The minimum diagonal element of a positive matrix

by

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Abstract. Properties of the minimum diagonal element of a positive matrix are exploited to obtain new bounds on the eigenvalues thus exhibiting a spectral bias along the positive real axis familiar in Perron-Frobenius theory.

The $(i, j)$th entry of an $n \times n$ matrix $T$ is written $[T]_{ij}$. The matrix $T$ is positive ($T \geq 0$) if $[T]_{ij} \geq 0 \quad (\forall i, j)$ while $T$ is strictly positive ($T > 0$) if $[T]_{ij} > 0 \quad (\forall i, j)$. The spectrum, or set of eigenvalues of $T$, is denoted $\sigma(T)$, the spectral radius $r(T)$ and the peripheral spectrum

$$\text{Per } \sigma(T) = \{ \lambda \in \mathbb{C} : \lambda \in \sigma(T), |\lambda| = r(T) \}.$$ 

The trace of $T$ will be written $\text{tr}(T)$, and the complex field $\mathbb{C}$.

This paper combines properties of the minimum diagonal element $\epsilon(T)$ of a positive matrix $T$,

$$\epsilon(T) = \min_{1 \leq i \leq n} [T]_{ii},$$

with elementary spectral theory to show that $\sigma(T)$ lies inside the disc centred at $(\epsilon(T), 0)$ with radius $r(T) - \epsilon(T)$ (Proposition 6 and Figure 1) generalising a result known for stochastic matrices ([2], III.3.4.1). Various improvements of this result are then considered.

We start with the elementary properties of $\epsilon(T)$ for positive $T$, and we note that $S \geq T \geq 0$ implies that $r(S) \geq r(T)$.

Lemma 1. (i) If $T \geq 0$ then $\epsilon(T) \leq n^{-1} \text{tr}(T) \leq r(T)$.

(ii) If $S, T \geq 0$ then $\epsilon(ST) \geq \epsilon(S) \epsilon(T)$.

Proof. Property (i) follows from the fact that the trace of $T$ is the sum of the eigenvalues of $T$ repeated according to multiplicity. For (ii) we have

$$[ST]_{ii} = \sum_{k=1}^{n} [S]_{ik}[T]_{ki} \geq [S]_{ii}[T]_{ii} \quad (\forall i),$$

hence $[ST]_{ii} \geq \epsilon(S) \epsilon(T)$, giving $\epsilon(ST) \geq \epsilon(S) \epsilon(T)$.

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Proposition 2. If $T \geq 0$ and $\varepsilon(T) > 0$ then $r(T) > 0$ and
\[\text{Per } \sigma(T) = \{r(T)\}.\]

Proof. Put $\varepsilon(T) = \varepsilon > 0$. Then $r(T) \geq \varepsilon > 0$ and $T \geq T - \varepsilon I \geq 0$.
Hence $r(T) \geq r(T - \varepsilon I)$, therefore
\[(1) \quad \sigma(T - \varepsilon I) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}.\]
But, by the spectral mapping theorem,
\[\sigma(T - \varepsilon I) = \sigma(T) - \varepsilon.\]

Now suppose $\pi/2 \leq \theta \leq 3\pi/2$ and that $\mu = e^{i\theta} r(T) \in \sigma(T)$. The point $\mu - \varepsilon$ lies strictly outside the disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\}$ contradicting (1). It follows that
\[\text{arc} \{e^{i\theta} r(T) : \pi/2 \leq \theta \leq 3\pi/2 \} \cap \sigma(T) = \emptyset.\]
If now $|\mu| = r(T)$ but $\mu \notin \sigma(T)$ then for some positive integer $m$,
\[\mu^m \in \text{arc} \{e^{i\theta} r(T^m) : \pi/2 \leq \theta \leq 3\pi/2 \}.\]
But $T^m \geq 0$ and, by Lemma 1,
\[\varepsilon(T^m) \geq \varepsilon(T)^m > 0.\]
Hence $\mu^m \notin \sigma(T^m)$, therefore $\mu \notin \sigma(T)$, which completes the proof.

Corollary 3. If $T > 0$ then $\text{Per } \sigma(T) = \{r(T)\}$.

Corollary 4. If $T \geq 0$ then $r(T) \in \text{Per } \sigma(T)$.

Proof. Suppose that $T \geq 0$ and $\delta > 0$. Then
\[\varepsilon(T + \delta I) \geq \delta > 0,\]
hence, by Proposition 2,
\[\text{Per } \sigma(T + \delta I) = \{r(T + \delta I)\}\]
and $r(T + \delta I) \geq r(T)$. Thus $T + \delta I$ has a real eigenvalue not less than $r(T)$, hence $T$ has a real eigenvalue not less than $r(T) - \delta$ for $0 \leq \delta \leq r(T)$. Since $\delta$ is arbitrary, $r(T) \in \text{Per } \sigma(T)$.

It is now clear that if $T \geq 0$ and $\delta \geq 0$ then $r(T + \delta I) = r(T) + \delta$. This result can be extended.

Lemma 5. If $T \geq 0$ then $r(T + \delta I) = r(T) + \delta$ for $\delta \geq -\varepsilon(T)$.

Proof. $T + \delta I \geq 0$ for $\delta \geq -\varepsilon(T)$ so, by Corollary 4, $r(T + \delta I)$ is the maximum point on the real axis in $\sigma(T + \delta I)$ for this range of values of $\delta$. Similarly $r(T)$ is the maximum point on the real axis in $\sigma(T)$. But now
\[r(T) + \delta \in \sigma(T) + \delta = \sigma(T + \delta I).\]
Thus $r(T) + \delta$ is the maximum point on the real axis in $\sigma(T + \delta I)$. Hence
\[r(T + \delta I) = r(T) + \delta \quad \text{for } \delta \geq -\varepsilon(T).\]

This result can often be improved upon. Consider the positive matrix
\[T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.\]
Since $\varepsilon(T) = 0$ the two spectral discs of Figure 1 coincide giving nothing new.
Observe however that, by the invariance of the trace and of the determinant under similarity, $T$ is similar to the positive matrix
\[S = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix},\]
where $0 \leq a \leq 1$, $0 \leq b, c$ and $bc = a(1 - a)$; and that all positive matrices similar to $T$ have this form. Further
\[\varepsilon(S) = \min_{0 \leq a \leq 1} \{a, 1 - a\},\]
and the maximum value which \( \varepsilon(S) \) can take is 1/2. Since the spectrum is invariant under similarity we see that for each \( S \geq 0 \) and similar to \( T \) we can replace \( \varepsilon(T) \) by \( \varepsilon(S) \) in Proposition 6 and Figure 1.

This suggests that, for \( T \geq 0 \), we introduce the subset of the positive real axis

\[
\Delta(T) = \{ \varepsilon(S) : S \geq 0 \text{ and similar to } T \}
\]

and put \( \eta(T) = \sup \Delta(T) \). Note that since the trace is invariant under similarity it follows from Lemma 1 that

\[
\eta(T) \leq n^{-1} \text{tr}(T).
\]

Observe that, in the previous example,

\[
\Delta(T) = [0, \frac{1}{2}] \quad \text{and} \quad \eta(T) = \frac{1}{2} = \frac{1}{2} \text{tr}(T).
\]

Our improved eccentric disc theorem states that \( \varepsilon(T) \) can be replaced by \( \eta(T) \) in Proposition 6 and Figure 1.

**Proposition 7.** If \( T \geq 0 \) then

\[
\sigma(T) \subset \{ \lambda \in \mathbb{C} : |\lambda - \eta(T)| \leq r(T) - \eta(T) \}.
\]

**Proof.** Let \( \mu \in \sigma(T) \), and suppose that

\[
|\mu - \eta(T)| > r(T) - \eta(T).
\]

Then there exists \( \delta > 0 \) such that

\[
|\mu - \eta(T)| = r(T) - \eta(T) + \delta.
\]

Choose \( S \geq 0 \) and similar to \( T \) such that

\[
\eta(T) - \varepsilon(S) < \delta/3.
\]

Then, by Proposition 6,

\[
|\mu - \eta(T)| \leq |\mu - \varepsilon(S)| + |\eta(T) - \varepsilon(S)| \leq r(S) - \varepsilon(S) + \delta/3 = r(T) - \eta(T) + \eta(T) - \varepsilon(S) + \delta/3 \leq r(T) - \eta(T) + 2\delta/3
\]

by (3). This contradicts equation (2), therefore our original assumption is false.

**Corollary 8.** If \( T \geq 0 \) and \( \text{Per } \sigma(T) \) contains two distinct eigenvalues of \( T \) then

\[
\varepsilon(T) = \eta(T) = 0.
\]

It is easy to see that the converse of Corollary 8 is false. Consider the positive matrix

\[
T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\]

Then \( \text{tr}(T) = 0 \) so \( \eta(T) = 0 \) but \( \sigma(T) = \{-1, 2\} \).

If \( S \) is a positive matrix we know that \( \eta(S) \leq n^{-1} \text{tr}(S) \). The next example shows that this inequality may be strict. Consider

\[
S = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

Then \( \text{Per } \sigma(S) = \{-1, 1\} \), thus, by Corollary 8, \( \eta(S) = 0 \). But \( \text{tr}(S) = 1 \) so \( \eta(S) \neq \frac{1}{3} \text{tr}(S) \).

**Problem.** For \( T \geq 0 \) is \( \Delta(T) \) a closed subinterval of the positive real axis?

**Remark.** The minimum diagonal element of a real matrix is an object which hitherto seems to have attracted surprisingly little attention. The class \( \mathcal{K} \), introduced by Fiedler and Pták [1], consists of matrices whose off-diagonal elements are negative (\( \leq 0 \)) and whose real eigenvalues are strictly positive; their inverses lie within the class of positive matrices (1).2.

Let \( \mathcal{N} \) denote the class of negative inverses matrices \( (N \in \mathcal{N} \Leftrightarrow [N]_{ij} \leq 0 \quad \forall i, j) \). Then a strong duality exists between classes \( \mathcal{N} \) and \( \mathcal{K} \) based on shifts (adding real multiples of the identity). When a sufficiently large positive shift is added to a matrix in \( \mathcal{N} \) the sum is in \( \mathcal{K} \). Conversely, a big enough negative shift added to a member of \( \mathcal{K} \) yields a member of \( \mathcal{N} \). More precisely, for real \( \lambda \),

(i) if \( T \in \mathcal{N} \) then \( T + \lambda I \in \mathcal{K} \Leftrightarrow \lambda > r(T) \); while

(ii) if \( T \in \mathcal{K} \) then \( T - \lambda I \in \mathcal{N} \Leftrightarrow \lambda \geq \max_{1 \leq i \leq n} |T|_{ii} \).

Now Perron–Frobenius theory for positive matrices may be employed to derive results for matrices in class \( \mathcal{K} \). For example our Proposition 6 quickly leads to 4.8 of [1].

**References**
