t-s_j| < 4^{-j}r^2$ and $2^{-j-1}r < x_n < 2^{-j}r\}$ in $\mathbb{R}^{n-1}\times\mathbb{R}^1_+\times\mathbb{R}^1_+$. Then the regions $\varphi(R_j) = \{(x',\varphi(x',t)+x_n,t) : (x',x_n,t)\in R_j\}$ give a disjoint cover of the region near the boundary, $T(\triangle_0)$, at $\triangle_0$, and form the usual decomposition of $T(\triangle_0)$ into subsets whose dimension is comparable to the distance from $\partial_p\Omega$, $\mathrm{Proj}_{\partial_p\Omega}\varphi(R_j) = \varphi(I_j)$. Here the dimension of $\varphi(R_j)$ is defined as $|\mathrm{vol}\,\varphi(R_j)|^{1/(n+2)}$.

The image sets $\varphi(I_j)$ retain the property of being either nested or disjoint. The fact that the usual Lebesgue measure of $\varphi(I_j)$ or the surface area of $\varphi(R_j)$ may be infinite does not cause a problem here: the cubes $\varphi(I_j)$ are

considered with respect to the parabolic measure ω_0 and the regions $\varphi(R_j)$ have volume $\sim \mathrm{vol}(R_j)$.

The stopping time proof can now be used with these regions in $\partial\Omega$ and Ω taking the place of the dyadic decomposition and dyadic approach regions used in the case of a cylinder domain.

If necessary to keep the "cone" apertures from becoming too large the regions $\varphi(R_j)$ can all be subdivided into a fixed number of subregions. The second Carleson condition

$$\int\limits_{P_{\delta/2}(x,t)} \frac{G_0(X_0,T_0;y,s)a(y,s)^2}{\delta(y,s)^2}\,dy\,ds \le c\varepsilon_0\omega_0(\triangle_{\delta/2}(x^*,t^*))$$

can be used on the regions $\varphi(R_j)$ and estimates for $G_0(X_0,T_0;y,s)$, $G_0(x,t;y,s)$, $G_0(x_j,t_j;y,s)$ are valid for

$$(y,s)\in\varphi(R_j),\quad (x_j,t_j)\in\Omega_j = \Psi_{2^j\delta(x,t)}(x^*,t^*),\quad (x,t)\in\Omega\setminus\bigcup_{j=1}^N\Omega_j$$

since $\delta(\varphi(R_j),\partial_p\Omega)\sim 2^j r$.

The stopping time argument gives (i) of Lemma 5.

The second estimate follows from the pointwise inequality

$$(*)\qquad [\widetilde{N}(\delta\nabla F)(Q,s)]^2 \le c(\widetilde{N}_\alpha(F)\widetilde{N}_\alpha(\delta\nabla F)(Q,s) + \varepsilon_0(\widetilde{N}_\alpha(F)+\widetilde{N}_\alpha(\delta\nabla F))S_\alpha(u_1)(Q,s))$$

which holds a.e. $d\omega_0$, hence a.e. $d\mu$. Γ_α is a "cone" of wider aperture than Γ. Given $(*)$ and using (i) $\widetilde{N}_\alpha(F)(Q,s) \le c\varepsilon_0 M_{\omega_0}(S_\beta(u_1))(Q,s)$ along with the inequality

$$\widetilde{N}_\alpha(\delta\nabla F)(Q,s) \lesssim N_\beta(u_0)(Q,s) + S_\alpha(u_1)(Q,s)$$

one obtains a pointwise inequality in terms of quantities whose $L^p(d\mu)$ norms can be bounded by $\|S(u_1)\|_{L^p(\partial_p\Omega,d\mu)}$ and $\|f\|_{L^p(\partial_p\Omega,d\mu)}$. If μ is not taken to be ω_0, the B^q condition must be used.

$(*)$ follows by the argument used to prove Lemma 1 in [CS] (derived originally from Lemma 2.9 of [FKP]) with only minor changes.

Proof of Lemma 7 (see [FKP], proof of Theorem 2.18). What follows is a standard saw-tooth domain argument.

The set $\{S(F)>\lambda\}$ is divided into Whitney (parabolic) cubes $\triangle_j(Q_j,s_j)$ and $E_j = E\cap\triangle_j(Q_j,s_j)$. Now fix j so that $\triangle_j = \triangle_{r_j}$, $E_j = E\cap\triangle_j$ and one can construct a $\mathrm{Lip}(1,1/2)$ region $W = \bigcup_{(Q,s)\in E}\Gamma_\alpha(Q,s)\cap\widehat{\Psi}_r$ as in [RB], p. 572. The estimates of Lemma 3.1 and in the proof of Lemma 3.11 in the same paper are used below. By the B^q condition it suffices to show that $\omega_0(E) \le c(\gamma\lambda)^2\omega_0(\triangle_r)$ to prove Lemma 7. For any τ, $0<\tau<1$ the cones $\Gamma(Q,s)$ can be truncated to obtain the estimate

$$(**)\qquad S_{\tau r}(F)(Q,s)>\lambda/2$$

for any $(Q,s)\in E$ if γ is chosen sufficiently small, where

$$S_{\tau r}(F)(Q,s)=\Big(\int\limits_{\Gamma_\alpha^{\tau r}(Q,s)}|\nabla F(x,t)|^2\delta(x,t)^{-n}\,dx\,dt\Big)^{1/2},$$

$$\Gamma_\alpha^{\tau r}(Q,s)=\Gamma_\alpha(Q,s)\cap\{(x,t):\delta(x,t;Q,s)<\tau r\}.$$

The proof of $(**)$ follows the proof of Lemma 1 in [DJK].

Write $S_{\tau r}(F)(Q,s)=S(F)(Q,s)-S_{U_1}(F)(Q,s)-S_{U_2}(F)(Q,s)$ where

$$U_1=\{(\vec{x},t):(\vec{x},t)\in\Gamma_{\tau r}(Q,s)^c\cap\Gamma(P^*)\cap\Gamma(Q,s)\},$$
$$U_2=\{(\vec{x},t):(\vec{x},t)\in\Gamma_{\tau r}(Q,s)^c\ \cap\Gamma(Q,s)\setminus\Gamma(P^*)\}.$$

Here $P^*\in\partial_p\Omega$ is a point in $\{S(F)>\lambda\}^c$ such that $\delta(P^*;Q,s)\sim\operatorname{diam}(\triangle)=r$, i.e. $\delta(P^*;Q,s)=cr$, for $c=c(M,n)$.

For simplicity assume $c\le 1$. Then

$$S_{U_1}(F)(Q,s)=\Big(\int\limits_{\Gamma(P^*)\cap(\Gamma(Q,s)\setminus\Gamma_{\tau r}(Q,s))}|\nabla F|^2\delta^{-n}\Big)^{1/2}$$
$$\le\Big(\int\limits_{\Gamma(P^*)}|\nabla F|^2\delta^{-n}\Big)^{1/2}\le\lambda$$

by definition of P^*.

To estimate $S_{U_2}(F)(Q,s)$ subdivide U_2 into the regions $R_j=\{(\vec{x},t):2^{j-1}\tau r<\delta(\vec{x},t;\partial_p\Omega)\le 2^j\tau r\}$. Then $R_j\cap U_2$ can be further subdivided into a bounded number of parabolic cubes (or partial cubes) that are of Whitney type with respect to $\partial_p\Omega$. The regions $R_j\cap U_2$ will be treated as if they were these cubes. Now

$$\int\limits_{U_2}|\nabla F(x,t)|^2\delta(x,t)^{-n}\,dx\,dt$$
$$=\sum_{j=0}^N\int\limits_{R_j\cap U_2}|\nabla F|^2\delta^{-n}\le c\sum_{j=0}^N\int\limits_{R_j\cap U_2}|\nabla F|(|\nabla u_1|+|\nabla u_0|)\delta^{-n}.$$

The argument that follows is identical in u_0 and u_1. A p-Caccioppoli estimate for solutions is used on these functions (see [GS]):

$$\sum_{j=0}^N\int\limits_{R_j\cap U_2}|\nabla F|\cdot|\nabla u_0|\delta^{-n}$$
$$\le\sum_{j=0}^N\Big(\frac{1}{|R_j|}\int\limits_{R_j\cap U_2}|\delta\nabla F|^2\Big)^{1/2}\Big(\int\limits_{R_j\cap U_2}|\nabla u_0|^2\delta^{-n}\Big)^{1/2}$$

$$\le\widehat{N}(\delta\nabla F)(Q,s)$$
$$\times\sum_{j=0}^N\Big(\int\limits_{c_1(2^{j-1}\tau r)^2}^{c_2(2^j\tau r)^2}\Big(\int\limits_{R_j\cap U_2\times t}|\nabla u_0|^{2p}\delta^{-pn}\,dx\Big)^{1/p}$$
$$\times\Big(\int\limits_{R_j\cap U_2}|\chi_{R_j}(x,t)|^q\,dx\Big)^{1/q}dt\Big)^{1/2}$$

$$\lesssim\widetilde{N}(\delta\nabla F)(Q,s)\sum_{j=0}^N(2^{j-1}\tau r)^{-n/2}\Big(\int\limits_{c_1(2^{j-1}\tau r)^2}^{c_2(2^j\tau r)^2}\Big(\int\limits_{R_j\cap U_2\times t}|\nabla u_0|^{2p}\,dx\Big)^{1/p}\Big)^{1/2}$$
$$\times\sup_{c_1(2^{j-1}\tau r)^2<t\le c_2(2^j\tau r)^2}|R_j\cap U_2\times t|^{1/(2q)}.$$

N has been chosen so that $2^N\tau r\sim 1$, i.e. $N\sim-\log(\tau r)/\log 2$. Also $|R_j\cap U_2\times t|\sim r(2^{j-1}\tau r)^{n-1}$ if $(2^{j-1}\tau r)^2\lesssim t\lesssim(2^j\tau r)^2$. By the p-Caccioppoli inequality the above is

$$\le c\widetilde{N}(\delta\nabla F)(Q,s)$$
$$\times\sum_{j=0}^N r^{1/(2q)}(2^{j-1}\tau r)^{-n/2+(n-1)/(2q)}(2^{j-1}\tau r)^{n/(2p)}\Big(\frac{1}{|R_j^*|}\int|u_0|^2\,dx\,dt\Big)^{1/2}$$

$$\le\widetilde{N}(\delta\nabla F)(Q,s)N_\beta(u_0)(Q,s)\sum_{j=0}^N(2^{j-1})^{-1/(2q)}\tau^{-1/(2q)}\le c(\gamma\lambda)^2c(\tau)$$

because $(Q,s)\in E$ and $1/p+1/q=1$.

If $c>1$, one can proceed as in [DJK] to break the region $\Gamma(Q,s)\cap\Psi_{\tau r}(Q,s)^c$ into three regions, one inside $\Gamma(P^*)$ as above, the other two being $\Gamma(Q,s)\cap\Psi_{\tau r}(Q,s)^c\cap\Psi_{tr}(Q,s)$ and $\Gamma(Q,s)\cap\Psi_{tr}(Q,s)^c$. Here t is chosen so that $\inf_{(x,\widehat{s})\in\Gamma(Q,s)\cap\Gamma(p^*)}\widehat{s}=t$. Just as in the elliptic case,

$$\int\limits_{\Gamma\cap P_{\tau r}^c\cap B_{tr}}|\nabla F|^2\delta^{-n}\le c\log\frac{2t}{\tau}(\gamma\lambda)^2.$$

The third region is estimated as above.

Consequently, for $(Q,s)\in E$,

$$S(F)-S_{U_1}(F)-S_{U_2}(F)\ge 2\lambda-c_1(\gamma\lambda)-c_2(\gamma\lambda)\ge\lambda/2$$

for γ sufficiently small.

Now

$$\omega_0(E)\le\frac{c}{\lambda^2}\int\limits_E S_{\tau r}^2(F)(Q,s)\,d\omega(Q,s)$$
$$\le\frac{c}{\lambda^2}\int\limits_W(\nabla F\cdot([a_{ij}]\nabla F))(y,s)G_0(X_0,T_0;y,s)\,dy\,ds.$$

Using identities with $\partial/\partial t - L_i$, $i=0,1$, and integration by parts the latter integral equals

$$\frac{c}{\lambda^2}\Big[\int_{\partial W} G_0 F([a_{ij}]\nabla F)\cdot\vec{n} - \frac{1}{2}\int_{\partial W} F([a_{ij}]\nabla G_0)^2\cdot\vec{n} - \int_W G_0 F \operatorname{div}([\varepsilon_{ij}]\nabla u_1) + \frac{1}{2}\int_W \frac{\partial}{\partial t}(F^2 G_0)\Big].$$

The last expression is only a formal one since the boundary integrals may not be defined unless the functions involved are smooth. Also since ∂W is a $\mathrm{Lip}(1,1/2)$ surface (see p. 572 of [RB]), the surface measure may not be finite. As in the proof of Theorem 2.18 of [FKP] both problems can be handled by using averaging over cones $\Gamma_\varrho(Q,s)$, $\alpha<\varrho<\widehat{\beta}$; this means that boundary integrals are replaced by integrals over solid regions inside Ω. The integration by parts formulas can be used on regions W_ϱ^n converging to $W_\varrho = \bigcup_{(Q,s)\in E}\Gamma_\varrho(Q,s)$ where initially ∂W_ϱ^n has finite surface measure. Then averaging over ϱ allows the integrals to be well-defined as $W_{\widehat{\beta}}^n\setminus W_\alpha^n \to W_{\widehat{\beta}}\setminus W_\alpha$. Notice that $F(\vec{x},t)=0$ on $\partial_{\rm p}\Omega$ so only regions interior to Ω are involved in the averaging. Specifically, one can estimate

$$\begin{aligned}\omega_0(E) &\le \frac{c}{\lambda^2}\Big[\frac{1}{\widehat{\beta}-\alpha}\int_\alpha^{\widehat{\beta}}\int_{W_\varrho}(\nabla F\cdot[a_{ij}]\nabla F)G_0\,dy\,ds\,d\varrho\Big]\\ &\le \frac{c'}{\lambda_2}\Big[\int_{W_{\widehat{\beta}}\setminus W_\alpha} G_0F([a_{ij}]\nabla F)\cdot\vec{n}_\varrho\,dy\,ds\\ &\quad - \frac{1}{2}\int_{W_{\widehat{\beta}}\setminus W_\alpha} F([a_{ij}]\nabla G_0)^2\cdot\vec{n}_\varrho\,dy\,ds\\ &\quad - \int_\alpha^{\widehat{\beta}}\int_{W_\varrho} FG_0\operatorname{div}([\varepsilon_{ij}]\nabla u_1)\,dy\,ds\,d\varrho + \frac{1}{2}\int_\alpha^{\widehat{\beta}}\int_{W_\varrho}\frac{\partial}{\partial t}[F^2G_0]\,dy\,ds\,d\varrho\Big]\end{aligned}$$

when F, G_0 and the coefficients of L_0 and L_1 are smooth. The functions can now approach rough coefficient operators and solutions in Sobolev space norm of $W^{1,2}(\Omega)$.

It is easy to see that showing that the following four integrals are bounded above by $c(\gamma\lambda)^2\omega_0(\triangle)$ will show that $\omega_0(E)\le c(\gamma\lambda)^2\omega_0(\triangle)$:

$$(1)\quad \int_{W_{\widehat{\beta}}\setminus W_\alpha}(G_0|F|\cdot|[a_{ij}]|\cdot|\nabla F|)(y,s)\,dy\,ds,$$

$$(2)\quad \int_{W_{\widehat{\beta}}\setminus W_\alpha}(F^2|[a_{ij}]|\cdot|\nabla G_0|)(y,s)\,dy\,ds,$$

$$(3)\quad \Big|\int_\alpha^{\widehat{\beta}}\int_{W_\varrho}(FG_0\operatorname{div}([\varepsilon_{ij}]\nabla u_1))(y,s)\,dy\,ds\,d\varrho\Big|,$$

$$(4)\quad \int_{W_{\widehat{\beta}}}(F^2G_0)(y,s)\,dy\,ds.$$

Certain further identities will be used on (3) so in fact weak convergence in the Sobolev space is used later.

The main fact for the estimates is that if $\alpha<\widehat{\beta}$ and $(Q,s)\in\triangle$ then $\Gamma_\alpha(Q,s)\cap W_{\widehat{\beta}}$ forms an area which will be contained inside some larger "cone" $\Gamma_\gamma(\widehat{Q},\widehat{s})$ where $(\widehat{Q},\widehat{s})\in E$. The proof of this fact is an easy application of the triangle inequality.

Now, we have

$$\begin{aligned}(1) &= \int_{W_{\widehat{\beta}}\setminus W_\alpha}(G_0|F|\cdot|A_0|\cdot|\nabla F|)(y,s)\,dy\,ds\\ &\le c\int_\triangle\Big(\int_{\Gamma(Q,s)\cap(W_{\widehat{\beta}}\setminus W_\alpha)}\frac{G_0(x_0,T_0;y,s')}{\omega_0(\triangle_\delta(y^*,s^*))}\\ &\qquad\times|F(y,s')|\cdot|\nabla F(y,s')|\,dy\,ds'\Big)\,d\omega_0(Q,s)\\ &\le c\int_\triangle\sum_{j=-\infty}^{N}\int_{\Gamma(Q,s)\cap(W_{\widehat{\beta}}\setminus W_\alpha)\cap R_j}|F|\cdot|\nabla F|\delta(y,s')^{-n}\,dy\,ds'\,d\omega_0(Q,s)\end{aligned}$$

where the regions R_j are as in the proof of the stopping time argument and are of dimension $\sim 2^jr$ in the x_n variable and $\delta(R_j,\partial_{\rm p}\Omega)\sim 2^jr$, $|x'-Q_0'|<c(M)r$; $|t-s_0|<c(M)r^2$ where (Q_0,s_0) is the center of the original Whitney cube $\triangle_r$. Now from $|\nabla F|=|\nabla(u_1-u_0)|\le|\nabla u_1|+|\nabla u_0|$ and Cauchy–Schwarz the last integral is

$$\begin{aligned}&\le c\int_\triangle\sum_{i=0,1}\sum_{j=-\infty}^{N}\Big(\frac{1}{|R_j\cap W_\beta|}\int_{\Gamma\cap(W_\beta\setminus W_\alpha)\cap R_j}|F|^2\Big)^{1/2}\\ &\qquad\times\delta_j^{-n/2+1}\Big(\int_{\Gamma\cap(W_\alpha\setminus W_{\beta})\cap R_j}|\nabla u_i|^2\Big)^{1/2}d\omega_0(Q,s)\\ &\le c\sum_{i=0,1}\int_\triangle\widetilde{N}_\beta(F)(\widehat{Q},\widehat{s})\Big(\sum_{j=-\infty}^{N}\delta_j^2\Big)^{1/2}\\ &\qquad\times\Big(\sum_{j=-\infty}^{N}\int_{\Gamma_\alpha\cap R_j\cap(W_\beta\setminus W_\alpha)}|\nabla u_i|^2\delta^{-n}\Big)^{1/2}d\omega_0(Q,s)\\ &\le c\int_\triangle\widetilde{N}_\beta(F)(\widehat{Q},\widehat{s})[S_\beta(u_1)(\widehat{Q},\widehat{s})+N_\beta(u_0)(\widehat{Q},\widehat{s})]\,d\omega_0(Q,s)\le c(\gamma\lambda)^2\omega_0(\triangle).\end{aligned}$$

Here N is a fixed constant depending on τ and β. $(\widehat{Q},\widehat{s})$ denotes a point in E, and the estimate $\Gamma(Q,s)\cap W_\beta\subseteq\Gamma_\gamma(\widehat{Q},\widehat{s})$ has been used several times.

(2) and (4) can be bounded similarly by

$$\int_\triangle c(\widehat{N}_\beta(F)(\widehat{Q},\widehat{s})^2+N_\beta(u_0)(\widehat{Q},\widehat{s}))^2\,d\omega_0(Q,s)\le c(\gamma\lambda)^2\omega_0(\triangle)$$

and finally

$$\begin{aligned}(3)\le&\int_{W_{\widehat\beta}\setminus W_\alpha}|F|\cdot|G_0|\cdot|[\varepsilon_{ij}]|\cdot|\nabla u_1|\,dy\,ds\\&+\int_{W_{\widehat\beta}}(|\nabla F|\cdot|G_0|\cdot|[\varepsilon_{ij}]|\cdot|\nabla u_1|)(y,s)\,dy\,ds\\&+\int_{W_{\widehat\beta}}(|F|\cdot|\nabla G_0|\cdot|[\varepsilon_{ij}]|\cdot|\nabla u_1|)(y,s)\,dy\,ds.\end{aligned}$$

The second integral is less than or equal to

$$\begin{aligned}&\int_\triangle\sum_{j=-\infty}^{N}\int_{R_j\cap W_{\widehat\beta}\cap\Gamma(Q,s)}(|\nabla F|\cdot|\varepsilon|\cdot|\nabla u_1|)(y,s)\delta(y,s)^{-n}\,dy\,ds\,d\omega_0\\
&\le\int_\triangle\sum_{j=-\infty}^{N}a(x_j,t_j)\bigg(\frac{1}{|R_j\cap\Gamma(Q,s)|}\int_{R_j\cap W_{\widehat\beta}\cap\Gamma(Q,s)}(|\delta\nabla F|)^2\bigg)^{1/2}\\
&\quad\times\bigg(\int_{R_j\cap W_{\widehat\beta}\cap\Gamma}|\nabla u_1|^2\,\delta^{-n}\bigg)^{1/2}d\omega_0\\
&\le\int_\triangle\bigg(\sum_j\bigg(\int_{R_j\cap\Gamma(Q,s)}\frac{a(y,s)^2}{\delta(y,s)^{n+2}}\bigg)^{\frac12\cdot2}\bigg)^{1/2}\widehat{N}_\beta(\delta\nabla F)(\widehat{Q},\widehat{s})\\
&\quad\times\bigg(\int_{\Gamma(Q,s)\cap W_\beta\subseteq\Gamma_\beta(\widehat{Q},\widehat{s})}|\nabla u_1|^2\,(y,s)\;\delta(y,s)^{-n}\,dy\,ds\bigg)^{1/2}d\omega_0(Q,s)\\
&\le\int_\triangle\bigg(\int_{\Gamma(Q,s)}\frac{a^2}{\delta^{n+2}}\bigg)^{1/2}N_\beta(\delta\nabla F)S_\beta(u_1)(\widehat{Q},\widehat{s})\,d\omega_0(Q,s)\\
&\le c(\gamma\lambda)^2\bigg(\int_\triangle\bigg(\int_{\Gamma(Q,s)}\frac{a(y,s)^2}{\delta(y,s)^{n+2}}\,dy\,ds\bigg)^{2/2}d\omega_0\bigg)^{1/2}\bigg(\int_\triangle d\omega_0\bigg)^{1/2}\\
&\le c(\gamma\lambda)^2\omega_0(\triangle)^{1/2}\omega_0(\triangle)^{1/2}=c(\gamma\lambda)^2\omega_0(\triangle)\end{aligned}$$

by the Carleson condition for a^2/δ^{n+2} which implies that $\int_\Gamma(a^2/\delta^{n+2})$ is BMO with respect to ω_0. The other two terms have similar bounds.

4. A result similar to the one in [CS], Theorem 1, for a "nondoubling" measure ω_1 found by using L^2 boundary data can be proved if the boundary function f is in $L^p(\partial D)$. In fact, one can obtain

$$\frac{\omega_1(E)}{\omega_1(\triangle_{4r}(Q,s))}\le c\bigg(\frac{\omega_0(E)}{\omega_0(\triangle_r(Q,s))}\bigg)^{1/p}\quad\text{for } E\subseteq\triangle_r(Q,s)$$

without assuming a center-doubling condition for ω_1. However, the B^q result of Theorem 4 cannot be proved unless both measures ω_0 and ω_1 satisfy center-doubling conditions. This condition is true for several parabolic-type measures: caloric measure associated with $\partial/\partial t-\Delta$ satisfies a center-doubling condition and so do the measures whose operators $\partial/\partial t-L$ have coefficients satisfying certain Lipschitz conditions [YH].

Consequently, the theorem in Section 3 holds for all center-doubling strictly elliptic parabolic-type measures. However, it is a fact that the norm inequality $\|N(u_1)\|_{L^2(\partial D,d\omega_0)}\le c\|f\|_{L^2(\partial D,d\omega_0)}$ can be proved given a Carleson condition using only the doubling property of the measure ω_0 and backwards Harnack for G_0. The properties needed for the second operator $\partial/\partial t-L_1$ are that its solutions satisfy Harnack's inequality and that the "measure" $\omega_1^{(x,t)}(\cdot)$ be a well-defined, nonnegative set function. For elliptic operators Chanillo and Wheeden [CW] give conditions on a weight $w(x)$ so that if $w(x)|\xi|^2\le\xi_ib^{ij}(x)\xi_j\le cw(x)|\xi|^2$ then $L_1=\sum_{i,j=1}^n(\partial/\partial x_i)(b_{ij}(x)\partial/\partial x_j)$ has solutions that satisfy a scale invariant version of Harnack's inequality. For any such operator on a domain where $\omega_1^{(x)}(\cdot)$ is well-defined the following result is valid:

THEOREM 8. *Suppose $a(y)=\sup_{x\in P_{\delta/2}(y)}\varepsilon(x)$, $\varepsilon(x)=\sup_{i,j}|a_{ij}(x)-b_{ij}(x)|$, and $G_0(x_0;y)$, the Green's function for L_0 on D, satisfy*

$$\sup_{r>0}\bigg(\frac{1}{\omega_0(\triangle_r(Q))}\int_{B_r(Q)\cap D}\frac{a(y)^2G_0(x_0;y)}{\delta(y)^2}\,dy\bigg)^{1/2}<c\varepsilon_0$$

for ε_0 sufficiently small. Then ω_1 is absolutely continuous with ω_0 on ∂D if ω_0 is a center-doubling measure.

L_0 must be assumed to be a strictly elliptic divergence form operator on D, but L_1 can be degenerate as described above.

The same result can be extended to a degenerate parabolic measure (or set function); in this case the coefficients of ellipticity are assumed to form measures $(w(x,t)dxdt)$ which compare with Lebesgue measure on approaching the boundary of the domain, as well as several other conditions (see [GW]).

An interesting open problem is to determine what kind of condition on the operators would yield absolute continuity of the associated measures when one of the operators is nonlinear.

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