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Weighted inequalities for one-sided maximal functions in Orlicz spaces

by

PEDRO ORTEGA SALVADOR (Málaga)

Abstract. Let M_g^+ be the maximal operator defined by

$$M_g^+ f(x) = \sup_{h>0} \frac{\int_x^{x+h} |f|g}{\int_x^{x+h} g},$$

where g is a positive locally integrable function on $\mathbb{R}$. Let Φ be an N-function such that both Φ and its complementary N-function satisfy Δ_2. We characterize the pairs of positive functions (u,w) such that the weak type inequality

$$u(\{x\in\mathbb{R} \mid M_g^+ f(x) > \lambda\}) \le \frac{C}{\Phi(\lambda)} \int_{\mathbb{R}} \Phi(|f|)w$$

holds for every f in the Orlicz space $L_\Phi(w)$. We also characterize the positive functions w such that the integral inequality

$$\int_{\mathbb{R}} \Phi(|M_g^+ f|)w \le \int_{\mathbb{R}} \Phi(|f|)w$$

holds for every $f\in L_\Phi(w)$. Our results include some already obtained for functions in L^p and yield as consequences one-dimensional theorems due to Gallardo and Kerman–Torchinsky.

1. Introduction and results. Let g be a positive locally integrable function on $\mathbb{R}$ and consider the maximal operator acting on measurable functions on $\mathbb{R}$ defined by

$$M_g^+ f(x) = \sup_{h>0} \frac{\int_x^{x+h} |f|g}{\int_x^{x+h} g}.$$

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The good weights for M_g^+ have been studied in [MOT] and, when $g = 1$, in [S]. The results obtained there were the following:

THEOREM A. *The operator M_g^+ is of weak type $(1,1)$ with respect to the measures $u\,dx$ and $w\,dx$ if and only if the couple (u,w) satisfies condition $A_1^+(g)$, which means that there exists $C > 0$ such that*

$$M_g^-(g^{-1}u) \le Cg^{-1}w \quad a.e.,$$

where M_g^- is the left maximal operator defined in the obvious way.

THEOREM B. *Let $1 < p < \infty$. The operator M_g^+ is of weak type (p,p) with respect to the measures $u\,dx$ and $w\,dx$ if and only if (u,w) satisfies condition A_p^+, which means that there exists $C > 0$ such that*

$$\int_a^b u\Big(\int_b^c g^{p'} w^{1-p'}\Big)^{p-1} \le C\Big(\int_a^c g\Big)^p$$

for every $a,b,c \in \mathbb{R}$ with $a < b < c$, where p' is the conjugate exponent of p.

THEOREM C. *Let $1 < p < \infty$ and let w be a nonnegative measurable function. The following statements are equivalent:*

(i) *M_g^+ is of weak type (p,p) with respect to the measure $w\,dx$.*
(ii) *M_g^+ is bounded in $L^p(w\,dx)$.*
(iii) *w satisfies $A_p^+(g)$ (i.e., (w,w) satisfies $A_p^+(g)$).*

Muckenhoupt's results for the Hardy–Littlewood maximal operator (see [M]) in the one-dimensional case are consequences of Theorems A, B and C.

It is interesting to ask whether generalizations of these theorems to Orlicz spaces are possible, as was done in [KT] and [G] for the Hardy–Littlewood maximal operator. The purpose of this paper is to give an affirmative answer to this question. In the proofs of our results we use arguments and techniques due to Gallardo [G], Kerman–Torchinsky [KT] and Martín Reyes [MR], whose new simple proofs of Theorems A, B and C have been fundamental.

Before giving the statements of the theorems we recall the basic definitions and results about N-functions and Orlicz spaces which will be used later. Detailed treatments can be found in [KR] and [Mu].

An *N-function* is a continuous and convex function $\Phi : [0,\infty) \to \mathbb{R}$ such that $\Phi(s) > 0$ if $s > 0$, $s^{-1}\Phi(s) \to 0$ as $s \to 0$ and $s^{-1}\Phi(s) \to \infty$ as $s \to \infty$.

Every N-function Φ admits a representation of the form $\Phi(s) = \int_0^s \phi(t)\,dt$, where $\phi : [0,\infty) \to \mathbb{R}$ is nondecreasing right-continuous with $\phi(0) = 0$, $\phi(s) > 0$ if $s > 0$ and $\phi(s) \to \infty$ as $s \to \infty$. The function ϕ is called the *density function* of Φ.

Associated with ϕ we have a function $\psi : [0,\infty) \to \mathbb{R}$ defined by $\psi(t) = \sup\{s : \phi(s) \le t\}$. The function ψ has the same properties as ϕ and is called

the *generalized inverse* of ϕ. The N-function Ψ defined by $\Psi(t) = \int_0^t \psi(s)\,ds$ is called the *complementary N-function* of Φ.

An N-function Φ satisfies *condition* Δ_2 in $[0,\infty)$ if $\sup_{s>0}\Phi(2s)/\Phi(s) < \infty$. If ϕ is the density function of Φ, then Φ satisfies Δ_2 if and only if there exists $\alpha > 1$ such that $s\phi(s) < \alpha\Phi(s)$ for every $s > 0$. Condition Δ_2 can also be expressed in the following equivalent way: for every $A > 0$ there exists $B > 0$ such that $\Phi(At) \le B\Phi(t)$ for every $t > 0$. Condition Δ_2 does not necessarily pass to the complementary N-function. A necessary and sufficient condition for Ψ to satisfy Δ_2 is that there exists $\beta > 1$ such that $\beta\Phi(s) < s\phi(s)$ for every $s > 0$.

If $(X,\mathcal{M},\mu)$ is a σ-finite measure space and Φ is an N-function, the *Orlicz spaces* $\widetilde{L}_\Phi$ and L_Φ are defined as follows:

$$\widetilde{L}_\Phi = \Big\{f : X \to \mathbb{R} \;\Big|\; f \text{ is measurable and } \int_X \Phi(|f|) < \infty\Big\}$$

and

$$L_\Phi = \{f : X \to \mathbb{R} \mid f \text{ is measurable and } fg \in L_1 \text{ for every } g \in \widetilde{L}_\Psi\},$$

where Ψ is the complementary N-function of Φ. We always have $\widetilde{L}_\Phi \subset L_\Phi$. If Φ satisfies Δ_2, we have $\widetilde{L}_\Phi = L_\Phi$.

The Orlicz space L_Φ is a Banach space with the norms

$$\|f\|_\Phi = \inf\Big\{\lambda > 0 \;\Big|\; \int_X \Phi(\lambda^{-1}|f|)\,d\mu \le 1\Big\}$$

and

$$\|f\|_{(\Phi)} = \sup\Big\{\int_X |fg|\,d\mu \;\Big|\; g \in S_\Psi\Big\},$$

where $S_\Psi = \{g \in L_\Psi \mid \int_X \Psi(|g|) \le 1\}$. These norms are called, respectively, the *Luxemburg norm* and the *Orlicz norm*. They are equivalent; in fact, the inequalities $\|f\|_\Phi \le \|f\|_{(\Phi)} \le 2\|f\|_\Phi$ hold.

The Hölder inequality in L^p spaces has a natural extension to Orlicz spaces: if $f \in L_\Phi$ and $g \in L_\Psi$, then $fg \in L^1$ and

$$\int_X |fg| \le \|f\|_{(\Phi)}\|g\|_\Psi \le 2\|f\|_\Phi\|g\|_\Psi.$$

When both Φ and Ψ satisfy Δ_2, the Banach space L_Φ is reflexive.

If Φ is an N-function, we can define its *upper* and *lower indices*, respectively, as follows:

$$\alpha_\Phi = \inf_{0<s<1} \frac{-\log h_\Phi(s)}{\log s} \quad \text{and} \quad \beta_\Phi = \sup_{s>1} \frac{-\log h_\Phi(s)}{\log s},$$

where $h_\Phi(s) = \sup_{t>0} \Phi^{-1}(t)/\Phi^{-1}(st)$.

The inequalities $0 \le \beta_\Phi \le \alpha_\Phi \le 1$ are always satisfied. If Φ satisfies Δ_2, then $\beta_\Phi > 0$ and if the complementary function of Φ satisfies Δ_2, then $\alpha_\Phi < 1$. The numbers $p_\Phi = \alpha_\Phi^{-1}$ and $q_\Phi = \beta_\Phi^{-1}$ are called, respectively, the *lower exponent* and *upper exponent* of Φ.

We will need, finally, the following interpolation theorem (see [G]):

THEOREM D. *Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{F}, \nu)$ be two σ-finite measure spaces. Let Φ be an N-function with complementary N-function Ψ. Suppose that Φ and Ψ satisfy Δ_2. Let p and q be, respectively, the lower and upper exponents of Φ. Let T be a sublinear operator which is of weak type (r,r) and of weak type (s,s), where $1 \le r < p$ and $q < s \le \infty$. Then T maps $L_\Phi(\mu)$ into $L_\Phi(\nu)$ and there exists $C > 0$ such that*

$$\int_Y \Phi(|Tf|)\,d\nu \le C \int_X \Phi(|f|)\,d\mu$$

for every $f \in L_\Phi(\mu)$.

In what follows, Φ will be an N-function with density ϕ, ψ will be the generalized inverse of ϕ and Ψ the complementary N-function of Φ. We will suppose that both Φ and Ψ satisfy Δ_2. Throughout the paper, C will stand for a positive constant, not necessarily the same at each occurrence. We will often use the notation $h(E)$ for the integral of the function h over the measurable set E.

DEFINITION. A couple (u, w) of positive functions on $\mathbb{R}$ satisfies $A_\Phi^+(g)$ if there exists $K > 0$ such that

$$\left(\frac{\int_a^b \varepsilon u}{\int_a^c g}\right)\phi\left(\frac{\int_b^c g\psi(g/(\varepsilon w))}{\int_a^c g}\right) \le K$$

for every $a, b, c \in \mathbb{R}$ with $a < b < c$ and every $\varepsilon > 0$.

It is clear that if $\Phi(t) = t^p$, $p > 1$, then $A_\Phi^+(g)$ is nothing but $A_p^+(g)$.

Our results are the following:

THEOREM 1. *The following statements are equivalent:*

(a) *The couple (u, w) satisfies $A_\Phi^+(g)$.*

(b) *There exists $C > 0$ such that*

$$u(\{x \in \mathbb{R} \mid M_g^+ f(x) > \lambda\}) \le \frac{C}{\Phi(\lambda)} \int_{\mathbb{R}} \Phi(|f|)w$$

for every $\lambda > 0$ and every $f \in L_\Phi(w)$.

THEOREM 2. *Let w be a positive measurable function. Let p^{-1} be the upper index of Φ. The following statements are equivalent:*

(a) *There exists $C > 0$ such that*

$$\int_{\mathbb{R}} \Phi(M_g^+ f)w \le C \int_{\mathbb{R}} \Phi(|f|)w$$

for every $f \in L_\Phi(w)$.

(b) *There exists $C > 0$ such that*

$$w(\{x \in \mathbb{R} \mid M_g^+ f(x) > \lambda\}) \le \frac{C}{\Phi(\lambda)} \int_{\mathbb{R}} \Phi(|f|)w$$

for every $\lambda > 0$ and every $f \in L_\Phi(w)$.

(c) *The function w satisfies $A_\Phi^+(g)$.*

(d) *The function w satisfies $A_p^+(g)$.*

It is clear that the results of [KT] and [G] are consequences of Theorems 1 and 2 in the one-dimensional case.

2. Proof of Theorem 1. (a)⇒(b). Let f be a measurable function and $\lambda > 0$. We may assume without loss of generality that f is a bounded nonnegative function with compact support. Let $O_\lambda = \{x \in \mathbb{R} \mid M_g^+ f(x) > \lambda\}$. Let $\{I_j\}$ be the sequence of the connected components of O_λ. Each I_j is a bounded open interval (a_j, b_j) with

$$\lambda \int_x^{b_j} g \le \int_x^{b_j} fg \quad \text{for every } x \in I_j. \tag{1}$$

Let $I = (a, b)$ be one of the connected components of O_λ and consider the following sequence: $x_0 = a$ and, given x_k, x_{k+1} is the real number such that

$$\int_{x_k}^{x_{k+1}} fg = \frac{1}{2} \int_{x_k}^{b} fg. \tag{2}$$

The sequence $\{x_k\}$ is increasing with limit b and satisfies

$$\int_{x_{k-1}}^{b} fg = 4 \int_{x_k}^{x_{k+1}} fg \quad \text{for every } k. \tag{3}$$

By (1) and (3) it follows that

$$\lambda \int_{x_{k-1}}^{b} g \le 4 \int_{x_k}^{x_{k+1}} fg \quad \text{for every } k. \tag{4}$$

Relation (4), monotonicity of Φ and condition Δ_2 for Φ give

$$1 \le C \frac{1}{\Phi(\lambda)} \Phi\left(\frac{\int_{x_k}^{x_{k+1}} fg}{\int_{x_{k-1}}^{b} g}\right) \quad \text{for every } k. \tag{5}$$

By the Hölder inequality, the right-hand side of (5) is smaller than

$$(6)\qquad C\frac{1}{\Phi(\lambda)}\Phi\bigg(\frac{1}{\int_{x_{k-1}}^{b} g}\|f\chi_{(x_k,x_{k+1})}\|_{\Phi,\varepsilon w}\bigg\|\frac{g\chi_{(x_k,x_{k+1})}}{\varepsilon w}\bigg\|_{\Psi,\varepsilon w}\bigg)\qquad \text{for all } \varepsilon>0.$$

Now, we are going to estimate the second norm which appears in (6). By definition,

$$\bigg\|\frac{g\chi_{(x_k,x_{k+1})}}{\varepsilon w}\bigg\|_{\Psi,\varepsilon w}=\inf\bigg\{\alpha>0\ \bigg|\ \int_{\mathbb{R}}\Psi\bigg(\frac{g\chi_{(x_k,x_{k+1})}}{\alpha\varepsilon w}\bigg)\varepsilon w\le 1\bigg\}.$$

The existence of $\beta_1>1$ such that $\beta_1\Psi(s)<s\psi(s)$ for every $s>0$ and $A_\Phi^+(g)$ give

$$(7)\qquad \int_{\mathbb{R}}\Psi\bigg(\frac{g\chi_{(x_k,x_{k+1})}}{\alpha\varepsilon w}\bigg)\varepsilon w\le\beta_1^{-1}\alpha^{-1}\int_{x_k}^{x_{k+1}} g\psi\bigg(\frac{g}{\alpha\varepsilon w}\bigg)$$
$$\le\beta_1^{-1}\alpha^{-1}\int_{x_{k-1}}^{x_{k+1}} g\psi\bigg(\frac{K\int_{x_{k-1}}^{x_{k+1}} g}{\alpha\varepsilon\int_{x_{k-1}}^{x_k} u}\bigg)\qquad\text{for every }\alpha>0.$$

Let

$$\alpha=K\int_{x_{k-1}}^{x_{k+1}} g\Phi^{-1}\bigg(\varepsilon^{-1}\Big(\int_{x_{k-1}}^{x_k} u\Big)^{-1}\bigg).$$

For this α, the last member of (7) equals

$$(8)\qquad K^{-1}\beta_1^{-1}\varepsilon\int_{x_{k-1}}^{x_k} u\frac{\varepsilon^{-1}(\int_{x_{k-1}}^{x_k} u)^{-1}}{\Phi^{-1}(\varepsilon^{-1}(\int_{x_{k-1}}^{x_k} u)^{-1})}\psi\bigg(\frac{\varepsilon^{-1}(\int_{x_{k-1}}^{x_k} u)^{-1}}{\Phi^{-1}(\varepsilon^{-1}(\int_{x_{k-1}}^{x_k} u)^{-1})}\bigg).$$

Since there exists $\beta_2>1$ such that $s\psi(s)\le\beta_2\Psi(s)$ for every $s>0$ and besides $s\le\Phi^{-1}(s)\Psi^{-1}(s)$, (8) is dominated by

$$(9)\qquad \beta_1^{-1}\beta_2K^{-1}\varepsilon\int_{x_{k-1}}^{x_k} u\Psi\bigg(\frac{\varepsilon^{-1}(\int_{x_{k-1}}^{x_k} u)^{-1}}{\Phi^{-1}(\varepsilon^{-1}(\int_{x_{k-1}}^{x_k} u)^{-1})}\bigg)\le\beta_1^{-1}\beta_2K^{-1}\le 1,$$

where the last inequality holds if we take $K\ge\beta_1^{-1}\beta_2$ from the beginning. Therefore, the definition of the Luxemburg norm gives

$$(10)\qquad \bigg\|\frac{g\chi_{(x_k,x_{k+1})}}{\varepsilon w}\bigg\|_{\Psi,\varepsilon w}\le K\int_{x_{k-1}}^{x_{k+1}} g\Phi^{-1}\bigg(\varepsilon^{-1}\Big(\int_{x_{k-1}}^{x_k} u\Big)^{-1}\bigg).$$

From (5), (6) and (10) we obtain

$$(11)\qquad 1\le C\frac{1}{\Phi(\lambda)}\Phi\bigg(K\|f\chi_{(x_k,x_{k+1})}\|_{\Phi,\varepsilon w}\frac{\int_{x_{k-1}}^{x_{k+1}} g}{\int_{x_{k-1}}^{b} g}\Phi^{-1}\bigg(\varepsilon^{-1}\Big(\int_{x_{k-1}}^{x_k} u\Big)^{-1}\bigg)\bigg)$$

for every $\varepsilon>0$. Set $\varepsilon=(\int_{x_k}^{x_{k+1}}\Phi(f)w)^{-1}$. Then $\|f\chi_{(x_k,x_{k+1})}\|_{\Phi,\varepsilon w}=1$. From (11) and Δ_2 it now follows that

$$(12)\qquad 1\le C\frac{1}{\Phi(\lambda)}\Phi\bigg(K\frac{\int_{x_{k-1}}^{x_{k+1}} g}{\int_{x_{k-1}}^{b} g}\Phi^{-1}\bigg(\frac{\int_{x_k}^{x_{k+1}}\Phi(f)w}{\int_{x_{k-1}}^{x_k} u}\bigg)\bigg)\le\frac{C}{\Phi(\lambda)}\cdot\frac{\int_{x_k}^{x_{k+1}}\Phi(f)w}{\int_{x_{k-1}}^{x_k} u},$$

i.e.,

$$(13)\qquad \int_{x_{k-1}}^{x_k} u\le\frac{C}{\Phi(\lambda)}\int_{x_k}^{x_{k+1}}\Phi(f)w.$$

Summing up over k and then over I_j we obtain (b).

(b)⇒(a). Let $a,b,c\in\mathbb{R}$ with $a<b<c$, let $\varepsilon>0$ and let $f=\chi_{(b,c)}\psi(g(\varepsilon w)^{-1})$. If $x\in(a,b)$, then

$$(14)\qquad M_g^+f(x)\ge\frac{\int_x^c\chi_{(b,c)}\psi(g(\varepsilon w)^{-1})g}{\int_x^c g}>\frac{\int_b^c g\psi(g(\varepsilon w)^{-1})}{\int_a^c g}.$$

This means that

$$(15)\qquad (a,b)\subset\bigg\{x\in\mathbb{R}\ \bigg|\ M_g^+f(x)>\frac{\int_b^c g\psi(g(\varepsilon w)^{-1})}{\int_a^c g}\bigg\}.$$

From the weak type inequality (b) and (15) we obtain

$$(16)\qquad \int_a^b\varepsilon u\le\frac{C}{\Phi((\int_a^c g)^{-1}\int_b^c g\psi(g(\varepsilon w)^{-1}))}\int_b^c\Phi(\psi(g(\varepsilon w)^{-1}))\varepsilon w.$$

Inequality (16) and the existence of $\beta_1>1$ such that $s\phi(s)<\beta_1\Phi(s)$ for every $s>0$ give

$$(17)\qquad \int_a^b\varepsilon u\le\frac{C}{((\int_a^c g)^{-1}\int_b^c g\psi(g(\varepsilon w)^{-1}))\phi((\int_a^c g)^{-1}\int_b^c g\psi(g(\varepsilon w)^{-1}))}$$
$$\times\int_b^c\Phi(\psi(g(\varepsilon w)^{-1}))\varepsilon w,$$

i.e.,

$$(18)\qquad \frac{\int_a^b\varepsilon u}{\int_a^c g}\phi\bigg(\frac{\int_b^c g\psi(g(\varepsilon w)^{-1})}{\int_a^c g}\bigg)\le C\frac{\int_b^c\Phi(\psi(g(\varepsilon w)^{-1}))\varepsilon w}{\int_b^c g\psi(g(\varepsilon w)^{-1})},$$

which, upon taking into account $\Phi(\psi(s))\le Cs\psi(s)$, implies $A_\Phi^+(g)$.

3. Proof of Theorem 2. The implication (a)⇒(b) is clear and (b)⇒(c) is already proved in Theorem 1.

(c)⇒(d). We will need several lemmas.

LEMMA 1. *The following statements are equivalent*:

(i) *The operator M_g^+ is of restricted weak type (p,p) with respect to the measure $w\,dx$.*

(ii) *There exists $C>0$ such that*

$$\frac{g(E)}{g(a,c)} \le C\left(\frac{w(E)}{w(a,b)}\right)^{1/p} \tag{19}$$

for every $a,b,c\in\mathbb{R}$ with $a<b<c$ and every measurable set $E\subseteq(b,c)$.

Proof. (i)⇒(ii). Let $a,b,c\in\mathbb{R}$ with $a<b<c$ and let $E\subset(b,c)$ be a measurable set with positive measure. If $x\in(a,b)$, we have

$$M_g^+\chi_E(x) \ge \frac{\int_x^c \chi_E g}{\int_x^c g} \ge \frac{g(E)}{g(a,c)}.$$

This implies that $(a,b)\subset\{x \mid M_g^+\chi_E(x) > g(E)/(2g(a,c))\}$. Then, by (i),

$$w(a,b) \le C\left(\frac{g(a,c)}{g(E)}\right)^p w(E),$$

as we wished to prove.

(ii)⇒(i). Let E be a measurable set. We may assume without loss of generality that E is bounded and has positive measure. Let $\lambda>0$ and $O_\lambda=\{x \mid M_g^+\chi_E(x)>\lambda\}$. We have $O_\lambda=\bigcup_j I_j$, where the intervals I_j are bounded, pairwise disjoint and every $I_j=(a_j,b_j)$ satisfies $\lambda\int_x^{b_j} g\le\int_x^{b_j}\chi_E g$ for every $x\in I_j$. Let $I=(a,b)$ be one of the component intervals and define the sequence $\{x_k\}$ by $x_0=a$ and by letting x_{k+1} be the only real number such that $\int_{x_k}^{x_{k+1}}\chi_E g=\int_{x_{k+1}}^b \chi_E g$. Then, for every $k\ge1$,

$$\frac{\int_{x_{k-1}}^b \chi_E g}{\int_{x_{k-1}}^b g} = \frac{4\int_{x_k}^{x_{k+1}}\chi_E g}{\int_{x_{k-1}}^b g} \ge \lambda.$$

From this and (ii) it follows that

$$\int\limits_{x_{k-1}}^{x_k} w \le C\frac{w(x_{k-1},x_k)(g(E\cap(x_k,x_{k+1})))^p}{\lambda^p(g(x_{k-1},b))^p}$$
$$\le \frac{C}{\lambda^p}w(x_{k-1},x_k)\left(\frac{g(E\cap(x_k,x_{k+1}))}{g(x_{k-1},x_{k+1})}\right)^p$$
$$\le \frac{C}{\lambda^p}w(E\cap(x_k,x_{k+1})) = \frac{C}{\lambda^p}\int\limits_{x_k}^{x_{k+1}}\chi_E w.$$

Summing up over k we obtain $\int_a^b w\le(C/\lambda^p)\int_a^b\chi_E w$, and since O_λ is the disjoint union of the intervals I_j, the restricted weak type inequality is proved.

LEMMA 2. *If $w\in A_\Phi^+(g)$, then $w\in A_r^+(g)$ for every $r>p$.*

Proof. First we prove that $w\in A_\Phi^+(g)$ implies (19). Let $a,b,c\in\mathbb{R}$ with $a<b<c$ and let $E\subseteq(b,c)$ be measurable with positive measure. If $w(E)/w(a,b)\ge1$, there is nothing to prove. Suppose that $w(E)/w(a,b)<1$. The Hölder inequality ensures that $g(E)\le\|\chi_E\|_{\Phi,\varepsilon w}\|g\chi_E\varepsilon^{-1}w^{-1}\|_{\Psi,\varepsilon w}$ for every $\varepsilon>0$. If we argue as in Theorem 1 to dominate $\|g\chi_E\varepsilon^{-1}w^{-1}\|_{\Psi,\varepsilon w}$ (the argument uses $A_\Phi^+(g)$), we obtain

$$\|g\chi_E\varepsilon^{-1}w^{-1}\|_{\Psi,\varepsilon w}\le Kg(a,c)\Phi^{-1}(\varepsilon^{-1}(w(a,b))^{-1}).$$

Then

$$\frac{g(E)}{g(a,c)}\le K\frac{\Phi^{-1}\left(\frac{1}{\varepsilon w(E)}\frac{w(E)}{w(a,b)}\right)}{\Phi^{-1}\left(\frac{1}{\varepsilon w(E)}\right)}$$

for every $\varepsilon>0$. From the definition of the upper index of Φ it follows that for every $s\in(0,1)$ there exists $t>0$ such that $\Phi^{-1}(st)/\Phi^{-1}(t)<2s^{1/p}$. If we take $s=w(E)/w(a,b)$ and $\varepsilon=1/tw(E)$, we obtain (19). Now, (19), Lemma 1 and the interpolation theorem of Stein and Weiss (Theorem 3.15 of [SW]) give $w\in A_r^+(g)$ for every $r>p$.

LEMMA 3. *Let $\delta>0$ and let $\psi_\delta(t)=(\psi(t))^{1+\delta}$. Let Ψ_δ be the N-function with density ψ_δ and let Φ_δ be the complementary N-function of Ψ_δ. Then the upper index of Φ_δ is greater than the upper index of Φ.*

Lemma 3 and its proof can be found in [KT].

LEMMA 4. *Let $w\in A_\Phi^+(g)$ and let $v(x)=\psi(g(x)/w(x))$. Then there exist two positive numbers α and β such that*

$$g(\{x\in(a,b) \mid v(x)>\beta\lambda\})>\alpha g(a,b) \tag{20}$$

for every $\lambda>0$ and every interval (a,b) satisfying $\lambda\int_x^b g\le\int_x^b gv$ for every $x\in(a,b)$.

Proof. Let $\lambda>0$ and let (a,b) be an interval with the above property. Let $\{x_k\}$ be the sequence defined by letting $x_0=a$ and x_{k+1} be the real number which satisfies $\int_{x_k}^b gv=2\int_{x_k}^{x_{k+1}}gv$. This implies $\int_{x_{k-1}}^b gv=4\int_{x_k}^{x_{k+1}}gv$. Then

$$g(\{x\in(a,b) \mid v(x)\le\beta\lambda\})=\sum_{k=1}^\infty g(\{x\in(x_{k-1},x_k) \mid v(x)\le\beta\lambda\}) \tag{21}$$
$$\le\sum_{k=1}^\infty g\left(\left\{x\in(x_{k-1},x_k)\;\middle|\; v(x)\le4\beta\frac{\int_{x_k}^{x_{k+1}}gv}{\int_{x_{k-1}}^b g}\right\}\right)=\sum_{k=1}^\infty g(E_k),$$

where

$$E_k=\Big\{x\in(x_{k-1},x_k)\ \Big|\ v(x)\le 4\beta\frac{\int_{x_k}^{x_{k+1}} gv}{\int_{x_{k-1}}^{b} g}\Big\}.$$

If we take $\beta<1/4$ and r with $1<r<p$, then the definition of E_k, the fact that the function $g(x)/\phi(v(x))$ is essentially equal to $w(x)$, the property $\Phi(st)\le Cs^r\Phi(t)$ ($0<s<1$, $t>0$), which implies $\phi(st)\le Cs^{r-1}\phi(t)$ ($0<s<1$, $t>0$), and $A_\Phi^+(g)$ give

$$\text{(22)}\quad \frac{g(E_k)}{g(x_{k-1},x_{k+1})}\le\frac{\int_{x_{k-1}}^{x_k}\frac{g(x)}{\phi(v(x))}\,dx}{\int_{x_{k-1}}^{x_{k+1}} g}\,\phi\Big(4\beta\frac{\int_{x_k}^{x_{k+1}} gv}{\int_{x_{k-1}}^{b} g}\Big)$$
$$\le C(4\beta)^{r-1}\frac{\int_{x_{k-1}}^{x_k} w}{\int_{x_{k-1}}^{x_{k+1}} g}\,\phi\Big(\frac{\int_{x_k}^{x_{k+1}} gv}{\int_{x_{k-1}}^{b} g}\Big)\le CK(4\beta)^{r-1}.$$

From (22) and (21) we get

$$\text{(23)}\quad g(\{x\in(a,b)\mid v(x)\le\beta\lambda\})\le CK(4\beta)^{r-1}\sum_{k=1}^{\infty} g(x_{k-1},x_{k+1})$$
$$\le 2CK(4\beta)^{r-1}g(a,b).$$

Finally, from (23) it follows that

$$g(\{x\in(a,b)\mid v(x)>\beta\lambda\})\ge g(a,b)-2CK(4\beta)^{r-1}g(a,b)$$
$$=(1-2CK(4\beta)^{r-1})g(a,b).$$

Taking β small enough, we are done.

LEMMA 5. *Let $w\in A_\Phi^+(g)$. Then there exist $\delta>0$ and $C>0$ such that*

$$\text{(24)}\quad \int_a^b g\psi_\delta\Big(\frac{g}{w}\Big)\le C\int_a^b g\psi\Big(\frac{g}{w}\Big)\Big(M_g^+\Big(\psi\Big(\frac{g}{w}\Big)\chi_{(a,b)}\Big)(a)\Big)^{\delta}$$

for every interval (a,b) and, therefore,

$$\text{(25)}\quad M_g^+\Big(\psi_\delta\Big(\frac{g}{w}\Big)\chi_{(a,b)}\Big)(a)\le C\Big(M_g^+\Big(\psi\Big(\frac{g}{w}\Big)\chi_{(a,b)}\Big)(a)\Big)^{1+\delta}.$$

The proof of Lemma 5 can be done by arguing as in Lemma 5 of [MR]. The main tool in the proof is Lemma 4.

LEMMA 6. *If $w\in A_\Phi^+(g)$, then there exists $C>0$ such that*

$$\text{(26)}\quad M_g^+\Big(\chi_I\psi\Big(\frac{g}{w}\Big)\Big)(x)\le C\psi\Big(M_w^+\Big(\frac{\chi_I g}{w}\Big)(x)\Big)$$

for every bounded interval I and every $x\in I$.

Proof. Since $w\in A_\Phi^+(g)$, the function $g\psi(g/w)$ is locally integrable. Let $I=(a,b)$ and $x\in I$. There exists $h>0$ with $x+h\in I$ such that

$$\text{(27)}\quad \frac{3}{4}M_g^+\Big(\chi_I\psi\Big(\frac{g}{w}\Big)\Big)(x)\le\frac{\int_x^{x+h} g\psi(g/w)}{\int_x^{x+h} g}.$$

For this h there exists t with $0<t<h$ such that $2\int_x^{x+t} g=\int_x^{x+h} g$. The number t satisfies

$$\text{(28)}\quad \frac{\int_x^{x+t} g\psi(g/w)}{\int_x^{x+t} g}\le M_g^+\Big(\chi_I\psi\Big(\frac{g}{w}\Big)\Big)(x).$$

From (27) and (28) we obtain

$$\text{(29)}\quad M_g^+\Big(\chi_I\psi\Big(\frac{g}{w}\Big)\Big)(x)\le\frac{4}{3}\frac{\int_x^{x+h} g\psi(g/w)}{\int_x^{x+h} g}$$
$$\le\frac{4}{3}\frac{\int_x^{x+t} g\psi(g/w)}{\int_x^{x+h} g}+\frac{4}{3}\frac{\int_{x+t}^{x+h} g\psi(g/w)}{\int_x^{x+h} g}$$
$$\le\frac{2}{3}M_g^+\Big(\chi_I\psi\Big(\frac{g}{w}\Big)\Big)(x)+\frac{4}{3}\frac{\int_{x+t}^{x+h} g\psi(g/w)}{\int_x^{x+h} g},$$

i.e.,

$$\text{(30)}\quad M_g^+\Big(\chi_I\psi\Big(\frac{g}{w}\Big)\Big)(x)\le 4\frac{\int_{x+t}^{x+h} g\psi(g/w)}{\int_x^{x+h} g}.$$

Finally, since $w\in A_\Phi^+(g)$, (30) gives

$$\text{(31)}\quad M_g^+\Big(\chi_I\psi\Big(\frac{g}{w}\Big)\Big)(x)\le C\psi\Big(\frac{\int_x^{x+h} g}{\int_x^{x+t} w}\Big)\le C\psi\Big(M_w^+\Big(\frac{\chi_I g}{w}\Big)(x)\Big),$$

which is the relationship we wished to prove.

LEMMA 7. *If $w\in A_\Phi^+(g)$, then there exists $\delta>0$ such that $w\in A_{\Phi_\delta}^+(g)$.*

Proof. Let $\delta>0$ be the number associated with w by Lemma 5. Let $a,b,c\in\mathbb{R}$ with $a<b<c$ and consider the finite sequence $x_0=b>x_1>\ldots>x_N\ge x_{N+1}=a$ defined by

$$\int_{x_k}^{c} g\psi_\delta\Big(\frac{g}{w}\Big)=2^k\int_b^c g\psi_\delta\Big(\frac{g}{w}\Big)\quad\text{if } k=0,1,\ldots,N$$

and

$$\int_a^{x_N} g\psi_\delta\Big(\frac{g}{w}\Big)<2^N\int_b^c g\psi_\delta\Big(\frac{g}{w}\Big).$$

Then the definition of M_g^+ and the property $\Phi_\delta(st) \le Cs^r\Phi_\delta(t)$ ($t>0$, $0<s<1$ and $r>1$ smaller than the lower exponent of Φ_δ) give

$$\begin{aligned}(32)\qquad \int_a^b w(x)\Phi_\delta\Big(\frac{\int_b^c g\psi_\delta(g/w)}{\int_a^c g}\Big)dx &= \sum_{k=0}^N \int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(\frac{\int_b^c g\psi_\delta(g/w)}{\int_a^c g}\Big)dx\\ &= \sum_{k=0}^N \int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(\frac{\int_{x_k}^c g\psi_\delta(g/w)}{2^k\int_a^c g}\Big)dx\\ &\le \sum_{k=0}^N \int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(\frac{\int_x^c g\psi_\delta(g/w)}{2^k\int_x^c g}\Big)dx\\ &\le \sum_{k=0}^N \int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(2^{-k}M_g^+\Big(\psi_\delta\Big(\frac{g}{w}\Big)\chi_{(x,c)}\Big)(x)\Big)dx\\ &\le C\sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(M_g^+\Big(\psi_\delta\Big(\frac{g}{w}\Big)\chi_{(x,c)}\Big)(x)\Big)dx.\end{aligned}$$

If we now apply Lemma 5 (inequality (25)), Lemma 6 and the definition of ψ_δ, then the last term of (32) is smaller than or equal to

$$\begin{aligned}(33)\qquad C\sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(\Big(M_g^+\Big(\psi\Big(\frac{g}{w}\Big)\chi_{(x,c)}\Big)(x)\Big)^{1+\delta}\Big)dx &\le \sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(\Big(\psi\Big(M_w^+\Big(\frac{\chi_{(x,c)}g}{w}\Big)(x)\Big)\Big)^{1+\delta}\Big)dx\\ &= \sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{x_k} w(x)\Phi_\delta\Big(\psi_\delta\Big(M_w^+\Big(\frac{\chi_{(x,c)}g}{w}\Big)(x)\Big)\Big)dx.\end{aligned}$$

The inequality $\Phi_\delta(\psi_\delta(s)) \le C\Psi_\delta(s)$, the boundedness of M_w^+ in $L_{\Psi_\delta}(w)$ and the property $\Psi_\delta(s)\le Cs\psi_\delta(s)$ allow us to dominate the last term of (33) by

$$\begin{aligned}(34)\qquad C\sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{x_k} w(x)\Psi_\delta\Big(M_w^+\Big(\frac{\chi_{(x,c)}g}{w}\Big)(x)\Big)dx &\le C\sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{c} w(x)\Psi_\delta\Big(M_w^+\Big(\frac{\chi_{(x_{k+1},c)}g}{w}\Big)(x)\Big)dx\\ &\le C\sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{c} w(x)\Psi_\delta\Big(\frac{g}{w}\Big)w\\ &\le C\sum_{k=0}^N 2^{-kr}\int_{x_{k+1}}^{c} w(x)\psi_\delta\Big(\frac{g}{w}\Big)g\\ &= C\sum_{k=0}^N 2^{-kr+k+1}2^{-(k+1)}\int_{x_{k+1}}^{c} w(x)\psi_\delta\Big(\frac{g}{w}\Big)g \le C\int_b^c g\psi_\delta\Big(\frac{g}{w}\Big).\end{aligned}$$

By (32)–(34) we get

$$(35)\qquad \int_a^b w(x)\Phi_\delta\Big(\frac{\int_b^c g\psi_\delta(g/w)}{\int_a^c g}\Big)dx \le C\int_b^c g\psi_\delta\Big(\frac{g}{w}\Big).$$

If we take into account in (35) that there exist $C_1>0$ and $C_2>0$ such that $C_1 s\phi_\delta(s)\le \Phi_\delta(s)\le C_2 s\phi_\delta(s)$ for all $s>0$, we obtain

$$\frac{\int_a^b w}{\int_a^c g}\phi_\delta\Big(\frac{\int_b^c g\psi_\delta(g/w)}{\int_a^c g}\Big)\le C,$$

and since the whole argument can be repeated replacing w by εw without changing the constants, it is proved that $w\in A_{\Phi_\delta}^+(g)$.

Now, the proof of (c)⇒(d) is easy. By Lemma 7, if $w\in A_\Phi^+(g)$ there exists $\delta>0$ such that $w\in A_{\Phi_\delta}^+(g)$. By Lemma 3, the upper index of Φ_δ, say r^{-1}, is greater than the upper index of Φ, which is p^{-1}. Finally, Lemma 2 ensures that $w\in A_s^+(g)$ for every $s>r$ and, since $p>r$, $w\in A_p^+(g)$.

(d)⇒(a). If $w\in A_p^+(g)$, there exists r with $1<r<p$ such that $w\in A_r^+(g)$. This means that M_g^+ is of weak type (r,r) with respect to $w\,dx$. On the other hand, M_g^+ is of weak type (q,q) with respect to $w\,dx$ for every $q>p$ and, in particular, for every $q>s$, where s is the upper exponent of Φ. Then Theorem D gives (a).

References

[G] D. Gallardo, *Weighted weak type integral inequalities for the Hardy–Littlewood maximal operator*, Israel J. Math. 67 (1989), 95–108.

[KT] R. A. Kerman and A. Torchinsky, *Integral inequalities with weights for the Hardy maximal function*, Studia Math. 71 (1981), 277–284.

[KR] M. A. Krasnosel'skiĭ and V. B. Rutitskiĭ, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen, 1961.

[MR] F. J. Martín Reyes, *New proofs of weighted inequalities for the one-sided Hardy–Littlewood maximal functions*, Proc. Amer. Math. Soc. 117 (1993), 691–698.

[MOT] F. J. Martín Reyes, P. Ortega Salvador and A. de la Torre, *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. 319 (1990), 517–534.
[M] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, ibid. 165 (1972), 207–226.
[Mu] J. Musielak, *Orlicz Spaces and Modular Spaces*, Springer, 1983.
[S] E. Sawyer, *Weighted inequalities for the one-sided Hardy–Littlewood maximal functions*, Trans. Amer. Math. Soc. 297 (1986), 53–61.
[SW] E. M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

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B^q for parabolic measures

by

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Abstract. If Ω is a Lip(1, 1/2) domain, μ a doubling measure on $\partial_{\mathrm{p}}\Omega$, $\partial/\partial t - L_i$, $i = 0, 1$, are two parabolic-type operators with coefficients bounded and measurable, $2 \le q < \infty$, then the associated measures ω_0, ω_1 have the property that $\omega_0 \in B^q(\mu)$ implies ω_1 is absolutely continuous with respect to ω_0 whenever a certain Carleson-type condition holds on the difference function of the coefficients of L_1 and L_0. Also $\omega_0 \in B^q(\mu)$ implies $\omega_1 \in B^q(\mu)$ whenever both measures are center-doubling measures. This is B. Dahlberg's result for elliptic measures extended to parabolic-type measures on time-varying domains. The method of proof is that of Fefferman, Kenig and Pipher.

A result of B. Dahlberg on two elliptic measures satisfying a $B^q(\mu)$ condition for μ a doubling measure is extended to parabolic-type measures on time-varying domains. The $B^q(\mu)$ condition for ω on $\partial\Omega$ is

$$\left(\frac{1}{\mu(\triangle_r(Q,s))}\int_{\Psi_r(Q,s)\cap\Omega}\left(\frac{d\omega}{d\mu}(\widehat{Q},\widehat{s})\right)^q d\mu(\widehat{Q},\widehat{s})\right)^{1/q} \le \frac{C}{\mu(\triangle_r)}\int_{\Psi_r\cap\Omega}\frac{d\omega}{d\mu}\,d\mu.$$

Here C is independent of (Q,s), $\triangle_r$ is a boundary cube in $\partial\Omega$, $\Psi_r(Q,s)$ is a cylinder of dimension r centered at (Q,s), and r is any real number with $0 < r < r_0$.

Dahlberg [D] proved that if one elliptic measure ω_0 is in $B^q(\mu)$ and if a certain Carleson-type condition holds for the difference function of the coefficients of two elliptic operators L_0, L_1 on a domain D with respect to a doubling measure μ on ∂D, then the second measure ω_1 is also in $B^q(\mu)$.

The main result of this paper is to obtain the preservation of the B^q condition for parabolic-type operators on Lip(1, 1/2) domains in $\mathbb{R}^{n+1}$. This result has been proved independently by Professor Kaj Nystrom [N].

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