STUDIA MATHEMATICA 131 (1) (1998)

The ratio and generating function of cogrowth coefficients of finitely generated groups

by

RYSZARD SZWARC (Wrocław)

Abstract. Let G be a group generated by r elements $g_1, \ldots, g_r$. Among the reduced words in $g_1, \ldots, g_r$ of length n some, say γ_n, represent the identity element of the group G. It has been shown in a combinatorial way that the $2n$th root of γ_{2n} has a limit, called the cogrowth exponent with respect to the generators $g_1, \ldots, g_r$. We show by analytic methods that the numbers γ_n vary regularly, i.e. the ratio $\gamma_{2n+2}/\gamma_{2n}$ is also convergent. Moreover, we derive new precise information on the domain of holomorphy of $\gamma(z)$, the generating function associated with the coefficients γ_n.

Every group G generated by r elements can be realized as a quotient of the free group $\mathbb{F}_r$ on r generators by a normal subgroup N of $\mathbb{F}_r$, in such a way that the generators of the free group $\mathbb{F}_r$ are sent to the generators of the group G. With the set of generators of $\mathbb{F}_r$ we associate the length function of words in these generators. The *cogrowth coefficients* $\gamma_n = \#\{x \in N \mid |x| = n\}$ were first introduced by Grigorchuk in [2]. They measure how large the group G is compared with $\mathbb{F}_r$. It has been shown that the quantities $\sqrt[2n]{\gamma_{2n}}$ have a limit, denoted by γ and called the *cogrowth exponent* of N in $\mathbb{F}_r$. Since the subgroup N can have at most $2r(2r-1)^{n-1}$ elements of length n, the cogrowth exponent γ can be at most $2r-1$. The famous Grigorchuk result, proved independently by J. M. Cohen in [1], states that the group G is amenable if and only if $\gamma = 2r-1$ (see also [6], [8]).

The main result of this note is that the coefficients γ_{2n} satisfy not only the Cauchy nth root test but also the d'Alembert ratio test.

THEOREM 1. *The ratio of two consecutive even cogrowth coefficients $\gamma_{2n+2}/\gamma_{2n}$ has a limit. Thus the ratio tends to γ^2, the square of the cogrowth exponent.*

1991 *Mathematics Subject Classification*: Primary 20F05, 20E05.

Key words and phrases: cogrowth of subgroups, free group, amenable groups.

Research partially supported by KBN (Poland).

Proof. Let $g_1,\dots,g_r$ be generators of G. Let μ be the measure equidistributed over the generators and their inverses according to the formula

$$\mu=\frac{1}{2\sqrt{q}}\sum_{i=1}^{r}(g_i+g_i^{-1}),$$

where $q=2r-1$. By an easy transformation of [6, Formula (*)] we obtain

$$\frac{z}{1-z^2}\sum_{n=0}^{\infty}\gamma_n z^n=\frac{1}{2\sqrt{q}}\sum_{n=0}^{\infty}\mu^{*n}(e)\left(\frac{2\sqrt{q}\,z}{qz^2+1}\right)^{n+1}, \tag{1}$$

for small values of $|z|$. Let ϱ denote the spectral radius of the random walk defined by μ, i.e.

$$\varrho=\lim_{n\to\infty}\sqrt[2n]{\mu^{*2n}(e)}.$$

We denote by $d\sigma(x)$ the spectral measure of this random walk. Hence

$$\mu^{*n}(e)=\int_{-\varrho}^{\varrho}x^n\,d\sigma(x). \tag{2}$$

Note that ϱ belongs to the support of σ. Combining (1) and (2) gives

$$\begin{aligned}\frac{z}{1-z^2}\sum_{n=0}^{\infty}\gamma_n z^n&=\frac{1}{2\sqrt{q}}\int_{-\varrho}^{\varrho}\sum_{n=0}^{\infty}x^n\left(\frac{2\sqrt{q}\,z}{qz^2+1}\right)^{n+1}d\sigma(x)\\&=\frac{1}{2\sqrt{q}}\int_{-\varrho}^{\varrho}\frac{z}{1-2\sqrt{q}\,xz+qz^2}\,d\sigma(x).\end{aligned} \tag{3}$$

By the well known formula for the generating function of the second kind Chebyshev polynomials $U_n(x)$ (see [4, (4.7.23), p. 82]) where

$$U_n\left(\tfrac12(t+t^{-1})\right)=\frac{t^{n+1}-t^{-n-1}}{t-t^{-1}}, \tag{4}$$

we have

$$\frac{1}{1-2\sqrt{q}\,xz+qz^2}=\sum_{n=0}^{\infty}U_n(x)q^{n/2}z^n.$$

Thus

$$\frac{z}{1-z^2}\sum_{n=0}^{\infty}\gamma_n z^n=z\sum_{n=0}^{\infty}q^{n/2}z^n\int_{-\varrho}^{\varrho}U_n(x)\,d\sigma(x).$$

Therefore for $n\ge 2$ we have

$$\gamma_n=q^{n/2}\int_{-\varrho}^{\varrho}\{U_n(x)-q^{-1}U_{n-2}(x)\}\,d\sigma(x). \tag{5}$$

Since $U_{2n}(-x)=U_{2n}(x)$ we get

$$\gamma_{2n}=q^n\int_0^{\varrho}\{U_{2n}(x)-q^{-1}U_{2n-2}(x)\}\,d\widetilde{\sigma}(x), \tag{6}$$

where $\widetilde{\sigma}(A)=\sigma(A)+\sigma(-A)$ for $A\subset(0,\varrho]$ and $\widetilde{\sigma}(\{0\})=\sigma(\{0\})$. Let

$$I_n=\int_0^{\varrho}\{U_{2n}(x)-q^{-1}U_{2n-2}(x)\}\,d\widetilde{\sigma}(x).$$

By [3, Corollary 2] we have $\varrho>1$. Hence we can split the integral I_n into two integrals: $I_{n,1}$ over $[0,\varrho_0]$, and $I_{n,2}$ over $[\varrho_0,\varrho]$, where $\varrho_0=(1+\varrho)/2$. By (4) we have $|U_m(x)|\le m+1$ for $x\in[0,1]$ and

$$|U_m(x)|\le(m+1)[x+\sqrt{x^2-1}]^m\quad\text{ for } x\ge 1.$$

Thus we get

$$\begin{aligned}I_{n,1}&\le 2(2n+1)(\varrho_0+\sqrt{\varrho_0^2-1})^{2n}\int_0^{\varrho_0}d\widetilde{\sigma}(x)\\&\le 2(2n+1)(\varrho_0+\sqrt{\varrho_0^2-1})^{2n}.\end{aligned} \tag{7}$$

Let us turn to estimating $I_{n,2}$. By (4) one can easily check that

$$\left|U_n(x)-\frac{(x+\sqrt{x^2-1})^{n+1}}{2\sqrt{x^2-1}}\right|=o(1)\quad\text{ as } n\to\infty,$$

uniformly on $[\varrho_0,\varrho]$. Hence

$$\left|U_{2n}(x)-q^{-1}U_{2n-2}(x)-(x+\sqrt{x^2-1})^{2n-1}\frac{(x+\sqrt{x^2-1})^2-q^{-1}}{2\sqrt{x^2-1}}\right|=o(1)$$

as $n\to\infty$, uniformly on $[\varrho_0,\varrho]$. This implies

$$I_{n,2}\approx\widetilde{I}_{n,2}=\int_{\varrho_0}^{\varrho}(x+\sqrt{x^2-1})^{2n}\frac{(x+\sqrt{x^2-1})^2-q^{-1}}{2\sqrt{x^2-1}(x+\sqrt{x^2-1})}\,d\widetilde{\sigma}(x). \tag{8}$$

Since the endpoint ϱ belongs to the support of $\widetilde{\sigma}$, we get

$$\widetilde{I}_{n,2}^{1/(2n)}\to\varrho+\sqrt{\varrho^2-1}. \tag{9}$$

By combining this with (7) and (8) we obtain

$$I_n=I_{n,1}+I_{n,2}=\widetilde{I}_{n,2}(1+o(1))\quad\text{ as } n\to\infty. \tag{10}$$

In view of (9) the integral $\widetilde{I}_{n,2}$ tends to infinity. Thus by (6) and (10) we have

$$\frac{\gamma_{2n+2}}{\gamma_{2n}}\approx q\frac{\widetilde{I}_{n+1,2}}{\widetilde{I}_{n,2}}.$$

LEMMA 1 ([7]). *Let f be a positive and continuous function on $[a,b]$, and μ be a finite measure on $[a,b]$. Then*

$$\lim_{n\to\infty}\frac{\int_a^b f(x)^{n+1}\,d\mu(x)}{\int_a^b f(x)^n\,d\mu(x)}=\max\{f(x)\mid x\in\operatorname{supp}\mu\}.$$

Applying Lemma 1 and using the fact that ϱ belongs to the support of $\widetilde{\sigma}$ gives

$$\frac{\gamma_{2n+2}}{\gamma_{2n}}\to qz\{\varrho+\sqrt{\varrho^2-1}\}^2. \blacksquare \tag{11}$$

THEOREM 2. *The generating function $\gamma(z)=\sum_{n=0}^{\infty}\gamma_n z^n$ can be decomposed into a sum of two functions $\gamma^{(0)}(z)$ and $\gamma^{(1)}(z)$ such that $\gamma^{(0)}(z)$ is analytic in the open disc of radius $q^{-1/2}$ (where $q=2r-1$), while $\gamma^{(1)}(z)$ is analytic in the whole complex plane with the two real intervals $[-\gamma q^{-1},\,-\gamma^{-1}]$ and $[\gamma^{-1},\gamma q^{-1}]$ removed. Moreover, $\gamma^{(1)}$ satisfies the functional equation*

$$\frac{z\gamma^{(1)}(z)}{1-z^2}=\frac{(q/z)\gamma^{(1)}(q/z)}{q/z}.$$

Proof. By (3) we have

$$\gamma(z)=(1-z^2)\int_{-\varrho}^{\varrho}\frac{1}{1-2\sqrt{q}\,xz+qz^2}\,d\sigma(x).$$

Let

$$\gamma^{(0)}(z)=(1-z^2)\int_{-1}^{1}\frac{1}{1-2\sqrt{q}\,xz+qz^2}\,d\sigma(x),$$

$$\gamma^{(1)}(z)=(1-z^2)\int_{1<|x|\le\varrho}\frac{1}{1-2\sqrt{q}\,xz+qz^2}\,d\sigma(x).$$

For $-1\le x\le 1$ the expression $1-2\sqrt{q}\,xz+qz^2$ vanishes only on the circle of radius $q^{-1/2}$. Thus $\gamma^{(0)}(z)$ has the desired property. For $1<|x|\le\varrho$ the same expression vanishes only on the intervals

$$\left[-\frac{\varrho+\sqrt{\varrho^2-1}}{\sqrt{q}},\,-\frac{\varrho-\sqrt{\varrho^2-1}}{\sqrt{q}}\right],\quad\left[\frac{\varrho-\sqrt{\varrho^2-1}}{\sqrt{q}},\,\frac{\varrho+\sqrt{\varrho^2-1}}{\sqrt{q}}\right].$$

By (11) we have $\gamma=q^{1/2}(\varrho+\sqrt{\varrho^2-1})$. This shows that $\gamma^{(1)}$ is analytic where required.

The functional equation follows immediately from the formula

$$\frac{z\gamma^{(1)}(z)}{1-z^2}=\int_{1<|x|\le\varrho}\frac{1}{z^{-1}-2\sqrt{q}\,x+qz}\,d\sigma(x). \blacksquare$$

REMARK. Combining (6) and (10) yields

$$\gamma_{2n}=q^n\left\{\int_{\varrho_0}^{\varrho}(x+\sqrt{x^2-1})^{2n}\frac{(x+\sqrt{x^2-1})^2-q^{-1}}{2\sqrt{x^2-1}\,(x+\sqrt{x^2-1})}\,d\widetilde{\sigma}(x)+o(1)\right\}.$$

We have

$$h(\varrho_0):=\frac{(\varrho_0+\sqrt{\varrho_0^2-1})^2-q^{-1}}{2\sqrt{\varrho_0^2-1}\,(\varrho_0+\sqrt{\varrho_0^2-1})}\ge\frac{(x+\sqrt{x^2-1})^2-q^{-1}}{2\sqrt{x^2-1}\,(x+\sqrt{x^2-1})},$$

$$\frac{\varrho+\sqrt{\varrho^2-1}}{\varrho}\,x\ge x+\sqrt{x^2-1}.$$

Therefore, in view of (2), we get

$$\gamma_{2n}\le q^n\left\{h(\varrho_0)\frac{(\varrho+\sqrt{\varrho^2-1})^{2n}}{\varrho^{2n}}\int_0^{\varrho}x^{2n}\,d\widetilde{\sigma}(x)+o(1)\right\}$$

$$=q^n h(\varrho_0)\left\{(\varrho+\sqrt{\varrho^2-1})^{2n}\frac{\mu^{*2n}(e)}{\varrho^{2n}}+o(1)\right\}.$$

Finally, we obtain

$$\frac{\gamma_{2n}}{\gamma^{2n}}\frac{\varrho^{2n}}{\mu^{*2n}(e)}=\frac{\gamma_{2n}}{\mu^{*2n}(e)}\left\{\frac{\varrho}{\sqrt{q}(\varrho+\sqrt{\varrho^2-1})}\right\}^{2n}\le h(\varrho_0)+o(1).$$

We conjecture that the opposite estimate also holds, i.e. the quantity on the left hand side is bounded away from zero, by a positive constant depending only on ϱ. This conjecture can be checked easily if the measure σ is smooth in the neighbourhood of ϱ and the density has a zero of finite order at ϱ.

Acknowledgements. For a long time I thought Theorem 1 followed from the following statement.

If $f(z)=\sum_{n=0}^{\infty}a_nz^n$ is analytic in the complex plane except the half line $[1,\infty)$, then the ratio a_{n+1}/a_n converges to 1.

I am grateful to Jacek Zienkiewicz from my Department for constructing a fine counterexample to this statement.

References

[1] J. M. Cohen, *Cogrowth and amenability of discrete groups*, J. Funct. Anal. 48 (1982), 301–309.

[2] R. I. Grigorchuk, *Symmetrical random walks on discrete groups*, in: Multicomponent Random Systems, R. L. Dobrushin and Ya. G. Sinai (eds.), Nauka, Moscow, 1978, 132–152 (in Russian); English transl.: Adv. Probab. Related Topics 6, Marcel Dekker, 1980, 285–325.

[3] H. Kesten, *Full Banach mean values on countable groups*, Math. Scand. 7 (1959), 149–156.

[4] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Providence, R.I., 1975.
[5] R. Szwarc, *An analytic series of irreducible representations of the free group*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 1, 87–110.
[6] —, *A short proof of the Grigorchuk-Cohen cogrowth theorem*, Proc. Amer. Math. Soc. 106 (1989), 663–665.
[7] S. Wagon, *Elementary problem E 3226*, Amer. Math. Monthly 94 (1987), 786–787.
[8] W. Woess, *Cogrowth of groups and simple random walks*, Arch. Math. (Basel) 41 (1983), 363–370.

Institute of Mathematics
Wrocław University
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: szwarc@math.uni.wroc.pl

Received November 3, 1997 (3988)

STUDIA MATHEMATICA 131 (1) (1998)

The minimum diagonal element of a positive matrix

by

M. R. F. SMYTH (Belfast) and T. T. WEST (Dublin)

Abstract. Properties of the minimum diagonal element of a positive matrix are exploited to obtain new bounds on the eigenvalues thus exhibiting a spectral bias along the positive real axis familiar in Perron–Frobenius theory.

The (i,j)th entry of an $n \times n$ matrix T is written $[T]_{ij}$. The matrix T is *positive* ($T \geq 0$) if $[T]_{ij} \geq 0$ $(\forall i,j)$ while T is *strictly positive* ($T > 0$) if $[T]_{ij} > 0$ $(\forall i,j)$. The spectrum, or set of eigenvalues of T, is denoted $\sigma(T)$, the spectral radius $r(T)$ and the peripheral spectrum

$$\operatorname{Per}\sigma(T) = \{\lambda \in \mathbb{C} : \lambda \in \sigma(T),\ |\lambda| = r(T)\}.$$

The trace of T will be written $\operatorname{tr}(T)$, and the complex field $\mathbb{C}$.

This paper combines properties of the minimum diagonal element $\varepsilon(T)$ of a positive matrix T,

$$\varepsilon(T) = \min_{1\leq i\leq n} [T]_{ii},$$

with elementary spectral theory to show that $\sigma(T)$ lies inside the disc centred at $(\varepsilon(T), 0)$ with radius $r(T) - \varepsilon(T)$ (Proposition 6 and Figure 1) generalising a result known for stochastic matrices ([2], III.3.4.1). Various improvements of this result are then considered.

We start with the elementary properties of $\varepsilon(T)$ for positive T, and we note that $S \geq T \geq 0$ implies that $r(S) \geq r(T)$.

LEMMA 1. (i) *If $T \geq 0$ then $\varepsilon(T) \leq n^{-1}\operatorname{tr}(T) \leq r(T)$.*

(ii) *If $S, T \geq 0$ then $\varepsilon(ST) \geq \varepsilon(S)\varepsilon(T)$.*

Proof. Property (i) follows from the fact that the trace of T is the sum of the eigenvalues of T repeated according to multiplicity. For (ii) we have

$$[ST]_{ii} = \sum_{k=1}^{n} [S]_{ik}[T]_{ki} \geq [S]_{ii}[T]_{ii} \quad (\forall i),$$

hence $[ST]_{ii} \geq \varepsilon(S)\varepsilon(T)$, giving $\varepsilon(ST) \geq \varepsilon(S)\varepsilon(T)$.

1991 *Mathematics Subject Classification*: Primary 15A48.